INTRODUCTION TO TRIGONOMETRY

Objectives

The following is a list of objectives for this section of the workbook.

By the time the student is finished with this section of the workbook, he/she should be able to…

- ***Give a graphical representation of any angle.***
- ***Find positive and negative coterminal angles.***
- ***Convert an angle measured in degrees to radians.***
- ***Convert an angle measured in radians to degrees.***
- ***Evaluate the basic trigonometric functions.***
- ***Use reference angles to evaluate the basic trigonometric functions.***
- ***Construct a unit circle.***
- ***Use the unit circle to evaluate basic trigonometric functions.***
- ***Use the unit circle to solve trigonometric equations.***

Math Standards Addressed

The following state standards are addressed in this section of the workbook.

Trigonometry

1.0 Students understand the notion of angle and how to measure it, in both degrees and radians. They can convert between degrees and radians.

2.0 Students know the definition of sine and cosine as *y*- and *x*- coordinates of points on the unit circle and are familiar with the graphs of the sine and cosine functions.

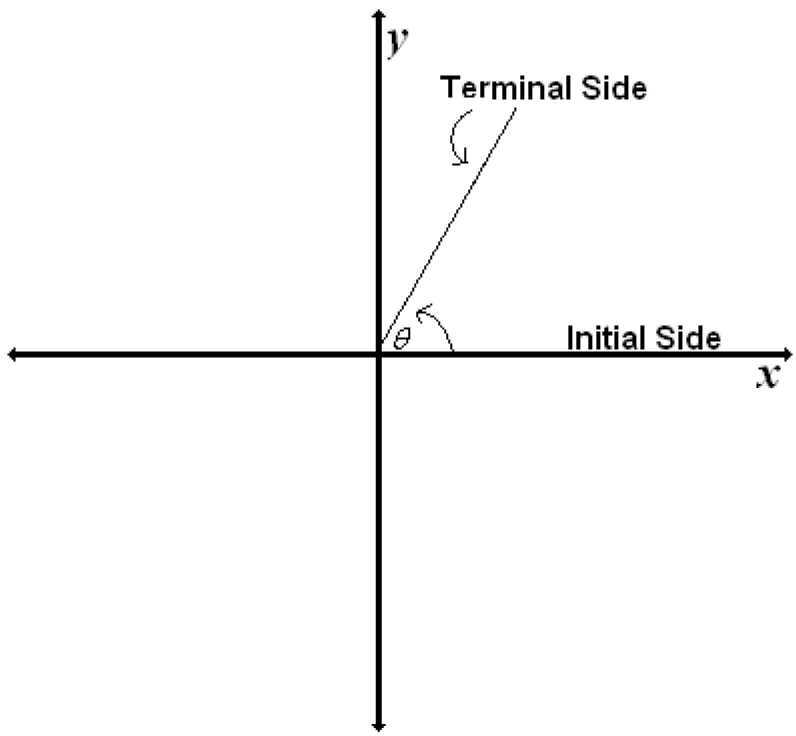
3.0 Students know the identity $\cos^2(x) + \sin^2(x) = 1$:

> 3.1 Students prove that this identity is equivalent to the Pythagorean theorem (i.e., students can prove this identity by using the Pythagorean theorem and, conversely, they can prove the Pythagorean theorem as a consequence of this identity).
>
> 3.2 Students prove other trigonometric identities and simplify others by using the identity $\cos^2(x) + \sin^2(x) = 1$. For example, students use this identity to prove that $\sec^2(x) = \tan^2(x) + 1$.

<u>Angles</u>

In trigonometry, we study angles and triangles. Before discussing angles, however, there are a few vocabulary terms that will be necessary. Each angle has an initial side and a terminal side. It will help to think of an angle in the following manner.

Begin by picturing a standard Cartesian Plane with two rays resting on the positive side of the x axis. As one of the sides moves in a counterclockwise direction, the other stays put. As the ray moves, an angle is being created at their vertex. The line segment that remains on the positive side of the x axis is called the <u>Initial Side</u> of the angle. The line segment that is moving is known as the <u>Terminal Side</u> of the angle.



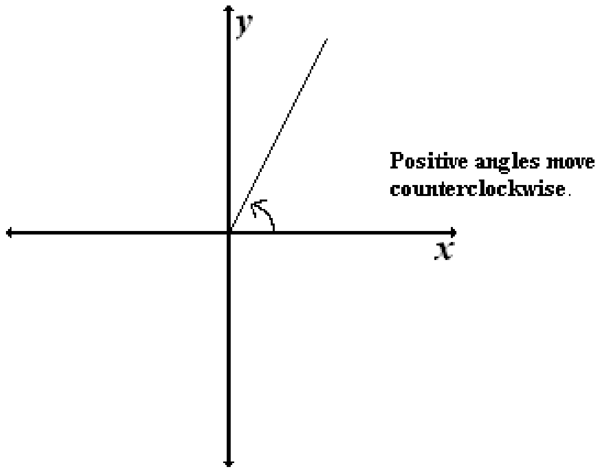
Notice the symbol used in the picture above. That symbol (θ) is the Greek letter theta. In trigonometry, Greek letters are often used to represent angles.

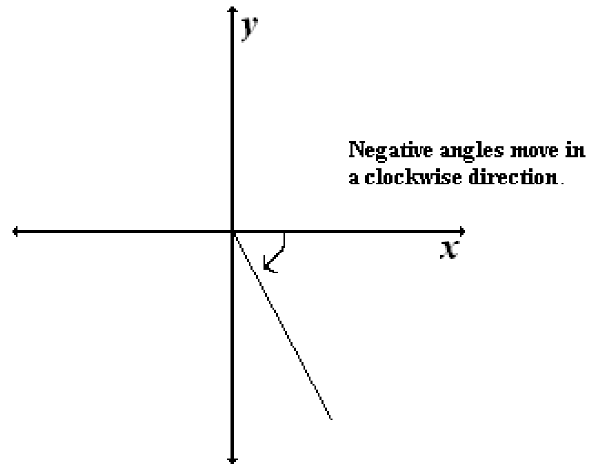
There are also some basic geometric terms that will be used in the study of trigonometry.

Recall that an <u>Acute Angle</u> is an angle that is less than 90 degrees, while an <u>Obtuse Angle</u> is an angle whose measure is between 90 and 180 degrees.

