RATIONAL FUNCTIONS

Objectives

The following is a list of objectives for this section of the workbook.

By the time the student is finished with this section of the workbook, he/she should be able to…

- ***Find the vertical asymptotes of a rational function.***
- ***Determine if the function has a horizontal or oblique asymptote.***
- ***Find the horizontal asymptote of a rational function if it exists.***
- ***Find the oblique asymptote of a rational function if it exists.***
- ***Find the domain of a rational function.***
- ***Find the x intercepts of a rational function.***
- ***Find the y intercept of a rational function.***
- ***Graph a rational function.***

Math Standards Addressed

The following state standards are addressed in this section of the workbook.

Algebra II

3.0 Students are adept at operations on polynomials, including long division.

4.0 Students factor polynomials representing the difference of squares, perfect square trinomials, and the sum and difference of two cubes.

7.0 Students add, subtract, multiply, divide, reduce, and evaluate rational expressions with monomial and polynomial denominators and simplify complicated rational expressions, including those with negative exponents in the denominator.

8.0 Students solve and graph quadratic equations by factoring, completing the square, or using the quadratic formula. Students apply these techniques in solving word problems. They also solve quadratic equations in the complex number system.

15.0 Students determine whether a specific algebraic statement involving rational expressions, radical expressions, or logarithmic or exponential functions is sometimes true, always true, or never true.

Mathematical Analysis

6.0 Students find the roots and poles of a rational function and can graph the function and locate its asymptotes.

Finding Asymptotes

Rational functions have various asymptotes. The following will aid in finding all asymptotes of a rational function. The first step to working with rational functions is to completely factor the polynomials. Once in factored form, find all zeros.

Vertical Asymptotes

- **The Vertical Asymptotes of a rational function are found using the zeros of the denominator.**

For Horizontal Asymptotes use the following guidelines.

- **If the degree of the numerator is greater than the degree of the denominator by more than one, the graph has no horizontal asymptote.(none)**
- **If the degree of the numerator is equal to the degree of the denominator, the horizontal asymptote is the ratio of the two leading coefficients.(y = #)**
- **If the degree of the numerator is less than the degree of the denominator, the horizontal asymptote is zero. (y = 0)**

Oblique Asymptotes

- **If the degree of the numerator is greater than the degree of the denominator by one, there is an oblique asymptote. The asymptote is the quotient numerator divided by the denominator.**

An asymptote is like an imaginary line that cannot be crossed. All rational functions have vertical asymptotes. A rational function may also have either a horizontal or oblique asymptote. A rational function will never have both a horizontal and oblique asymptote. It is either one or the other. Horizontal asymptotes are the only asymptotes that may be crossed. The vertical asymptotes come from zeroes of the denominator.

$$f_{(x)}=\frac{x}{(x+2)(x-3)}$$

Here is a rational function in completely factored form.

$$x=-2 \quad and \quad x=3$$

The zeros of the denominator are -2 and 3. Therefore, these are the vertical asymptotes of the function.

Since an x value of -2 or 3 would create a zero in the denominator, the function would be undefined at that location. As a result, these are the vertical asymptotes for this function.

In this same function, the degree of the numerator is less than the degree of the denominator, therefore, the horizontal asymptote is $y = 0$.

When finding the oblique asymptote, find the quotient of the numerator and denominator. If there are any remainders, disregard them. You only need the quotient. The graph of the function can have a either a horizontal asymptote, or an oblique asymptote. You can not have one of each. This particular function does not have an oblique asymptote.

Here is an example with an oblique asymptote.

Find the oblique asymptote of the rational function $f_{(x)} = \dfrac{x^2 + 8x - 20}{x - 1}$.

$$
\begin{array}{r|l}
 & \quad\quad\quad x + 9 \\
x - 1 & \overline{x^2 + 8x - 20} \\
 & \underline{-x^2 + x}\quad\quad\quad \\
 & \quad\quad 9x - 20 \\
 & \quad\quad \underline{-9x + 9} \\
 & \quad\quad\quad\quad -11
\end{array}
$$

Dividing the polynomials, the quotient x+9 is found.

$y = x + 9$

This is the equation for the oblique asymptote of the function. Notice the remainder of the division problem is disregarded. It plays no part in the equation for the oblique asymptote.

Finally, let us look at a rational function where the degree of the numerator is equal to the degree of the denominator.

Find the horizontal asymptote for the rational function $f_{(x)} = \dfrac{2x^2 - 4x + 8}{3x^2 - 27}$.

$f_{(x)} = \dfrac{2x^2 - 4x + 8}{3x^2 - 27}$

Notice the degree of the numerator is the same as the degree of the denominator.

$y = \dfrac{2}{3}$

Since the degree of the numerator equals that of the denominator, the equation for the horizontal asymptote is the ratio of the two leading coefficients.

Find all asymptotes of the following functions. **(Do not graph these)**

A) $f_{(x)} = \dfrac{x-7}{x+5}$

B) $f_{(x)} = \dfrac{3}{x^2-2}$

C) $f_{(x)} = \dfrac{x^2}{x-5}$

D) $f_{(x)} = \dfrac{2x^2-5x+3}{x-1}$

E) $f_{(x)} = \dfrac{7x^2+5x-2}{2x^2-18}$

F) $f_{(x)} = \dfrac{2x^2-5x+5}{x-2}$

G) $f_{(x)} = \dfrac{1}{3-x}$

H) $f_{(x)} = \dfrac{x^2-4}{x^4-81}$

I) $f_{(x)} = \dfrac{x^3-2x^2+5}{x^2}$

The Domain

The domain of a rational function is found using only the vertical asymptotes. As previously noted, rational functions are undefined at vertical asymptotes. The rational function will be defined at all other x values of the domain.

$$f_{(x)} = \frac{x}{(x+2)(x-3)}$$

Here is a rational function in completely factored form.

$$x = -2 \quad and \quad x = 3$$

Since the zeros of the denominator are -2 and 3, these are the vertical asymptotes of the function.

