POLYNOMIAL FUNCTIONS

Objectives

The following is a list of objectives for this section of the workbook.

By the time the student is finished with this section of the workbook, he/she should be able to…

- ***Find the quotient of a division problem involving polynomials using the polynomial long division method.***
- ***Find the quotient of a division problem involving polynomials using the synthetic division method.***
- ***Use the rational zero test to determine all possible rational zeros of a polynomial function.***
- ***Use the rational zero test to determine all possible roots of a polynomial equation.***
- ***Use Descarte's Rule of Signs to determine the possible number of positive or negative roots of a polynomial equation.***
- ***Find all zeros of a polynomial function.***
- ***Use the remainder theorem to evaluate the value of functions.***
- ***Write a polynomial in completely factored form.***
- ***Write a polynomial as a product of factors irreducible over the reals.***
- ***Write a polynomial as a product of factors irreducible over the rationals.***
- ***Find the equation of a polynomial function that has the given zeros.***
- ***Determine if a polynomial function is even, odd or neither.***
- ***Determine the left and right behaviors of a polynomial function without graphing.***
- ***Find the local maxima and minima of a polynomial function.***
- ***Find all x intercepts of a polynomial function.***
- ***Determine the maximum number of turns a given polynomial function may have.***
- ***Graph a polynomial function.***

Math Standards Addressed

The following state standards are addressed in this section of the workbook.

Algebra II

3.0 Students are adept at operations on polynomials, including long division.

4.0 Students factor polynomials representing the difference of squares, perfect square trinomials, and the sum and difference of two cubes.

5.0 Students demonstrate knowledge of how real and complex numbers are related both arithmetically and graphically. In particular, they can plot complex numbers as points in the plane.

6.0 Students add, subtract, multiply, and divide complex numbers.

7.0 Students add, subtract, multiply, divide, reduce, and evaluate rational expressions with monomial and polynomial denominators and simplify complicated rational expressions, including those with negative exponents in the denominator.

8.0 Students solve and graph quadratic equations by factoring, completing the square, or using the quadratic formula. Students apply these techniques in solving word problems. They also solve quadratic equations in the complex number system.

10.0 Students graph quadratic functions and determine the maxima, minima, and zeros of the function.

Mathematical Analysis

4.0 Students know the statement of, and can apply, the fundamental theorem of algebra.

Polynomial Division

There are two methods used to divide polynomials. This first is a traditional long division method, and the second is synthetic division. Using either of these methods will yield the correct answer to a division problem. There are restrictions, however, as to when each can be used.

Synthetic division can only be used if the divisor is a first degree binomial.
For the division problem $\frac{x^3+4x^2-2x+1}{2x-1}$***, the divisor,*** $2x-1$***, is a first degree binomial, so you may use synthetic division.***

There are no restrictions as to when polynomial long division may be used. The polynomial long division method may be used at any time. If the divisor is a polynomial greater than first degree, polynomial long division must be used.

The Division Algorithm

When working with division problems, it will sometimes be necessary to write the solution using The Division Algorithm.

The Division Algorithm: $f_{(x)} = d_{(x)} \cdot q_{(x)} + r_{(x)}$

Simply put, the function = divisor · quotient + remainder

Is 12 divisible by 4?

Is 18 divisible by 3?

Is 15 divisible by 2?

Is 32 divisible by 8?

Based on your observations from the previous questions, what determines divisibility?

How can you determine whether or not the polynomial x^2-3x+2 **is a factor of** x^4+10x^2-4**?**

Find the quotient of each of the following. You may use synthetic or long division, but you need to know when to use each.

A) $\dfrac{6x^3-16x^2+17x-6}{3x-2}$

B) $\dfrac{3x^3-17x^2+15x-25}{x-5}$

C) $\dfrac{x^4+3x^2+1}{x^2-2x+3}$

D) $\dfrac{x^4-x^3-12x^2-2x+8}{x-4}$

E) $\dfrac{6x^3+10x^2+x+8}{2x^2+1}$

F) $\dfrac{x^3-1}{x-1}$

G) $\dfrac{x^5-4x^4+4x^3-13x^2+3x-1}{x^2+3}$

H) $\dfrac{2x^3+5x^2+2x+15}{2x^2-x+5}$

I) $\dfrac{3x^3-16x^2-72}{x-6}$

When dividing polynomials using the long division method, how do you know when you are finished?

Is $x+2$ **a factor of** x^3+8**?**

Is $x-6$ **a factor of** $3x^3-16x^2-72$**?**

Describe the manner in which you determined whether or not the given binomials above were factors of their respective polynomials.

The Rational Zero Test

The ultimate objective for this section of the workbook is to graph polynomial functions of degree greater than 2. The first step in accomplishing this will be to find all real zeros of the function. As previously stated, the zeros of a function are the x intercepts of the graph of that function. Also, the zeros of a function are the roots of the equation that makes up that function. You should remember, the only difference between an polynomial equation and a polynomial function is that one of them has $f_{(x)}$.