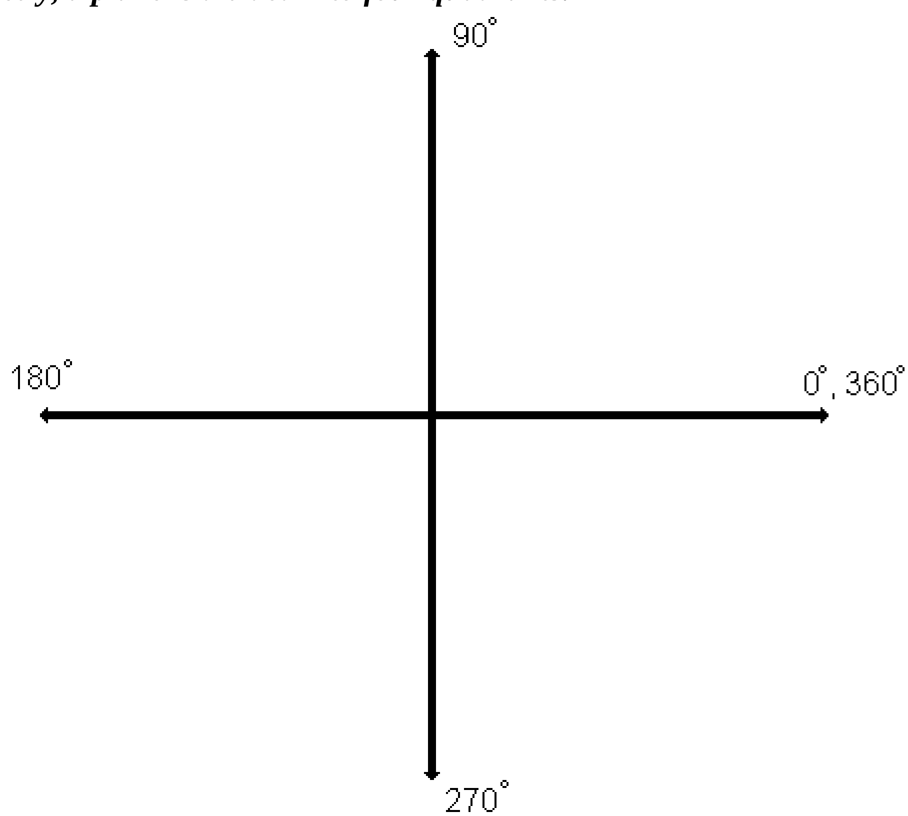
<u>Supplementary Angles</u> are two angles whose sum is 180 degree. <u>ComplimentaryAngles</u> are angles whose sum is 90 degrees.
In trigonometry, a plane is divided into four quadrants.

An angle whose initial side is on the positive side of the x axis is said to be in <u>Standard Position</u>. An angle is positive if the terminal side is moving in a counterclockwise direction. An angle is negative if the terminal side is moving in a clockwise direction.





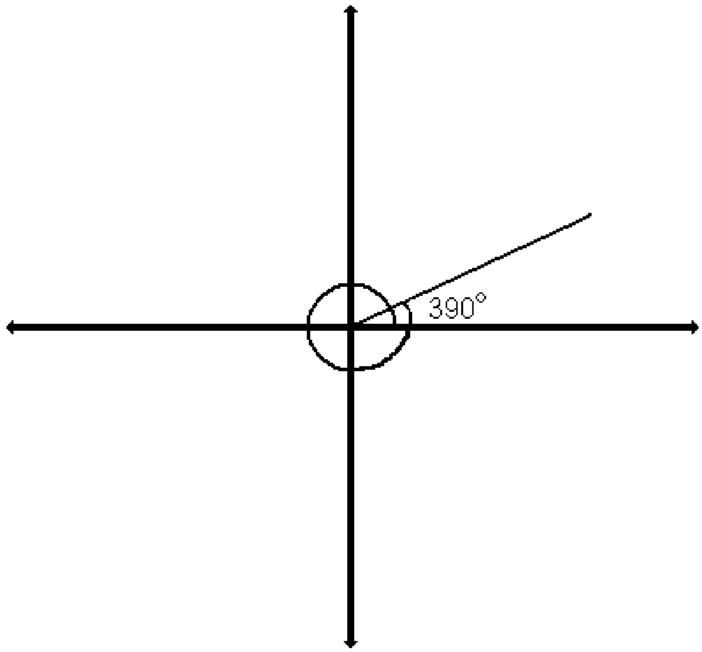
In trigonometry, a plane is divided into four quadrants.



According to the diagram above, the terminal side of a 20° angle would reside in quadrant I. However, an angle that measures 380° would also share the same terminal side. The only difference being, the terminal side of the 380° angle makes a complete revolution before finally coming to a stop.

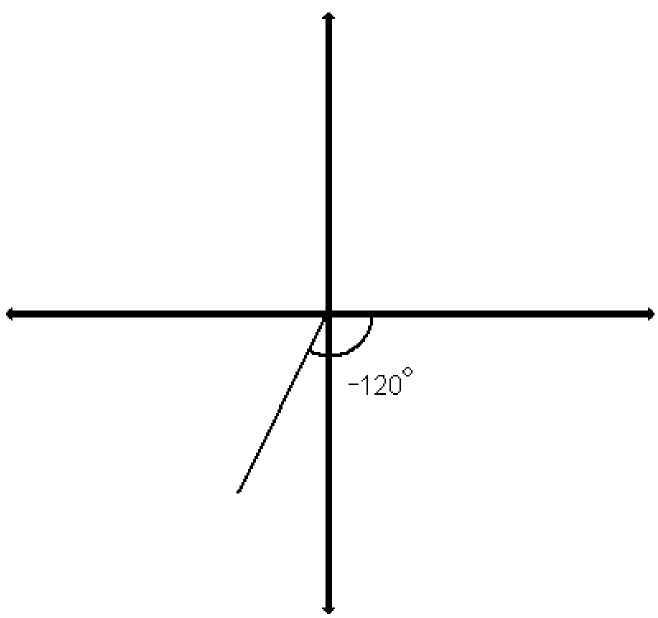
Here are a couple of examples of how to give a graphical representation of an angle.

Give a graphical representation of an angle that measures 390°.



In the above example, a 390° angle moves in a counterclockwise direction, and makes one complete revolution where the terminal side ends up in quadrant I.

Give a graphical representation of an angle that measures -120°.



In the above example, a -120° moves in a clockwise direction, and the terminal side resides in quadrant III.

Give a graphical representation of each of the following angles.

A) $130°$

B) $45°$

C) $-30°$

D) $-172°$

E) $200°$

F) $135°$

G) $-700°$

H) $-300°$

I) $290°$

J) $695°$

K) $-200°$

L) $-540°$

M) $-382°$

N) $810°$

O) $-830°$

Determine the quadrant in which the terminal side of each of the following angles resides.