Therefore, the domain of this function is $(-\infty,-2)\cup(-2,3)\cup(3,\infty)$. ***Notice there are two vertical asymptotes, and the domain is split into three parts. This pattern will repeat. If there are 4 vertical asymptotes, the domain of that function will be split into 5 parts.***

Find the domain of each of the following rational functions.

A) $f_{(x)} = \dfrac{x-7}{x+5}$

B) $f_{(x)} = \dfrac{3}{x^2-4}$

C) $f_{(x)} = \dfrac{x^2}{x-5}$

D) $f_{(x)} = \dfrac{2x^2-5x+3}{x-1}$

E) $f_{(x)} = \dfrac{x-8}{x^3-x^2-12x}$

F) $f_{(x)} = \dfrac{x^3}{x^2-7x+12}$

G) $f_{(x)} = \dfrac{1}{3-x}$

H) $f_{(x)} = \dfrac{x^2-4}{x^4-81}$

I) $f_{(x)} = \dfrac{x^3-2x^2+5}{x^2}$

Finding Intercepts

We have found that the zeros of the denominator of a rational function are the vertical asymptotes of the function. The zeros of the numerator on the other hand, are the x intercepts of the function.

Find all x and y intercepts of the function $f_{(x)} = \dfrac{x^2-9}{x-1}$.

$$f_{(x)} = \frac{(x+3)(x-3)}{x-1}$$

Write out the function in completely factored form.

Now, find the zeros of the numerator

$$x=-3 \quad and \quad x=3$$

These are the x intercepts of the function.

Look at the original function.

$$f_{(x)} = \frac{x^2-9}{x-1}$$

From here, substitute zero for x, and find the y intercept, which in this case will be the ratio of the two constants.

$$y=9$$

This is the y intercept of the function. In this case, it is the ratio of the two remaining constants once zero is substituted in for x. If there is no constant in the denominator, then there will be no y intercept as x=0 is a vertical asymptote and the graph is undefined at the y axis.

The x intercepts are $(-3,0)\, and\, (3,0)$

The y intercept is $(0,9)$

As demonstrated above, the y intercept of a rational function is the ratio of the two constants. Like always, substitute zero for x, and solve for y to find the y intercept.

Find the x and y intercepts of each rational function.

A) $f_{(x)} = \dfrac{x-7}{x+5}$

B) $f_{(x)} = \dfrac{3}{x^2-4}$

C) $f_{(x)} = \dfrac{x^2}{x-5}$

D) $f_{(x)} = \dfrac{2x^2-5x+3}{x-1}$

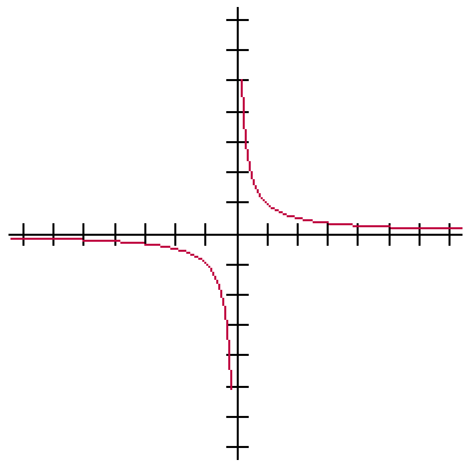
E) $f_{(x)} = \dfrac{x-8}{x^3-x^2-12x}$

F) $f_{(x)} = \dfrac{x^3}{x^2-7x+12}$

<u>Graphing Rational Functions</u>

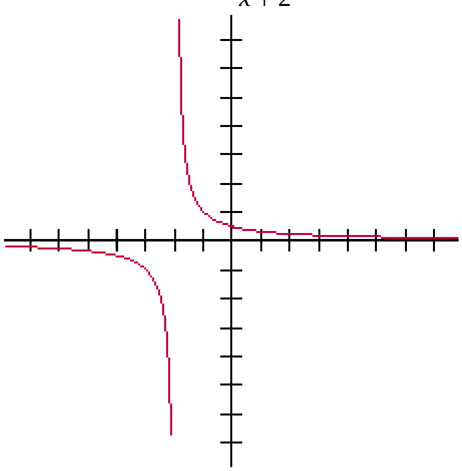
We really have no standard form of a rational function to look at, so we will concentrate on the parent function of $f_{(x)} = \frac{1}{x}$. The following pages illustrate the effects of the denominator, as well as the behavior of $-f_{(x)}$. A graphing calculator may be used to help get the overall shape of these functions. <u>DO NOT</u>, however, just copy the picture the calculator gives you.

$$f_{(x)} = \frac{1}{x}$$



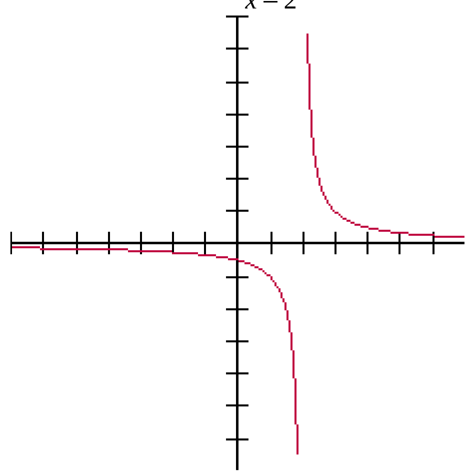
Here, the vertical asymptote is at x=0, and the horizontal asymptote is y=0.

$$f_{(x)} = \frac{1}{x+2}$$



The graph of this function shifts left 2.