You will be given a polynomial equation such as $2x^4+7x^3-4x^2-27x-18=0$***, and be asked to find all roots of the equation.***

The Rational Zero Test states that all possible rational zeros are given by the factors of the constant over the factors of the leading coefficient.

$$\frac{factors \quad of \quad the \quad constant}{factors \quad of \quad the \quad leading \quad coefficient} = all \quad possible \quad rational \quad zeros$$

Let's find all possible rational zeros of the equation $2x^4+7x^3-4x^2-27x-18=0$.

We begin with the equation $2x^4+7x^3-4x^2-27x-18=0$.

The constant of this equation is 18, while the leading coefficient is 2. We do not care about the (-) sign in front of the 18.

Writing out all factors of the constant over the factors of the leading coefficient gives the following.

$$\frac{\pm 1, \quad \pm 2, \quad \pm 3, \quad \pm 6, \quad \pm 9, \quad \pm 18}{\pm 1, \quad \pm 2}$$

These are not all possible rational zeros. To actually find them, take each number on top, and write it over each number in the bottom. If one such number occurs more than once, there is no need to write them both.

$$\pm 1, \quad \pm 2, \quad \pm 3, \quad \pm 6, \quad \pm 9, \quad \pm 18, \quad \pm\frac{1}{2}, \quad \pm\frac{3}{2}, \quad \pm\frac{9}{2}$$

These are all possible rational zeros for this particular equation.

The order in which you write this list of numbers is not important. The rational zero test is meant to assist in the overall objective of finding all zeros to the polynomial equation $2x^4+7x^3-4x^2-27x-18=0$. ***Each of these numbers is a potential root of the equation. Therefore, each will eventually be tested.***

Using the rational zero test, list all <u>possible</u> rational zeros of the following functions.

A) $f_{(x)} = 2x^4 - 6x^2 + 5x - 15$

B) $f_{(x)} = 3x^5 - 6x^4 + 2x^2 - 6x + 12$

C) $f_{(x)} = 8x^3 - 2x + 24$

D) $f_{(x)} = 10x^3 - 15x^2 - 16x + 12$

E) $f_{(x)} = -6x^3 + 5x^2 - 2x + 18$

F) $f_{(x)} = 4x^4 - 16x^3 + 12x - 30$

G) $f_{(x)} = 4x^4 + 3x^3 - 2x^2 + 5x - 12$

H) $f_{(x)} = x^5 - 6x^4 + 12x^2 - 8x + 36$

It is important to understand, these lists of possible zeros for each of the polynomial functions above, are also lists of possible roots for the polynomial equations contained therein.

Descarte's Rule of Signs

When solving these polynomial equations use the rational zero test to find all possible rational zeros first. Synthetic division will then be used to test each one of these possible zeros, until some are found that work. When we found all possible rational zeros of the equation $2x^4 + 7x^3 - 4x^2 - 27x - 18 = 0$***, there were 18 possible solutions to this equation. It could take a very long time to test each one. Luckily there is a rule to help narrow down these choices. Descarte's Rule of Signs can help to narrow the search of possible solutions to the equation.***

Descarte's Rule of Signs

- **The number of positive zeros can be found by counting the number of sign changes in the problem. The number of positive zeros is that number, or less by an even integer.**

- **The number of negative zeros can be found by evaluating $f_{(-x)}$. Count the number of sign changes, and the number of negative zeros is that number, or less by an even integer.**

When using Descarte's Rule of Signs, "less by an even integer," means subtract by two until there is 1 or 0 possible zeros.

Here is an example of how to use Descarte's rule of Signs to determine the possible number of positive and negative zeros for the equation $2x^4 + 7x^3 - 4x^2 - 27x - 18 = 0$.

To find the number of positive roots, count the number of sign changes in $2x^4 + 7x^3 - 4x^2 - 27x - 18 = 0$.

The signs only change once in the original equation, so there is only 1 positive zero.

Evaluating $f_{(-x)}$ ***results in*** $2x^4 - 7x^3 - 4x^2 + 27x - 18 = 0$. ***To evaluate*** $f_{(-x)}$, ***substitute –x for x. When this is done, only the terms were variables are being raised to odd powers change signs.***

Here, the signs changed 3 times. That means there are either 3 or 1 negative zeros.

Knowledge of complex roots will be used in conjunction with Descarte's Rule of Signs to create a table of possible combinations. Remember, COMPLEX NUMBERS ALWAYS COME IN CONJUGATE PAIRS when solving equations.

Using Descarte's Rule of Signs, state the possible number of positive zeros for each of the following functions.

A) $f_{(x)} = 3x^4 - 6x^3 + 2x^2 - x + 2$

B) $f_{(x)} = -x^5 + 2x^4 - 3x^3 - 7x + 2$

C) $f_{(x)} = 3x^6 - 2x^5 + 7x^4 + 5x^3 - x^2 + 2x - 1$

D) $f_{(x)} = -6x^4 - 5x^2 - 8$

E) $f_{(x)} = -5x^5 + 6x^4 - 3x^2 + x - 15$

F) $f_{(x)} = \frac{1}{2}x^6 - 3x^5 + 7x^3 - 5x^2 + x$

G) $f_{(x)} = x^5 - 3x^4 + 2x^3 + 4x^2 + 5x - 12$

H) $f_{(x)} = \frac{2}{3}x^4 - x^3 + 5x^2 - 3x + 2$

Using Descarte's Rule of Signs, state the possible number of negative zeros for each of the following functions.