A) $172°$

B) $-315°$

C) $718°$

D) $415°$

E) $-63°$

F) $135°$

G) $-700°$

H) $1020°$

I) $-284°$

J) $615°$

K) $-200°$

L) $-540°$

M) $-450°$

N) $-700°$

O) $840°$

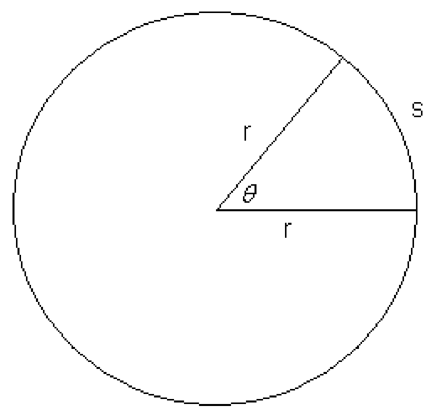
Coterminal Angles

***Coterminal Angles** are angles who share the same initial side and terminal sides. Finding coterminal angles is as simple as adding or subtracting $360°$ or 2π to each angle, depending on whether the given angle is in degrees or radians. There are an infinite number of coterminal angles that can be found. Following this procedure, all coterminal angles can be found. This is the basis for solving trigonometric equations which will be done in the future.*

Radians are often used in trigonometry to represent angle measures. Radian measures are very common in calculus, so it is important to have an understanding of what a radian is.

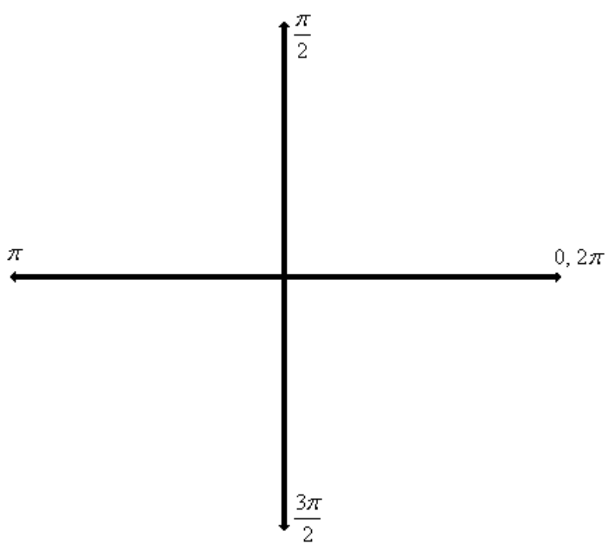
Definition of a Radian

A radian is the measure of a central angle θ that intercepts an arc s equal in length to the radius r of the circle. There are 2π, or approximately 6.28318, radians in a complete circle. Thus, one radian is about 57.296 angular degrees.



In other words, if we were to take the length of the radius of a circle, and lay in on the edge of a circle, that length would be one radian.

The number π is often used when describing radian measure. The approximate value of π is 3.14159... A plane, in trigonometry, can not only be divided into quadrants using degree measures, but radian as well. Observe the following moving in a counterclockwise direction.



When studying trigonometry, angles are usually measured in radians.

In relation to degrees, 180° is π radians. This means 2π radians is 360°. Since the approximate value of π is 3.14159..., it follows that 360° is approximately 6.28318...radians. ***When evaluating angles in trigonometry or calculus, always be aware of whether the question is given in terms of degrees or radians. If no degree symbol is given, the problem is in radians.***

Examples of finding coterminal angles

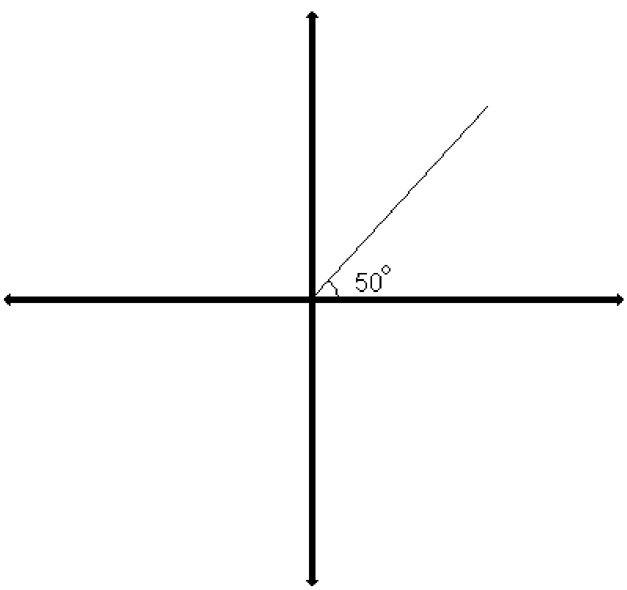
Find one positive angle that is coterminal to 50°.

Since the terminal side of a 50° angle resides in quadrant I, the terminal side of its coterminal angle must share that side. This means the new angle would make one complete revolution before having its terminal side come to rest at the same place.

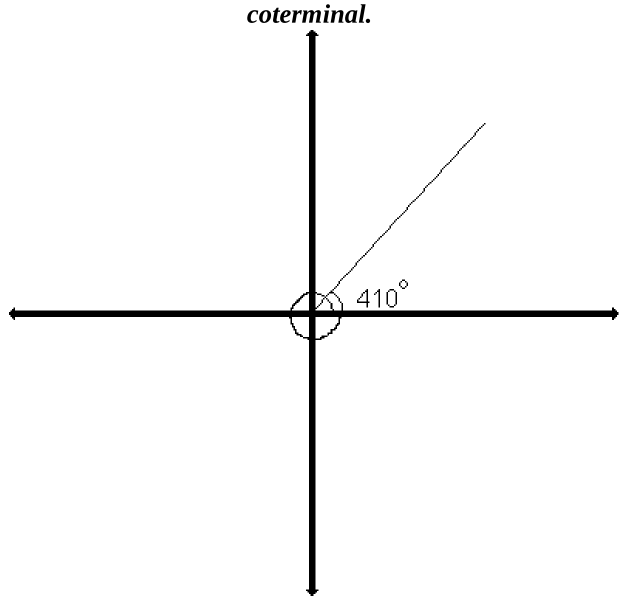
Therefore, to find the coterminal angle to a 50° angle, just add 360°.

$$50° + 360° = 410°$$

Below is the graphical representation of a 50° angle. Since its coterminal angle must share the same terminal side, it is reasonable to create a new angle that makes one complete revolution and ends up in the same place.



To find the coterminal angle of a 50° angle, add 360°. It would follow that 50°+360°=410°. A 410° angle is illustrated below. From the graphical representation of the angle, we can conclude that these two angle do indeed share the same terminal side, meaning they are coterminal.



Find one positive angle that is coterminal to 110°.

110° + 360° = 470°

Find two positive angles that are coterminal to -30°.

-30°+ 360° = 330°
330°+ 360° = 690°
In this case, the two positive coterminal angles to -30° are 330° and 690°.