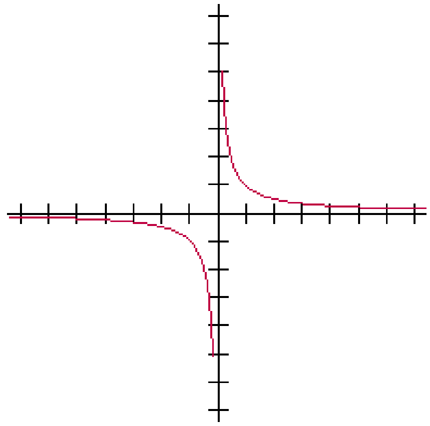
$$f_{(x)} = \frac{1}{x-2}$$



The graph of this function shifts right 2.

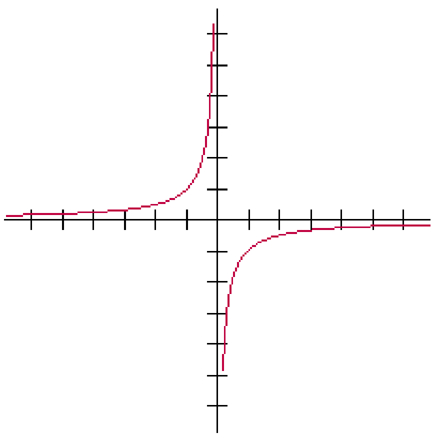
The range for each of these functions is $(-\infty, 0) \cup (0, \infty)$. There is no way to tell what the range of a rational function will be until it is graphedt. Remember, the curve may cross the horizontal axis.

$$f_{(x)} = \frac{1}{x}$$



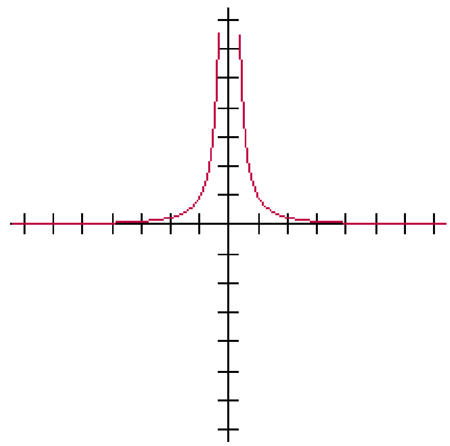
Here, the vertical asymptote is at x=0, and the horizontal asymptote is y=0.

$$f_{(x)} = -\frac{1}{x}$$



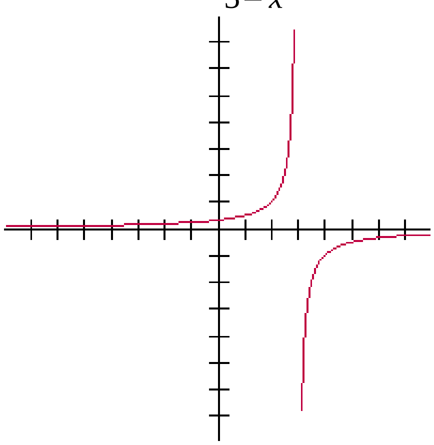
The graph of this function is a reflection of the parent function.

$$f_{(x)} = \frac{1}{x^2}$$



Notice how x^2 affects the function. Normally, one side of the function would go up, and the other would go down. Since there is no way to get a negative number in the denominator, both sides are going in the same direction.

$$f_{(x)} = \frac{1}{3-x}$$



The graph of the function to the left flips upside down, similar to $f_{(x)} = -\frac{1}{x}$, and shifts right 3. What happens here is a -1 is factored out of the denominator, changing the function to the following.

$$f_{(x)} = \frac{1}{3-x} \Rightarrow \frac{1}{-(x-3)} \Rightarrow -\frac{1}{x-3}$$

As a result, this graph is a combination of shifting the graph and reflecting it about the horizontal asymptote.

<u>Example</u>

Graph the function $f_{(x)} = \dfrac{x-3}{x^2 - x - 12}$**. Be sure to find all asymptotes, x and y intercepts, and the range and domain.**

$$f_{(x)} = \frac{x-3}{(x+3)(x-4)}$$

The first step is to completely factor the rational function.

Vertical Asymptotes are $x = -3$ **and** $x = 4$

The zeros of the denominator are the vertical asymptotes of the function.

Looking at the original function, the horizontal asymptote is $y = 0$**.**

If the degree of the numerator is less than the degree of the denominator, the horizontal asymptote is $y = 0$**.**

The x intercept is $(3, 0)$

The zero of the numerator is the x intercept of the function.

The y intercept is $\left(0, \frac{1}{4}\right)$**.**

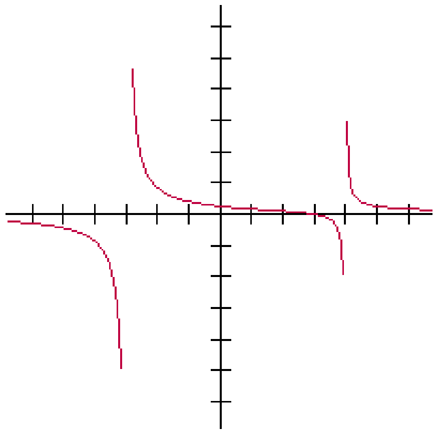
Substituting zero for x and evaluating the ratio of the two constants, -3 and -12. Yields a y intercept of $\frac{1}{4}$**.**

The domain of the function is $(-\infty, -3) \cup (-3, 4) \cup (4, \infty)$

The domain is found using the vertical asymptotes. The domain here is all real numbers except -3 and 4.

The range should not be found without first graphing the function.

$$f_{(x)} = \frac{x-3}{x^2 - x - 12}$$



Due to limitations in the graphing software, the graph to the left is incomplete. Each of these lines is continuous. Before attempting to graph the functions, graph <u>all</u> asymptotes using broken lines. This will ensure that a vertical asymptote is not crossed. To get the general shape of the equation, use a combination of the x and y intercepts that were found, and plug in values for x close to the vertical asymptote. Based on the properties of all rational functions, it should be obvious how these curves behave on the outer intervals of these functions. They will always ride along the asymptotes in these areas.