A) $f_{(x)} = 3x^4 - 6x^3 + 2x^2 - x + 2$

B) $f_{(x)} = -x^5 + 2x^4 - 3x^3 - 7x + 2$

C) $f_{(x)} = 3x^6 - 2x^5 + 7x^4 + 5x^3 - x^2 + 2x - 1$

D) $f_{(x)} = -6x^4 - 5x^2 - 8$

E) $f_{(x)} = -5x^5 + 6x^4 - 3x^2 + x - 15$

F) $f_{(x)} = \frac{1}{2}x^6 - 3x^5 + 7x^3 - 5x^2 + x$

G) $f_{(x)} = x^5 - 3x^4 + 2x^3 + 4x^2 + 5x - 12$

H) $f_{(x)} = \frac{2}{3}x^4 - x^3 + 5x^2 - 3x + 2$

The Remainder Theorem

When trying to find all zeros of a complex polynomial function, use the rational zero test to find all possible rational zeros. Each possible rational zero should then be tested using synthetic division. If one of these numbers work, there will be no remainder to the division problem. For every potential zero that works, there may be others that do not. Are these just useless? The answer is no. Every time synthetic division is attempted, we are actually evaluating the value of the function at the given x coordinate. When there is no remainder left, a zero of the function has just been found. This zero is an x intercept for the graph of the function. If the remainder is any other number, a set of coordinates on the graph has just been found. These coordinates would aid in graphing the function.

Let $P_{(x)}$ be a polynomial of positive degree n. Then for any number c,

$$P_{(x)} = Q_{(x)} \cdot (x-c) + P_{(c)},$$

Where $Q_{(x)}$ is a polynomial of degree n-1.

This simply means that if a polynomial $P_{(x)}$ is divided by $(x-c)$ using synthetic division, the resultant remainder is $P_{(c)}$.

When trying to find the zeros of the function $f_{(x)} = 2x^4 + 7x^3 - 4x^2 - 27x - 18$, first find all possible rational zeros. Then evaluate each one. Here is one particular example.

$$\begin{array}{r|rrrrr} -2 & 2 & 7 & -4 & -27 & -18 \\ & 0 & -4 & -6 & 20 & 14 \\ \hline & 2 & 3 & -10 & -7 & -4 \end{array}$$

In this example, (-2) is evaluated using synthetic division to see if it was a zero of the function. It turns out that (-2) is not a zero of the function, because there is a remainder of (-4).

Therefore, Using the Remainder Theorem, it can be stated that $f_{(-2)} = -4$.

You already saw that dividing by (-2) yields a result of (-4), giving us the statement $f_{(-2)} = -4$. This can be proven algebraically as follows.

$$f_{(-2)} = 2(-2)^4 + 7(-2)^3 - 4(-2)^2 - 27(-2) - 18$$

$$f_{(-2)} = 32 - 56 - 16 + 54 - 18$$

$$f_{(-2)} = -4$$

Given the polynomial function $f_{(x)} = 3x^3 + 2x^2 - 5x + 2$**, use synthetic division to evaluate each of the following.**

A) $f_{(2)} =$ **B)** $f_{(-2)} =$ **C)** $f_{(-3)} =$

Given the polynomial function $f_{(x)} = -2x^3 - 4x^2 - 5x + 12$**, use synthetic division to evaluate each of the following.**

D) $f_{(-1)} =$ **E)** $f_{(-2)} =$ **F)** $f_{(4)} =$

Given the polynomial function $f_{(x)} = -5x^2 - 4x + 6$**, use synthetic division to evaluate each of the following.**

G) $f_{(-1)} =$ **H)** $f_{(-2)} =$ **I)** $f_{(4)} =$

Given the polynomial function $f_{(x)} = 3x^4 - 2x^3 + x^2 - 4x + 1$**, use synthetic division to evaluate each of the following.**

J) $f_{(3)} =$ **K)** $f_{(1)} =$ **L)** $f_{(-2)} =$

Finding all Zeros of a Polynomial Function

When solving polynomial equations, use the rational zero test to find all possible rational zeros, then use Descarte's Rule of Signs to help narrow down the choices if possible. The fundamental theorem of Algebra plays a major role in this.

The Fundamental Theorem of Algebra

Every polynomial equation of degree n with complex coefficients has n roots in the complex numbers.

In other words, if you have a 5th degree polynomial equation, it has 5 roots.

Example: Find all zeros of the polynomial function $f_{(x)} = 2x^4 + 7x^3 - 4x^2 - 27x - 18$.

$$2x^4 + 7x^3 - 4x^2 - 27x - 18 = 0$$

Begin by setting the function equal to zero.