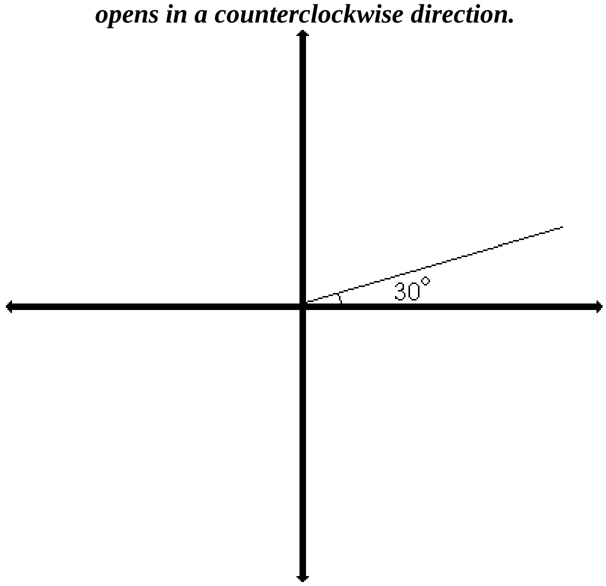
If more than one positive coterminal angle needs to be found, simply add another 360°. This would essentially make the new angle complete two full revolutions before its terminal side comes to rest.

Find one negative angle that is coterminal to 30°.

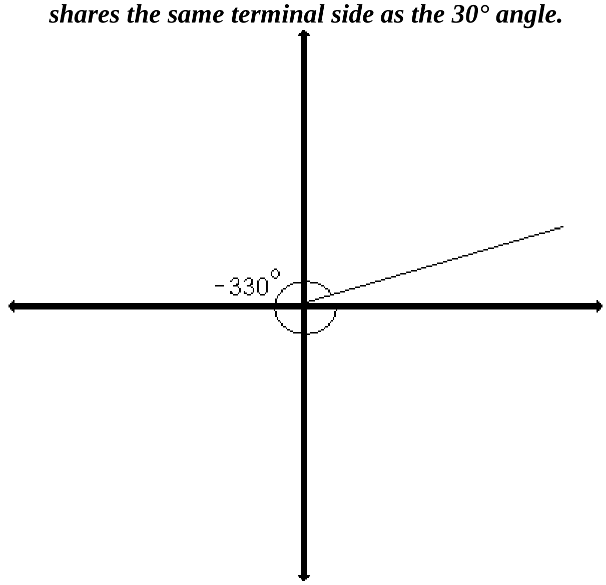
A negative angle moves in a clockwise direction. In this case, to find the negative coterminal angle, subtract 360° from 30°.

30° - 360° = -330°

Below is a 30° angle in standard position. This angle opens in a counterclockwise direction.



Here is a -330° angle. As the angle opens clockwise, it shares the same terminal side as the 30° angle.



Find one negative angle that is coterminal to 150°.

150° - 360° = -210°

Find one negative angle that is coterminal to 415°.

415° - 360° = 55°

Although 55° is a coterminal angle to 415°, this is not a solution to the problem. The problem specifically asked for a negative angle, so the process needs to take place one more time.

55° - 360° = -305°

These were all examples of finding coterminal angles. If the initial angle is given in the form or radians, add or subtract 2π instead of 360°.

Find a positive and negative angle that is coterminal to an angle that is $\frac{\pi}{6}$ radians.

$$\frac{\pi}{6}+2\pi$$
$$\frac{\pi}{6}+\frac{12\pi}{6}$$
$$\frac{13\pi}{6}$$

Adding 2π to the original angle yields the positive coterminal angle.

$$\frac{\pi}{6}-2\pi$$
$$\frac{\pi}{6}-\frac{12\pi}{6}$$
$$-\frac{11\pi}{6}$$

By subtracting 2π from the original angle, the negative coterminal angle has been found.

Find two positive angles that are coterminal to an angle that is $\frac{11\pi}{2}$ radians.

$$\frac{11\pi}{2}-2\pi$$
$$\frac{11\pi}{2}-\frac{4\pi}{2}$$
$$\frac{7\pi}{2}$$

$$\frac{7\pi}{2}-2\pi$$
$$\frac{7\pi}{2}-\frac{4\pi}{2}$$
$$\frac{3\pi}{2}$$

Since $\frac{11\pi}{2}$ is more than one complete revolution, 2π was subtracted from the initial angle yielding a coterminal angle of $\frac{7\pi}{2}$. This is still at least one full revolution, so 2π was subtracted yet again. This process resulted in the two positive coterminal angles of $\frac{7\pi}{2}$ and $\frac{3\pi}{2}$.

Find one positive and one negative coterminal angle of each of the following. There is no need to graph the angles.

A) $30°$

B) $-40°$

C) $150°$

D) $220°$

E) $-330°$

F) $\dfrac{\pi}{3}$

G) $\dfrac{5\pi}{2}$

H) $-\dfrac{2\pi}{3}$

I) $-\dfrac{5\pi}{6}$

J) $\dfrac{5\pi}{3}$

K) $-\dfrac{4\pi}{3}$

L) $300°$

M) $700°$

N) $-\dfrac{17\pi}{6}$

O) $\dfrac{7\pi}{3}$

P) $-410°$

Q) $1000°$

R) $\dfrac{31\pi}{6}$

S) $-\dfrac{15\pi}{4}$

T) $\dfrac{5\pi}{6}$

Conversions

Since both degrees and radian measures will be dealt with in trigonometry, it will sometimes be necessary to convert degrees to radians, or radians to degrees.

The following formulas are used for such conversions.

Degrees to Radians	***Radians to Degrees***
$\text{Degrees} \times \dfrac{\pi}{180°}$	$\text{Radians} \times \dfrac{180°}{\pi}$

Example

Convert 80° to radians.

$$80° \cdot \frac{\pi}{180°}$$
$$\frac{80\pi}{180}$$
$$\frac{4\pi}{9}$$

Once the final product is reduced, it is evident that 80° is equal to $\frac{4\pi}{9}$ radians.

Convert $-\frac{3\pi}{4}$ radians to degrees.

$$-\frac{3\pi}{4} \cdot \frac{180°}{\pi}$$
$$-\frac{540°}{4}$$
$$-135°$$

Notice the negative sign is kept in the problem. Once the final product is reduced, it can be concluded that $-\frac{3\pi}{4}$ radians is equal to -135°

Convert 3 radians to degrees.

$$3 \cdot \frac{180°}{\pi} = \frac{540°}{\pi} \approx 171.887°$$

In this example, there is no degree symbol next to the 3. This means we are looking at 3 radians. Once 3 is multiplied by 180°, divide the result by π. This will result in a decimal estimate of the measure of the angle.

<u>Why are these conversion formulas necessary?</u>

There are certain formulas used in trigonometry such as the, arc-length formula, where the angle used in the calculations must be in radians. These conversion formulas will allow this to be done. Also, it is sometimes difficult to tell in which quadrant the terminal side of an angle lies when it is written in radians. Converting from radians to degrees will make this process easier.

Convert each of the following to Radians.