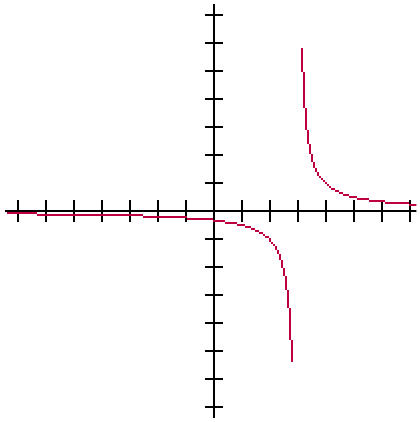
V.A.: $x = -3$, $x = 4$
H.A.: $y = 0$
x-int: $(3, 0)$
y-int: $\left(0, \frac{1}{4}\right)$
Range: $(-\infty, \infty)$
Domain: $(-\infty, -3) \cup (-3, 4) \cup (4, \infty)$

Here is a list of all required information needed for each rational function. Since the graph of the function crossed the horizontal asymptote in the interval $(-3, 4)$***, the range of this function is all real numbers.***

<u>These procedures must be used when graphing any rational function.</u>

Match the appropriate graph with its equation below. Explain why each of the solutions is true.

A



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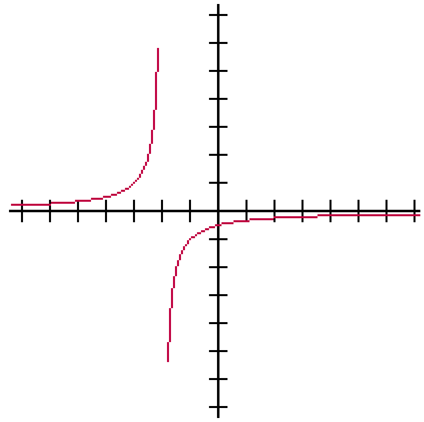
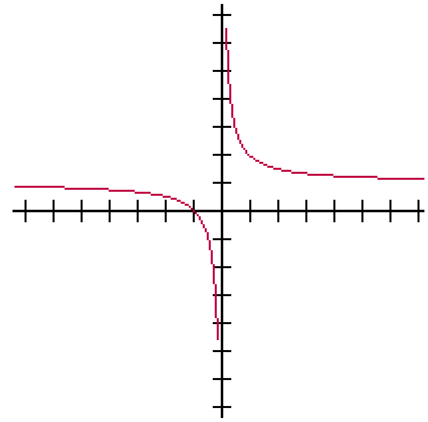
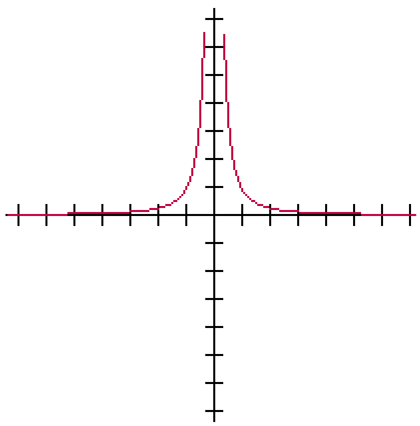