Find all possible rational zeros.

$$\pm 1,\quad \pm 2,\quad \pm 3,\quad \pm 6,\quad \pm 9,\quad \pm 18,\quad \pm\frac{1}{2},\quad \pm\frac{3}{2},\quad \pm\frac{9}{2}$$

Once again, there are 18 possible zeros to the function. If Descarte's Rule of Signs is used, it may or may not help narrow down the choices for synthetic division.

For this equation, there is 1 possible positive zero, and either 3 or 1 possible negative zeros.

This information was found in a previous example. Based on this, a chart may be constructed showing the possible combinations. Remember, this is a 4th degree polynomial, so each row must add up to 4.

+	-	*i*
1	**3**	**0**
1	**1**	**2**

$$\pm 1,\quad \pm 2,\quad \pm 3,\quad \pm 6,\quad \pm 9,\quad \pm 18,\quad \pm\frac{1}{2},\quad \pm\frac{3}{2},\quad \pm\frac{9}{2}$$

Now set up a synthetic division problem, and begin checking each zero until a root of the equation is found..

$$\begin{array}{|rrrrr} 2 & 7 & -4 & -27 & -18 \\ 0 & & & & \\ \hline \end{array}$$

$$\begin{array}{r|rrrrr} -1 & 2 & 7 & -4 & -27 & -18 \\ & 0 & -2 & -5 & 9 & 18 \\ \hline & 2 & 5 & -9 & -18 & 0 \end{array}$$

-1 works as a zero of the function. There are now 3 zeros left. We can continue to test each zero, but we need to first rewrite the new polynomial.

$2x^3 + 5x^5 - 9x - 18$

The reason this must be done is to check using the rational zero test again. Using the rational zero test again could reduce the number of choices to work with, or the new polynomial may be factorable.

$$\begin{array}{r|rrrr} -3 & 2 & 5 & -9 & -18 \\ & 0 & -6 & 3 & 18 \\ \hline & 2 & -1 & -6 & 0 \end{array}$$

Here, we found that -3 works. The reason negative numbers are being used first here is because of the chart above. The chart says there is a greater chance of one of the negatives working rather than a positive, since there are potentially 3 negative zeros here and only one positive. Notice the new equation was used for the division.

$$2x^2 - x - 6$$
$$(2x+3)(x-2) = 0$$
$$x = -3/2 \quad and \quad x = 2$$

This is now a factorable polynomial. Solve by factoring.

We now have all zeros of the polynomial function. They are $-3/2,\ -1,\ -3\ \ and\ \ 2$

Be aware, the remaining polynomial may not be factorable. In that case, it will be necessary to either use the quadratic formula, or complete the square.

Find all <u>real</u> zeros of the following functions (no complex numbers). **Remember, if there is no constant with which to use the rational zero test, factor out a zero first, then proceed.**

A) $f_{(x)} = x^3 - 6x^2 + 11x - 6$

B) $f_{(x)} = x^3 - 9x^2 + 27x - 27$

C) $f_{(x)} = x^3 - 9x^2 + 20x - 12$

D) $f_{(x)} = x^4 - 7x^2 + 12$

E) $f_{(x)} = x^5 - 7x^4 + 10x^3 + 14x^2 - 24x$

F) $f_{(x)} = x^4 - 13x^2 - 12x$

There is a difference between the questions: "find all real zeros" and "find all zeros." Be careful to pay attention to which is being asked.

The following questions, denoted by *, show up in the "**Writing a Polynomial Function in Factored Form"** ***topic of the workbook. Since these functions show up later, the zeros of these functions only need to be found once. As we move on to that topic, on page 326, the answers found below will be used to rewrite the polynomial in the manner indicated.***

***Find <u>all</u> zeros of the following functions. (Include any complex solutions)**

A) $f_{(x)} = x^4 - 81$

B) $f_{(x)} = x^4 - 7x^2 + 12$

C) $f_{(x)} = x^3 - x + 6$

D) $f_{(x)} = x^6 + 4x^4 - 41x^2 + 36$

E) $f_{(x)} = x^4 + 10x^2 + 9$

F) $f_{(x)} = x^4 - x^3 + 25x^2 - 25x$

G) $f_{(x)} = x^4 - x^3 - 2x^2 - 4x - 24$

H) $f_{(x)} = x^4 - x^3 - 29x^2 - x - 30$

I) $f_{(x)} = x^3 - x^2 - 3x + 3$

J) $f_{(x)} = x^4 - 7x^2 + 10$

K) $f_{(x)} = x^3 - 6x^2 + 13x - 10$

L) $f_{(x)} = x^5 + 15x^3 - 16x$

Writing a Polynomial Function in Factored Form

Once all zeros of a polynomial function are found, the function can be rewritten in one of several different ways.

A polynomial function may be written in one of the following ways.

- ***As a product of factors that are irreducible over the rationals.***
  This means only rational numbers may be used in the factors.