A) $120°$ **B)** $210°$ **C)** $-60°$ **D)** $420°$

E) $-110°$ **F)** $330°$ **G)** $-45°$ **H)** $150°$

I) $300°$ **J)** $-135°$ **K)** $450°$ **L)** $-210°$

M) $720°$ **N)** $315°$ **O)** $-30°$ **P)** $60°$

Q) $-15°$ **R)** $45°$ **S)** $225°$ **T)** $360°$

Convert each of the following from Radians to Degrees.

A) $\frac{\pi}{6}$ **B)** $\frac{5\pi}{3}$ **C)** $-\frac{\pi}{2}$ **D)** $\frac{3\pi}{4}$

E) $-\frac{\pi}{4}$ **F)** $-\frac{5\pi}{6}$ **G)** $\frac{7\pi}{6}$ **H)** $-\frac{\pi}{6}$

I) 2.3 **J)** $\frac{11\pi}{6}$ **K)** -1.28 **L)** $-\frac{2\pi}{3}$

M) $\frac{7\pi}{4}$ **N)** $-\frac{4\pi}{3}$ **O)** $\frac{5\pi}{4}$ **P)** π

Q) $\frac{13\pi}{6}$ **R)** 4.3 **S)** 2π **T)** $-\frac{11\pi}{3}$

U) $-\frac{9\pi}{4}$ **V)** $\frac{31\pi}{6}$ **W)** $\frac{11\pi}{2}$ **X)** $\frac{23\pi}{6}$

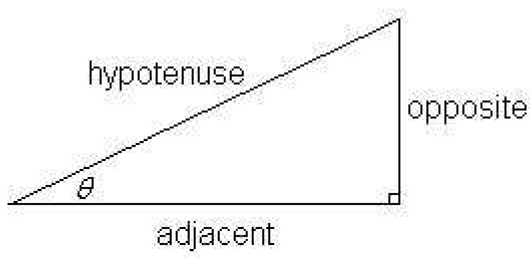
Determine the quadrant in which the terminal side of following angles resides.

A) $\frac{\pi}{6}$	**B)** $\frac{5\pi}{3}$	**C)** $-\frac{\pi}{2}$	**D)** $\frac{3\pi}{4}$
E) $-\frac{\pi}{4}$	**F)** $-\frac{5\pi}{6}$	**G)** $\frac{7\pi}{6}$	**H)** $-\frac{\pi}{6}$
I) 2.3	**J)** $\frac{11\pi}{6}$	**K)** -1.28	**L)** $-\frac{2\pi}{3}$
M) $\frac{7\pi}{4}$	**N)** $-\frac{4\pi}{3}$	**O)** $\frac{5\pi}{4}$	**P)** π
Q) $\frac{13\pi}{6}$	**R)** 4.3	**S)** 2π	**T)** $-\frac{11\pi}{3}$
U) $-\frac{9\pi}{4}$	**V)** $\frac{31\pi}{6}$	**W)** $\frac{11\pi}{2}$	**X)** $\frac{23\pi}{6}$

Basic Trigonometric Functions

There are six basic trigonometric functions used in trigonometry.
Below are the names of the six functions and their three letter abbreviation.

Sine (sin)	**Cosecant (csc)**
Cosine(cos)	**Secant (sec)**
Tangent (tan)	**Cotangent (cot)**



These six trigonometric functions are used to evaluate acute angles in a right triangle. The ratio of the lengths of two sides of a right triangle will be used to evaluate a given angle θ. We will go back to something introduced in geometry for this.

Soh-Cah-Toa

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

The second set of trigonometric functions, are reciprocal functions.

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

Therefore, it follows that...

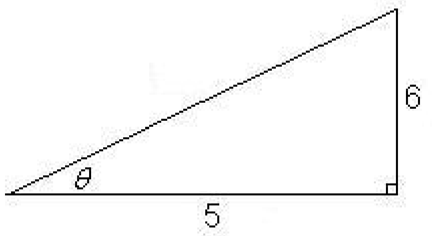
$$\csc\theta = \frac{\text{hyp}}{\text{opp}} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}}$$

Exercises that require finding the exact value of the six trigonometric functions follow on the next few pages. Sometimes, only two of the three sides of a triangle will be given requiring the student to find the third.

When given the lengths of two sides of a right triangle, how can the length of the third side be found?

Example

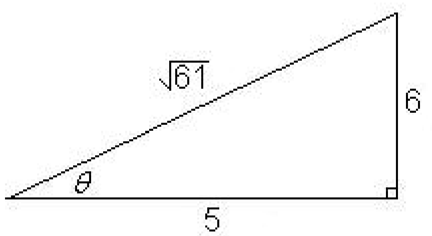
Find the exact values of the six trigonometric functions of θ.



In this example, the length of the side adjacent to the angle θ measures 5 units, while the opposite side measures 6 units. In order to find the missing side, the Pythagorean Theorem must be used.

$$a^2+b^2=c^2$$
$$(5)^2+(6)^2=c^2$$
$$25+36=c^2$$
$$61=c^2$$
$$\sqrt{61}=c$$

Using the Pythagorean Theorem, the length of the hypotenuse is $\sqrt{61}$ units.



Once the value of the hypotenuse is found, we can find the exact value of the six trigonometric functions of the angle θ.

$$\sin\theta=\frac{6}{\sqrt{61}}=\frac{6\sqrt{61}}{61} \qquad \csc\theta=\frac{\sqrt{61}}{6}$$
$$\cos\theta=\frac{5}{\sqrt{61}}=\frac{5\sqrt{61}}{61} \qquad \sec\theta=\frac{\sqrt{61}}{5}$$
$$\tan\theta=\frac{6}{5} \qquad \cot\theta=\frac{5}{6}$$

Evaluate the first three functions using Soh-Cah-Toa.

$$\sin\theta=\frac{\text{opp}}{\text{hyp}} \qquad \cos\theta=\frac{\text{adj}}{\text{hyp}} \qquad \tan\theta=\frac{\text{opp}}{\text{adj}}$$

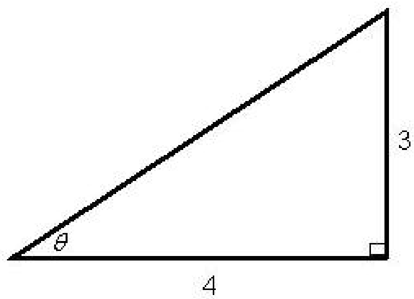
Evaluate second set of functions by finding the reciprocals of the first three.
Do not forget to rationalize any denominators if needed.

$$\sin\theta=\frac{6\sqrt{61}}{61} \qquad \csc\theta=\frac{\sqrt{61}}{6}$$
$$\cos\theta=\frac{5\sqrt{61}}{61} \qquad \sec\theta=\frac{\sqrt{61}}{5}$$
$$\tan\theta=\frac{6}{5} \qquad \cot\theta=\frac{5}{6}$$

Here are the exact values of the six trigonometric functions of the angle θ. Radicals are left in the solutions because we need the <u>exact</u> values, not estimates.