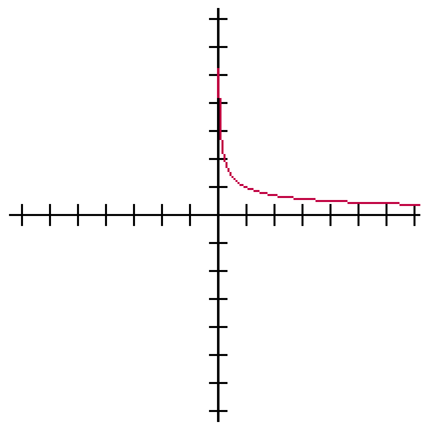
C



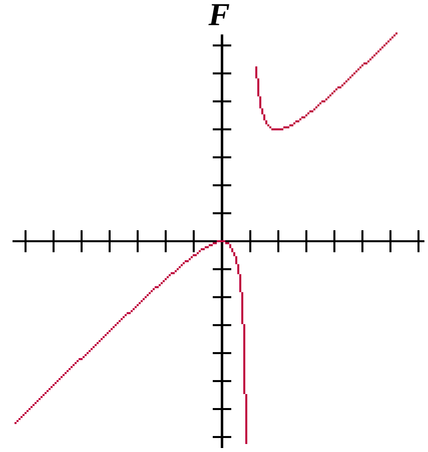
D



E



F



1) $f_{(x)} = \dfrac{1}{x^2}$

2) $f_{(x)} = \dfrac{1}{x-3}$

3) $f_{(x)} = \dfrac{1}{\sqrt{x}}$

4) $f_{(x)} = -\dfrac{1}{x+2}$

5) $f_{(x)} = \dfrac{x^2}{x-1}$

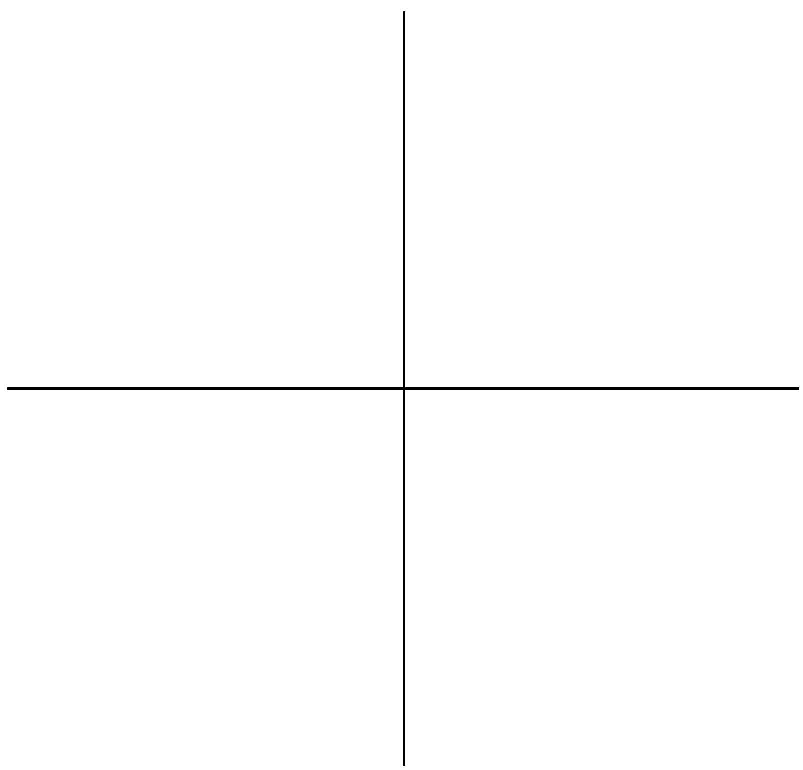
6) $f_{(x)} = \dfrac{1}{x}+1$

A graphing calculator may be used to help get a picture of the curve that will be created, but simply copying the picture shown in the calculator is unwise.

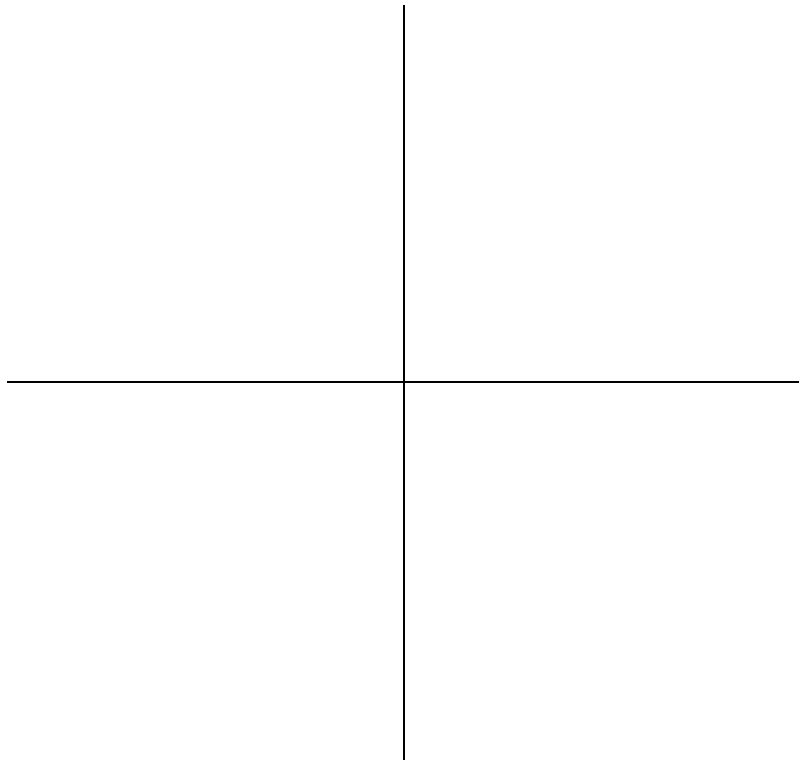
What is the problem with the picture of rational functions in graphing calculators?

Sketch the graph of each of the following functions. Be sure to find all asymptotes, x and y intercepts, and the range and domain of each of the following.