- ***As a product of factors that are irreducible over the reals.***
  Irrational numbers may be used as long as they are real, i.e. $\left(x+\sqrt{3}\right)\left(x-\sqrt{3}\right)$.

- ***In completely factored form. This may also be written as a product of linear factors.***
  Complex numbers may be used in the factors, i.e. $(x+2i)(x-2i)$.

This section involves writing polynomials in one of the factored forms illustrated above. These are the same problems that were solved on the previous page, so there is no need to solve them again. Use the solutions previously found, to write the polynomial in the desired form.

For example, a polynomial function that has zeros of 3 ***and*** $2\pm\sqrt{3}$ ***would look like the following; in completely factored form.***

$$f_{(x)}=(x-3)\left(x-2+\sqrt{3}\right)\left(x-2-\sqrt{3}\right)$$

Notice each variable x is to the first power, so these are linear factors.

When polynomial functions are written like this, it is obvious where the x intercepts lie.

***Write the polynomial function as a product of factors that are irreducible over the reals.**

A) $f_{(x)} = x^4 - 81$ **B)** $f_{(x)} = x^4 - 7x^2 + 12$ **C)** $f_{(x)} = x^3 - x + 6$

D) $f_{(x)} = x^6 + 4x^4 - 41x^2 + 36$ **E)** $f_{(x)} = x^4 + 10x^2 + 9$ **F)** $f_{(x)} = x^4 - x^3 + 25x^2 - 25x$

G) $f_{(x)} = x^4 - x^3 - 2x^2 - 4x - 24$ **H)** $f_{(x)} = x^4 - x^3 - 29x^2 - x - 30$ **I)** $f_{(x)} = x^3 - x^2 - 3x + 3$

J) $f_{(x)} = x^4 - 7x^2 + 10$ **K)** $f_{(x)} = x^3 - 6x^2 + 13x - 10$ **L)** $f_{(x)} = x^5 + 15x^3 - 16x$

***Write the polynomial function as a product of factors that are irreducible over the rationals.**

A) $f_{(x)} = x^4 - 81$ **B)** $f_{(x)} = x^4 - 7x^2 + 12$ **C)** $f_{(x)} = x^3 - x + 6$

D) $f_{(x)} = x^6 + 4x^4 - 41x^2 + 36$ **E)** $f_{(x)} = x^4 + 10x^2 + 9$ **F)** $f_{(x)} = x^4 - x^3 + 25x^2 - 25x$

G) $f_{(x)} = x^4 - x^3 - 2x^2 - 4x - 24$ **H)** $f_{(x)} = x^4 - x^3 - 29x^2 - x - 30$ **I)** $f_{(x)} = x^3 - x^2 - 3x + 3$

J) $f_{(x)} = x^4 - 7x^2 + 10$ **K)** $f_{(x)} = x^3 - 6x^2 + 13x - 10$ **L)** $f_{(x)} = x^5 + 15x^3 - 16x$

***Write the polynomial functions in completely factored form. (Remember, this can also be asked in the form "Write polynomial as a product of linear factors.")**

A) $f_{(x)} = x^4 - 81$

B) $f_{(x)} = x^4 - 7x^2 + 12$

C) $f_{(x)} = x^3 - x + 6$

D) $f_{(x)} = x^6 + 4x^4 - 41x^2 + 36$

E) $f_{(x)} = x^4 + 10x^2 + 9$

F) $f_{(x)} = x^4 - x^3 + 25x^2 - 25x$

G) $f_{(x)} = x^4 - x^3 - 2x^2 - 4x - 24$

H) $f_{(x)} = x^4 - x^3 - 29x^2 - x - 30$

I) $f_{(x)} = x^3 - x^2 - 3x + 3$

J) $f_{(x)} = x^4 - 7x^2 + 10$

K) $f_{(x)} = x^3 - 6x^2 + 13x - 10$

L) $f_{(x)} = x^5 + 15x^3 - 16x$

Write $x^4 - 81$ **as a product of linear factors.**

Write $x^4 - 16$ **as a product of linear factors.**

How can you identify linear factors?

Finding the Equation of a Polynomial Function

In this section we will work backwards with the roots of polynomial equations or zeros of polynomial functions. As we did with quadratics, so we will do with polynomials greater than second degree. Given the roots of an equation, work backwards to find the polynomial equation or function from whence they came. Recall the following example.

Find the equation of a parabola that has x intercepts of $(-3,0)$ *and* $(2,0)$.

$(-3,0)$ *and* $(2,0)$.	***Given x intercepts of -3 and 2***
$x=-3 \qquad x=2$	***If the x intercepts are -3 and 2, then the roots of the equation are -3 and 2. Set each root equal to zero.***
$(x+3) \quad (x-2)$	***For the first root, add 3 to both sides of the equal sign.*** ***For the second root, subtract 2 to both sides of the equal sign.***
x^2+x-6	***Multiply the results together to find a quadratic expression.***
$y=x^2+x-6$	***Set the expression equal to y, or*** $f_{(x)}$***, to write as the equation of a parabola.***

The exercises in this section will result in polynomials greater than second degree. <u>Be aware, you may not be given all roots with which to work</u>.