Find the exact values of the six trigonometric functions of θ.

A)

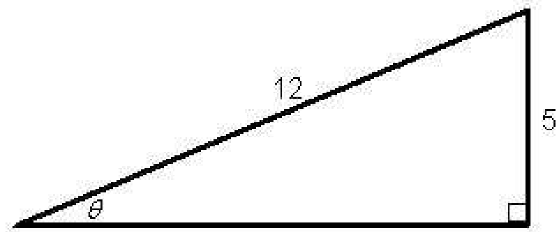


$\sin\theta =$ $\csc\theta =$

$\cos\theta =$ $\sec\theta =$

$\tan\theta =$ $\cot\theta =$

B)

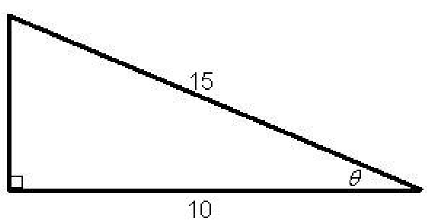


$\sin\theta =$ $\csc\theta =$

$\cos\theta =$ $\sec\theta =$

$\tan\theta =$ $\cot\theta =$

C)

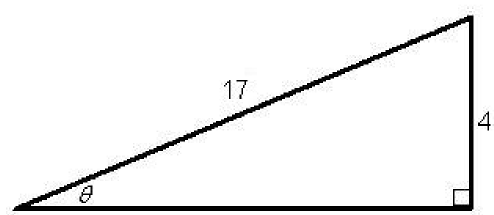


$\sin\theta =$ $\csc\theta =$

$\cos\theta =$ $\sec\theta =$

$\tan\theta =$ $\cot\theta =$

D)

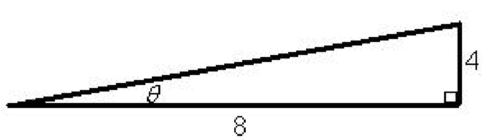


$\sin\theta =$ $\csc\theta =$

$\cos\theta =$ $\sec\theta =$

$\tan\theta =$ $\cot\theta =$

E)



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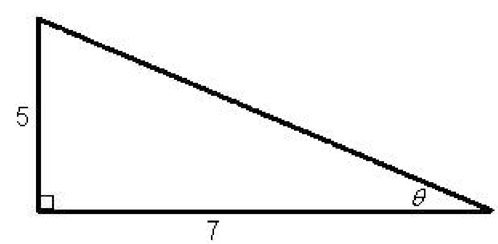

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$\cos\theta =$ $\sec\theta =$

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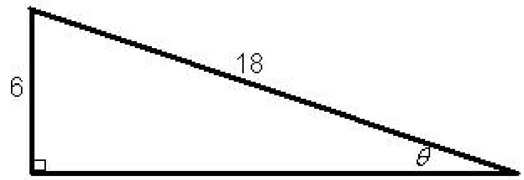
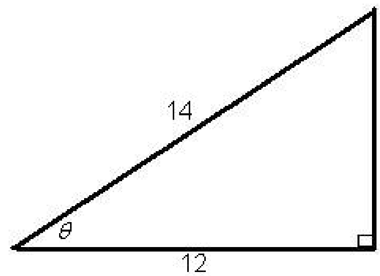
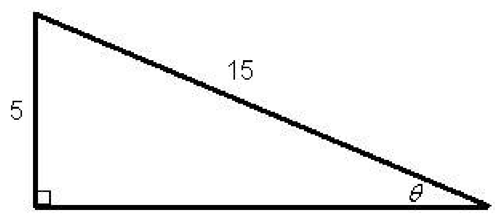
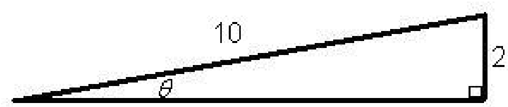
G) 

H) 

$\sin\theta =$	$\csc\theta =$
$\cos\theta =$	$\sec\theta =$
$\tan\theta =$	$\cot\theta =$

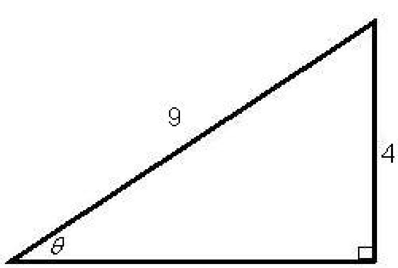
$\sin\theta =$	$\csc\theta =$
$\cos\theta =$	$\sec\theta =$
$\tan\theta =$	$\cot\theta =$

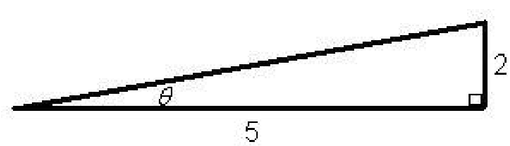
I) 

J) 

$\sin\theta =$	$\csc\theta =$
$\cos\theta =$	$\sec\theta =$
$\tan\theta =$	$\cot\theta =$

$\sin\theta =$	$\csc\theta =$
$\cos\theta =$	$\sec\theta =$
$\tan\theta =$	$\cot\theta =$

K) 

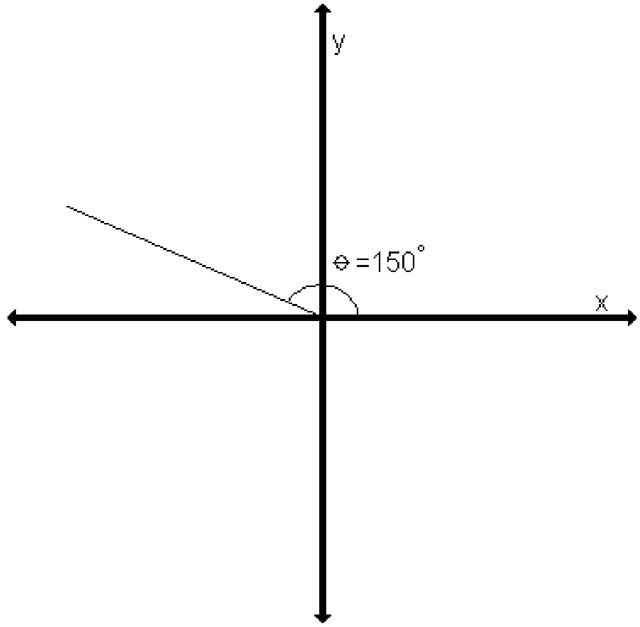
L) 

$\sin\theta =$	$\csc\theta =$
$\cos\theta =$	$\sec\theta =$
$\tan\theta =$	$\cot\theta =$

$\sin\theta =$	$\csc\theta =$
$\cos\theta =$	$\sec\theta =$
$\tan\theta =$	$\cot\theta =$

Reference Angles

The angles we will evaluate in trigonometry will always rest between the terminal side of that angle, and the x axis. Below is a graphical representation of a 150° angle.