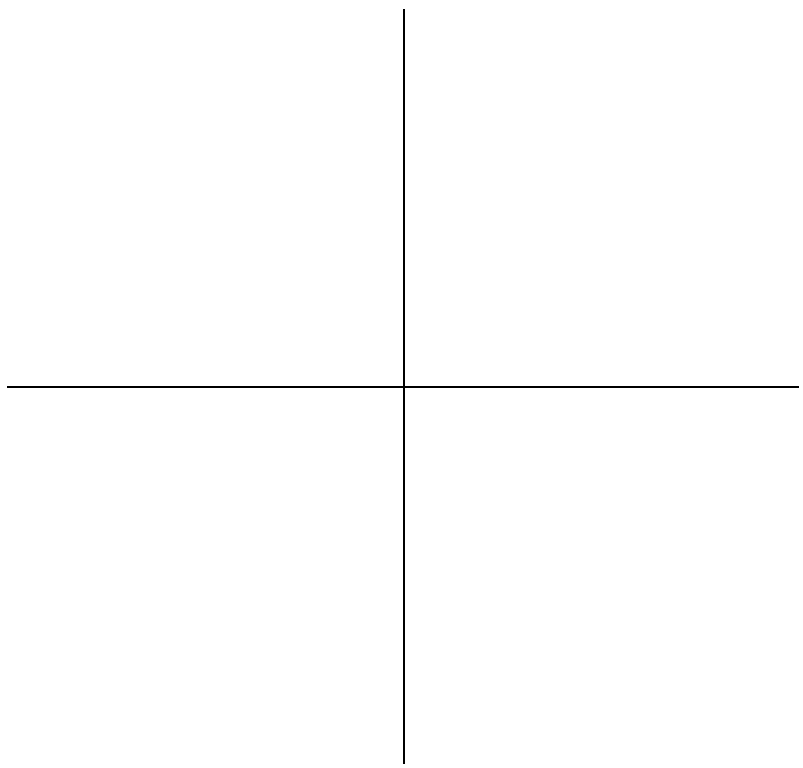
A) $f_{(x)} = \dfrac{1}{x-4}$



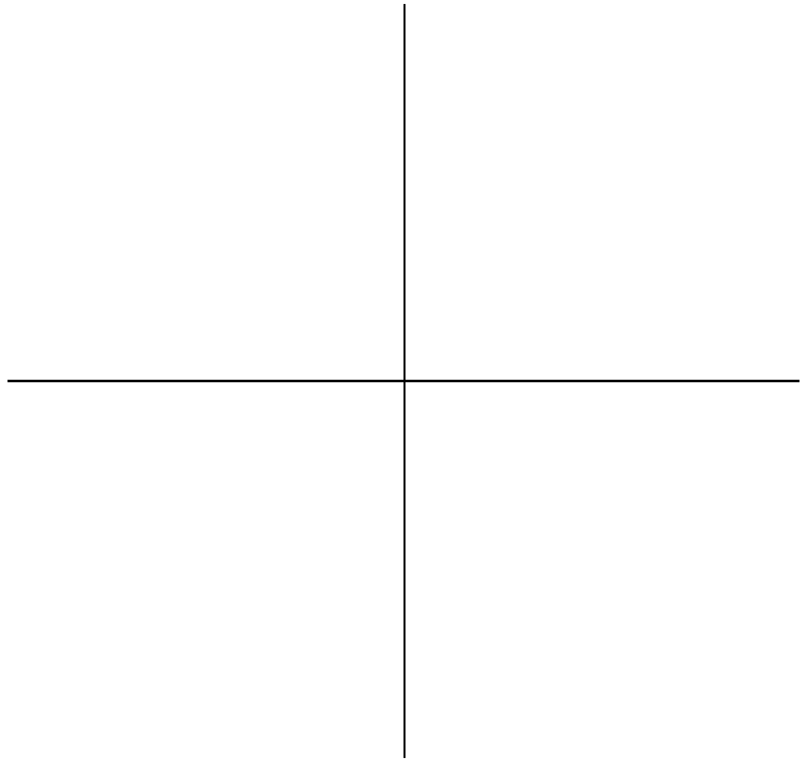
B) $f_{(x)} = -\dfrac{1}{x+2}$



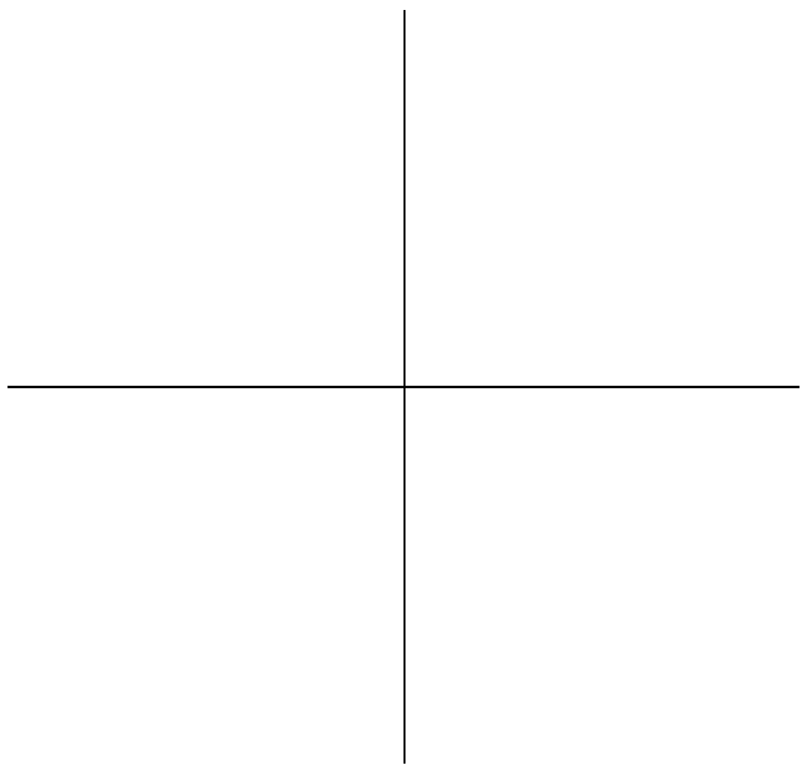
C) $f_{(x)} = \dfrac{x-3}{x+2}$



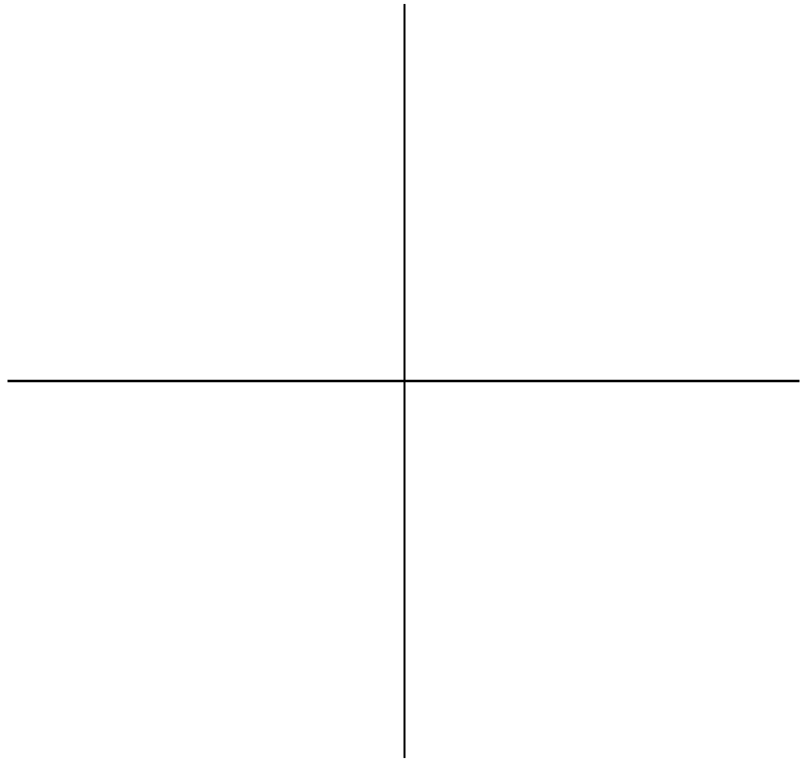
D) $f_{(x)} = \dfrac{x+2}{3x-9}$



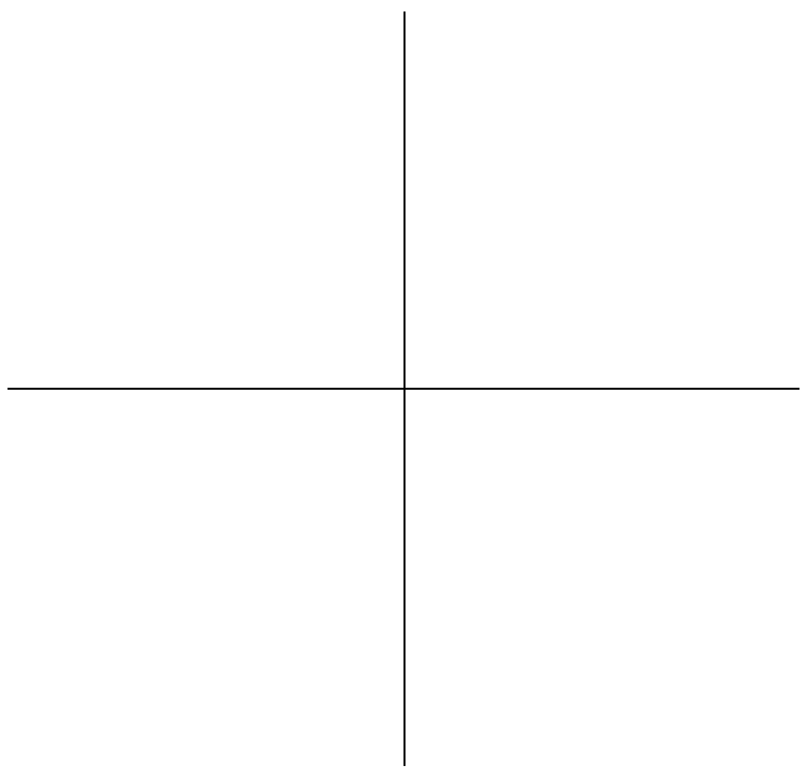
E) $f_{(x)} = \dfrac{2x^2+1}{x}$



F) $f_{(x)} = \dfrac{x+2}{x^2-9}$



G) $f_{(x)} = \dfrac{x+2}{x-1}$



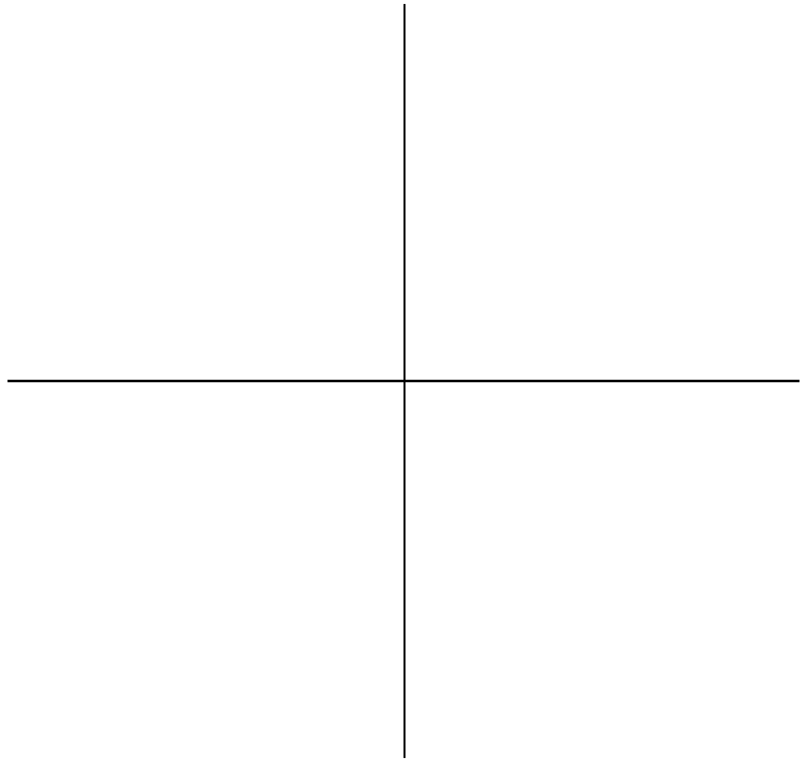
H) $f_{(x)} = \dfrac{2x^2}{x^2 - 4}$

I) $f_{(x)} = -\dfrac{x^3}{x^2 - 9}$