Consider the following example:

Find a polynomial function that has zeros of $0,\ 3$ *and* $2+3i$***. Although only three zeros are given here, there are actually four. Since complex numbers always come in conjugate pairs,*** $2-3i$ ***must also be a zero. Using the fundamental theorem of algebra, it can be determined that his is a 4th degree polynomial function.***

Take the zeros of $0,\ 3,\ 2\pm 3i$***, and work backwards to find the original function.***

$x=0$	$x=3$	$x=2\pm 3i$
$x=0$	$x=\ \ 3$ $-3 \quad -3$ $x-3=0$	$x=2\pm 3i$ $x-2=\pm 3i$ $x^2-4x+4=9i^2$ $x^2-4x+4=-9$ $x^2-4x+13=0$
x	$(x-3)$	$(x^2-4x+13)$

The polynomial function with zeros of $0,\ 3,\ 2\pm 3i$***, is equal to*** $f_{(x)}=x(x-3)(x^2-4x+13)$***. Multiplying this out will yield the following.***

$$f_{(x)}=x^4-7x^3+25x^2-39x$$

Find a polynomial function that has the following zeros.

A) -3, 2, 1

B) -4, 0, 1, 2

C) $\pm 1, \pm\sqrt{2}$

D) 0, 2, 5

E) 2, $1 \pm \sqrt{3}$

F) $\pm 4, 0, \pm\sqrt{2}$

G) -2, -1, 0, 1, 2

H) $1 \pm \sqrt{2}, \pm\sqrt{3}$

I) 0, -3

Here is a little practice with complex numbers.

Find a polynomial function that has the given zeros.

A) $0, 3, \pm 2i$

B) $\pm 2i, \ \pm 3i, \ 4$

C) $-2, \ 3, \ 3i$

D) $-i, \ 2i, \ -3i$

E) $-4, \ 1 \pm 2i$

F) $1-i, \ 1+3i, \ 0$

Complex solutions always come in ____________ __________.

What is the standard form of a complex number?

Even vs. Odd Functions

One of your many tasks in future mathematics courses will be to determine whether a function is even, odd or neither. This is very simple to do.

A function is even if $f_{(-x)} = f_{(x)}$

This means if a (-x) is substituted into the problem, and no signs change, the function is even.

A function is odd if $f_{(-x)} = -f_{(x)}$

In this case, a (-x) is substituted into the problem, and all signs change. If all signs change, this is an odd function.

If only some of the signs change, the function is neither even nor odd.

Even functions are symmetrical to the y axis.
Odd functions are symmetrical to the origin.

Based on this statement, can we conclude that all parabolas are even functions? Explain your answer.

Determine whether each of the following functions is even, odd or neither.

A) $f_{(x)} = 2x^4 - 6x^2 + 5x - 15$

B) $f_{(x)} = -4x^7 + 3x^5 + 2x^3 - 6x$

C) $f_{(x)} = 8x^4 - 2x^2 + 24$

D) $f_{(x)} = 10x^3 - 15x^2 - 16x + 12$

E) $f_{(x)} = -6x^3 + 5x^2 - 2x + 18$

F) $f_{(x)} = 4x^5 - 16x^3 + 12x$

G) $f_{(x)} = 4x^6 + 3x^4 - 2x^2 - 12$

H) $f_{(x)} = x^5 - 6x^3 - 8x$

Left and Right Behaviors of Polynomial Functions

As we graphed various functions, you should have noticed something about the graph of a polynomial function of an even degree versus the graph of a polynomial function of an odd degree. Think of a parabola versus a cubic function. The left and right behaviors of polynomial functions are pretty simple to memorize.

If the degree of the polynomial is even, the graph of the function will have either "both sides up", or "both sides down."

If the degree of the polynomial is odd, the graph of the function will have one side up and one side down.

As to which side is up and which is down, that all depends on the leading coefficient.

Refer to the following.

	Even Degree	*Odd Degree*
+ leading coefficient	↑↑	↓↑
- leading coefficient	↓↓	↑↓

Therefore, a 7th degree polynomial function having a leading coefficient that is negative, will rise on the left, and fall to the right.

In contrast, if the 7th degree polynomial has a positive leading coefficient, the graph of the function will fall on the left, and rise on the right hand side.

These rules are for polynomial functions in a single variable only!

When we begin to graph these polynomial functions, the first step will be to find all zeros of the function. The x intercepts have been found, plot them on the x axis, and refer to the two intercepts on the ends. At this point, use the rules for left and right behaviors of functions to draw a portion of the graph.

Describe the left and right behaviors of the following polynomial functions.