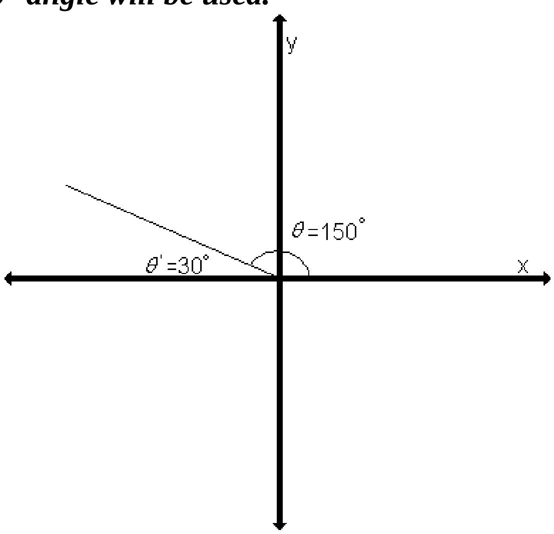


In the illustration above, the terminal side of a 150° angle, in standard position, resides in quadrant II. In order to evaluate the six trigonometric functions of an angle θ, we are required to use a <u>reference angle</u>.

A <u>reference angle</u> is the acute angle θ' (read as theta prime) formed by the terminal side of the angle θ, and the x axis.

REFERENCE ANGLES ARE <u>ALWAYS</u> DRAWN IN RELATION TO THE X AXIS.

Therefore, to evaluate the six trigonometric functions of a 150° angle in standard position, a 30° angle will be used.



Since the horizontal in quadrant II represents 180°, evaluate 180° - 150° to find θ', which is this case is 30°. This is <u>not</u> the same method that will be used for every angle.

If for example, we need to find a reference angle of a 200° angle, evaluate 200° - 180° = 20°.

Here are some guidelines for finding reference angles. The method used to find a reference angle depends on the quadrant in which the terminal side of the angle resides.

- **If the terminal side of an angle θ rests in quadrant I, $\theta' = \theta$.**
- **If the terminal side of an angle θ rests in quadrant II, $\theta' = 180° - \theta$ or $\theta' = \pi - \theta$.**
- **If the terminal side of an angle θ rests in quadrant III, $\theta' = \theta - 180°$ or $\theta' = \theta - \pi$.**
- **If the terminal side of an angle θ rests in quadrant IV, $\theta' = 360° - \theta$ or $\theta' = 2\pi - \theta$.**

Example 1

The angle $\theta = -150°$. Find the reference angle θ'.

Begin by finding the positive coterminal angle to a -150° angle.

-150° + 360° = 210°

The terminal side of a 210° angle resides in quadrant III. Therefore, to find the reference angle use $\theta' = \theta - 180°$.

210° - 180° = 30°

$$\theta' = 30°$$

Example 2

The angle $\theta = 2.5$. Find the reference angle θ'.

In this case, there is no degree symbol. This means the measure of angle θ is 2.5 radians.

First use the formula to convert radians to degrees.

$$2.5 \cdot \frac{180°}{\pi} \approx 143.239°$$

Since the question was asked in terms of radians, the answer must be given in the same way. Converting the angle measure to degrees allows us to get a clear picture of where the terminal side of the angle will lie. In this case, the terminal side of an angle that measures 2.5 radians lies in quadrant II.

To find the reference angle, evaluate $\theta' = \pi - \theta$.

$$\theta' = \pi - 2.5$$

$$\theta' \approx 0.642 \quad radians$$

As illustrated in example 1 on the previous page, it is sometimes necessary to find a coterminal angle first. If θ is negative, first find the coterminal angle, then use that to find the reference angle. If the measure of the original angle is given in degrees, its reference angle must also be in degrees. If the measure of the original angle is given in radians, then the reference angle found must also be in radians. Exact solutions should be found whenever possible. In example 2 on the previous page, it was impossible to give an exact solution, because the measure of angle θ did not include π. Therefore, a decimal approximation had to be made.

For each of the following, find the reference angle $\theta^{'}$.

A) $\theta = 57°$ **B)** $\theta = 113°$ **C)** $\theta = \frac{7\pi}{6}$ **D)** $\theta = \frac{5\pi}{3}$

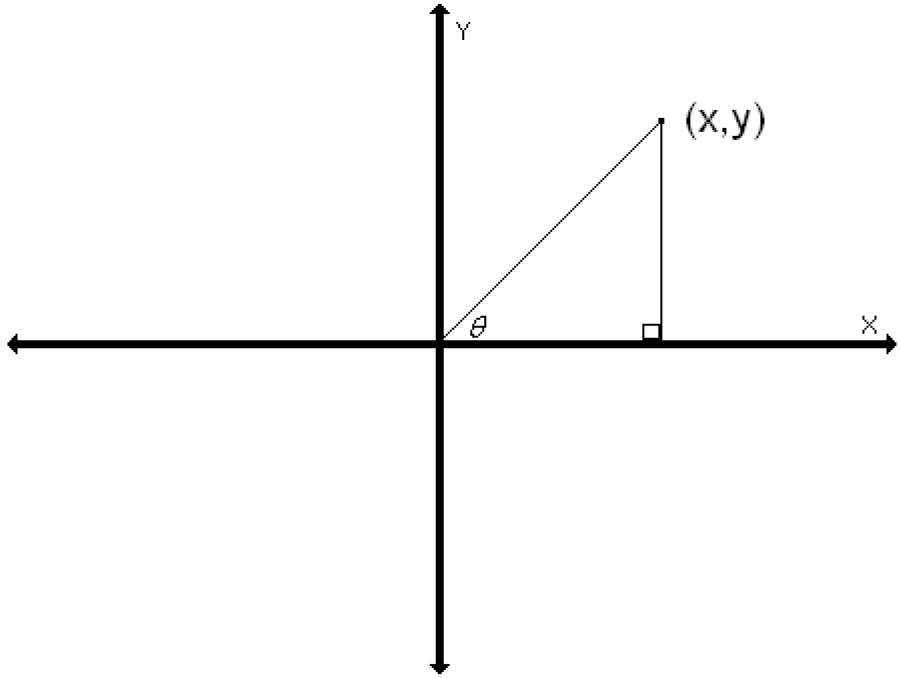
E) $\theta = -\frac{2\pi}{3}$ **F)** $\theta = -230°$ **G)** $\theta = 300°$ **H)** $\theta = 2.3$

I) $\theta = 280°$ **J)** $\theta = 1.2$ **K)** $\theta = 420°$ **L)** $\theta = -60°$

M) $\theta = -2$ **N)** $\theta = 100°$ **O)** $\theta = -\frac{4\pi}{3}$ **P)** $\theta = \frac{11\pi}{6}$

Q) $\theta = -135°$ **R)** $\theta = \frac{17\pi}{6}$ **S)** $\theta = \frac{\pi}{3}$ **T)** $\theta = -\frac{5\pi}{3}$

Trigonometric Functions of any Angle



When evaluating any angle θ, in standard position, whose terminal side is given by the coordinates (x,y), a reference angle is always used. Notice how a right triangle has been created. This will allow us to evaluate the six trigonometric functions of any angle.