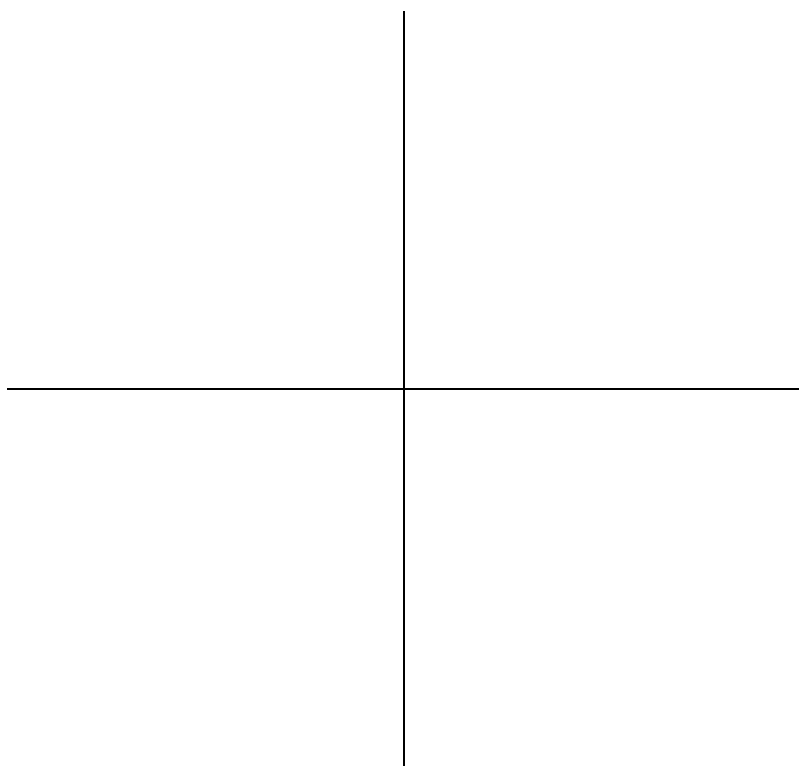
J) $f_{(x)} = -\dfrac{x+6}{x-2}$

K) $f_{(x)} = \dfrac{x^2-4}{x}$

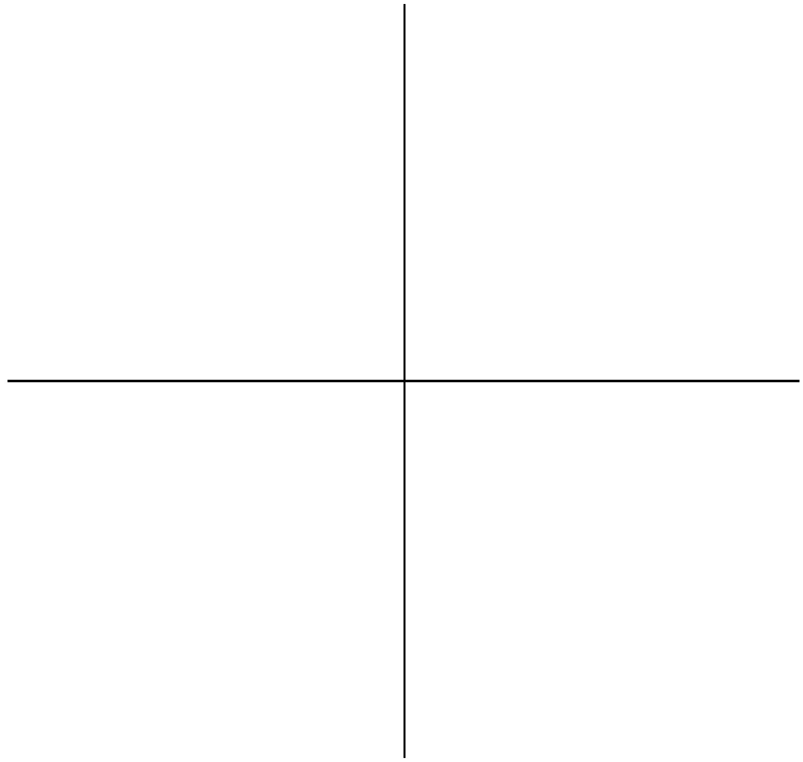
L) $f_{(x)} = \dfrac{1}{x+1} + 1$



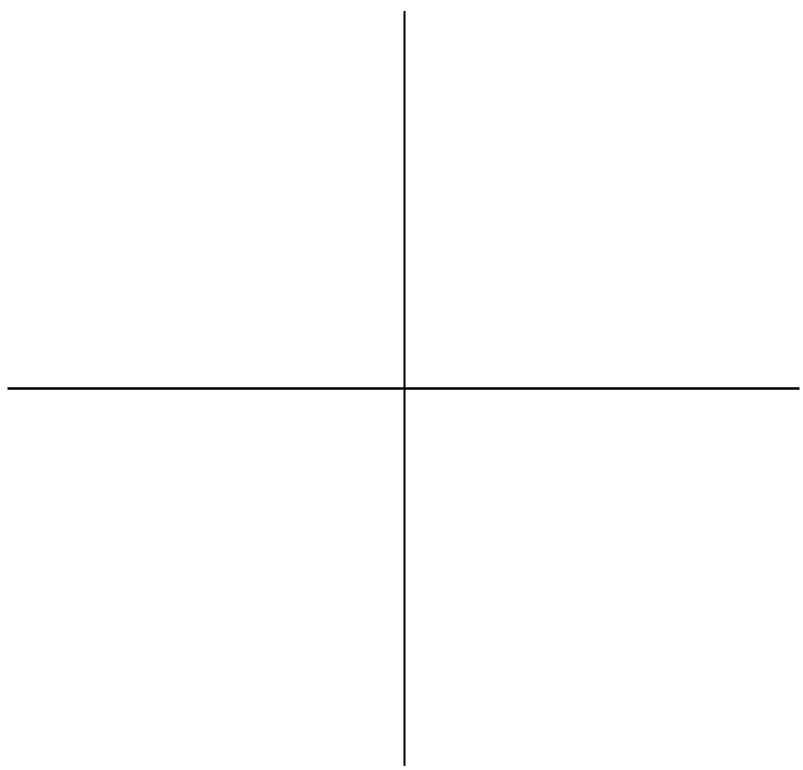
M) $f_{(x)} = \dfrac{1}{x+2} + 2$



N) $f_{(x)} = \dfrac{1}{(x-2)^2}$



O) $f_{(x)} = \dfrac{1}{\sqrt{x+2}}$



Checking Progress

You have now completed the "Rational Functions" section of the workbook. The following is a checklist so that you may check your progress. Check off each of the objectives you have accomplished.

The student should be able to…

- ☐ ***Find the vertical asymptotes of a rational function.***
- ☐ ***Determine if a rational function has a horizontal or oblique asymptote.***
- ☐ ***Find the horizontal asymptote of a rational function if it exists.***
- ☐ ***Find the oblique asymptote of a rational function if it exists.***
- ☐ ***Find the domain of a rational function.***
- ☐ ***Find the x intercepts of a rational function.***
- ☐ ***Find the y intercept of a rational function.***
- ☐ ***Graph a rational function.***