A) $f_{(x)} = 3x^3 + 6x^2 - 5$

B) $f_{(x)} = x^6 + 5x^3 + 7x^2 - x - 2$

C) $f_{(x)} = x(x+2)^3(x-3)(x+1)$

D) $f_{(x)} = -x^4 + 5x - 6$

E) $f_{(x)} = -\frac{1}{2}x^3 + 2x^2 - 3x + 5$

F) $f_{(x)} = 7x^8 - 6x^6 + 2x^4 - 8$

G) $f_{(x)} = 2x^5 + 3x^2 - x + 5$

H) $f_{(x)} = 14x^2 + 7x^3$

I) $f_{(x)} = -(x+3)^3(x-4)$

J) $f_{(x)} = -0.2x^5 + 6x^3 - x^2$

K) $f_{(x)} = -x^3 + 3x^2 + 12x^6 - 8$

L) $f_{(x)} = (x+2)^2(x-5)^5$

Graphing Polynomial Functions

When graphing a polynomial function, there are a series of steps to follow.

1. ***Find all zeros of the function. This will give the x intercepts of the function***
2. ***Plot all x intercepts for the function on the x axis.***
3. ***Using the properties of polynomial functions, determine the left and right behaviors of the function, and draw those segments.***
4. ***Substitute zero for x, and find the y intercept of the function.***
5. ***Using the graphing calculator, find the maxima and minima between each x intercept.***
6. ***Draw the rest of the function, making sure the maxima and minima are in their appropriate locations.***

The first step is the most time consuming, as the rational zero test and other methods are used to find all real zeros of the function. Depending on the degree of the function, there may be quite a few x intercepts to find. Imaginary solutions to the equation are not x intercepts. They will not be on the graph of the function.

The two most important properties of polynomial used to graph, are the left and right behaviors of a polynomial function, and the rule regarding the number of turns of a polynomial function.

	Even Degree	***Odd Degree***
+ leading coefficient	↑↑	↓↑
- leading coefficient	↓↓	↑↓

Any polynomial function to the nth degree, has <u>at most</u> n-1 turns.

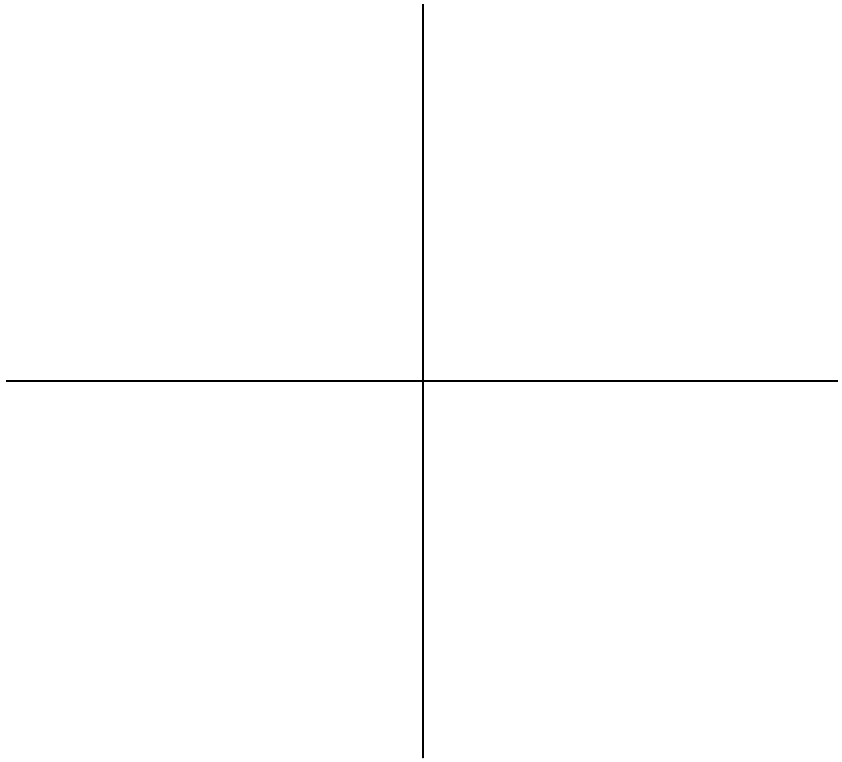
This means a 5th degree polynomial function will have at most 4 turns. It does not have to have that many, but it can have no more than 4 turns.

Sketch the graph of each of the following polynomial functions. **(Label all x intercepts, y intercepts, maxima, minima, and identify the range and domain.) Scale the graphs as needed.**

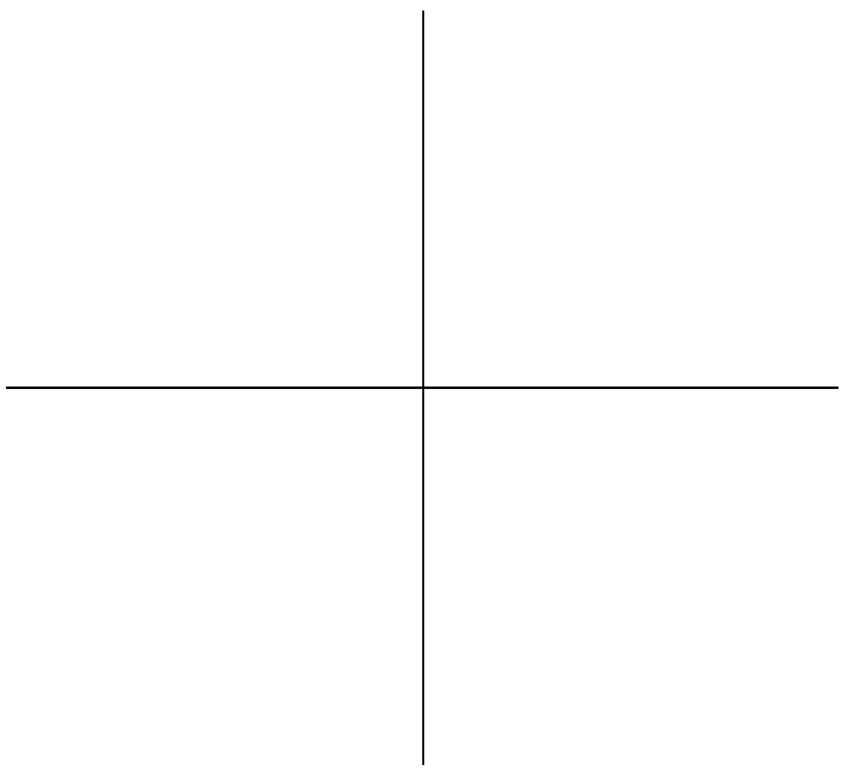
A) $f_{(x)} = x^4 - 10x^2 + 24$

B) $f_{(x)} = \frac{1}{2}\left(x^2 - 2x + 15\right)$

C) $f_{(x)} = -(x-1)^2(x+4)$



D) $f_{(x)} = \frac{1}{2}x(x-4)(x+7)$



E) $f_{(x)} = -\left(x+3\right)^3\left(x-1\right)\left(x-5\right)$

F) $f_{(x)} = -3x\left(x-2\right)^2\left(x+5\right)$

G) $f_{(x)} = \frac{1}{4}(x+5)^3(x+1)(x-3)$

H) $f_{(x)} = -x^2(x+3)(x-2)(x-6)^2$

I) $f_{(x)} = -(x+3)^3(x-5)^2$

J) $f_{(x)} = x^3 - 6x^2 - 9x + 54$

K) $f_{(x)} = -x^4 + 37x^2 - 24x - 180$

L) $f_{(x)} = x^6 - 25x^4 - 16x^2 + 400$

Word Problems

Refer to the following diagram for problems A-D.

A) You have a rectangular piece of wood that is 16 in by 8 in. A small length, x, is cut from each of the 4 sides yielding a surface area of $48\,in^2$. Find the value of x.

B) You have a rectangular piece of steel whose dimensions are 20 inches by 16 inches. You are required to cut out the four corners of the rectangle so that you may fold up the sides to create a box. Write the function you would use to find the volume of the box if x represents the length of the cuts.

C) You are given a rectangular sheet of metal that is 32 inches by 24 inches. You are required to cut a length from each corner of the sheet so that you may fold up the ends and create a box. What is the domain of the function you will use to find the volume of the box? Explain your answer.

D) You are given a 14 inch by 8 inch rectangular sheet of metal from which you are to construct a box. You are to cut a length, x, from each corner of the sheet of metal so that you can fold up the sides creating a box. Find the value of x that will yield the maximum volume of the box. Round your solution to 3 significant digits.

E) A rectangular field is twice as long as it is wide. If 3 feet are taken from the width, and 4 feet taken from the length, the resultant are of the field is $180\,\text{ft}^2$. Find the area of the original field.

Checking Progress

You have now completed the "Polynomial Functions" section of the workbook. The following is a checklist so that you may check your progress. Check off each of the objectives you have accomplished.

The student should be able to…

- ☐ ***Find the quotient of a division problem involving polynomials using the polynomial long division method.***
- ☐ ***Find the quotient of a division problem involving polynomials using the synthetic division method.***
- ☐ ***Use the rational zero test to determine all possible rational zeros of a polynomial function.***
- ☐ ***Use the rational zero test to determine all possible roots of a polynomial equation.***
- ☐ ***Use Descarte's Rule of Signs to determine the possible number of positive or negative roots of a polynomial equation.***
- ☐ ***Find all zeros of a polynomial function.***
- ☐ ***Use the remainder theorem to evaluate the value of functions.***
- ☐ ***Write a polynomial in completely factored form.***
- ☐ ***Write a polynomial as a product of factors irreducible over the reals.***
- ☐ ***Write a polynomial as a product of factors irreducible over the rationals.***
- ☐ ***Find the equation of a polynomial function that has the given zeros.***
- ☐ ***Determine if a polynomial function is even, odd or neither.***
- ☐ ***Determine the left and right behaviors of a polynomial function without graphing.***
- ☐ ***Find the local maxima and minima of a polynomial function.***
- ☐ ***Find all x intercepts of a polynomial function.***
- ☐ ***Determine the maximum number of turns a given polynomial function may have.***
- ☐ ***Graph a polynomial function.***