Notice the side opposite the angle θ has a length of the y value of the given coordinates. The adjacent side has a length of the x value of the coordinates. The length of the hypotenuse is given by $\sqrt{x^2+y^2}$.

Lets say, for the sake of argument, the length of the hypotenuse is 1 unit. This would mean the following would be true.

$$\sin\theta = y \qquad \csc\theta = \frac{1}{y}$$

$$\cos\theta = x \qquad \sec\theta = \frac{1}{x}$$

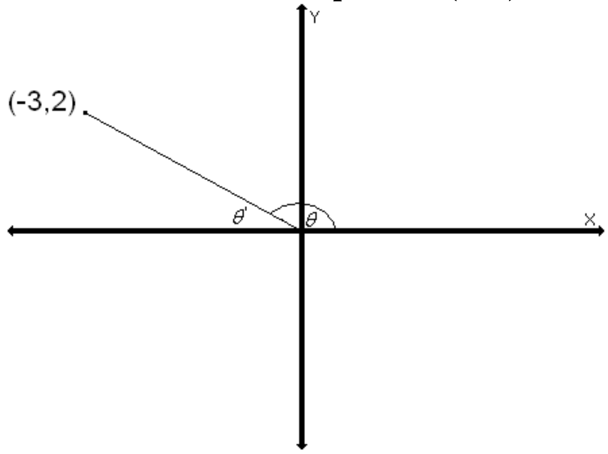
$$\tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{x}{y}$$

You must think of the sine function as giving you the y value, whereas the cosine function yields the x value. This is how we will determine whether the sine, cosine, tangent, cosecant, secant or cotangent of a given angle is a positive or negative value.

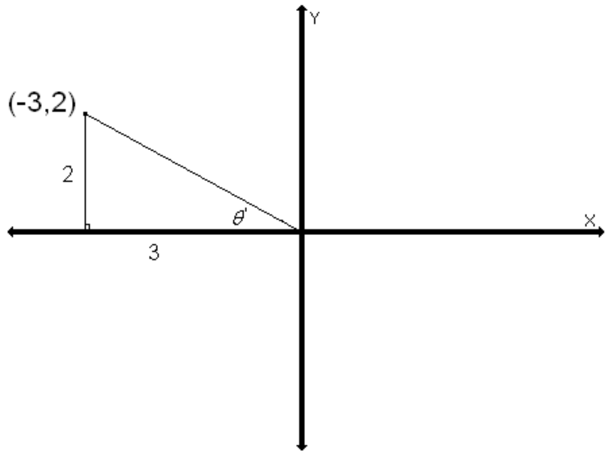
If the angle to be evaluated is in quadrant IV, for instance, the sine of the angle θ will be negative. The cosine of θ, in this instance, will be positive, while the tangent of the angle θ will be negative.

Example

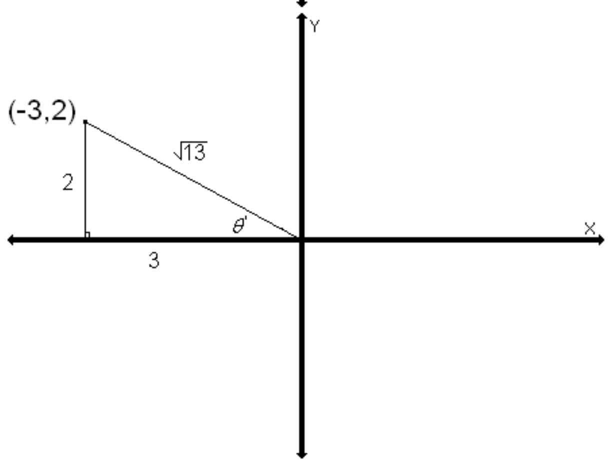
Evaluate the six trigonometric functions of an angle θ, in standard position, whose terminal side has an endpoint of (-3,2).



The angle with terminal side is first drawn. Remember, in order to evaluate the six trigonometric functions for θ, use the reference angle θ'.



From the endpoint of the terminal side of the angle, a line is drawn to the x axis. This is the reason reference angles are always drawn in relation to the x axis. It will always create a right triangle with which to work. Now all that is needed to solve the problem, is to find the length of the hypotenuse then the values of the six trigonometric functions can be found.



Using the Pythagorean Theorem, the length of the hypotenuse may be found.

$$2^2 + 3^2 = c^2$$

$$4 + 9 = c^2$$

$$13 = c^2$$

$$\sqrt{13} = c$$

The first three functions are evaluated using Soh-Cah-Toa.

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} \qquad \cos\theta = \frac{\text{adj}}{\text{hyp}} \qquad \tan\theta = \frac{\text{opp}}{\text{adj}}$$

To find the second set of functions take the reciprocals of the first three. Rationalize any denominators if needed. Note the terminal side to this angle is in quadrant II. This means cosine, tangent, secant and cotangent are all negative values.

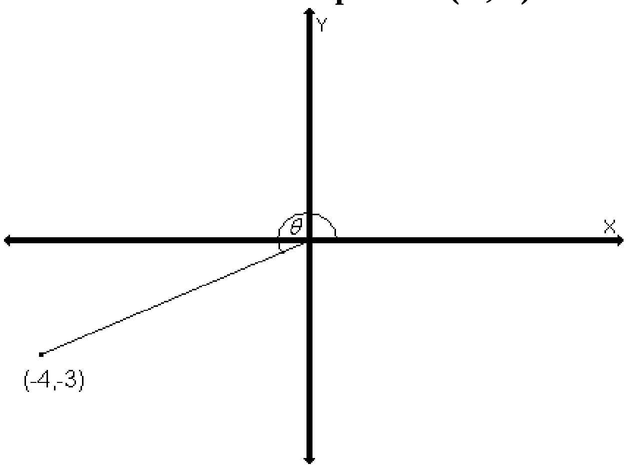
$$\sin\theta = \frac{2}{\sqrt{13}} = \frac{2\sqrt{13}}{13} \qquad \csc\theta = \frac{\sqrt{13}}{2}$$

$$\cos\theta = -\frac{3}{\sqrt{13}} = -\frac{3\sqrt{13}}{13} \qquad \sec\theta = -\frac{\sqrt{13}}{3}$$

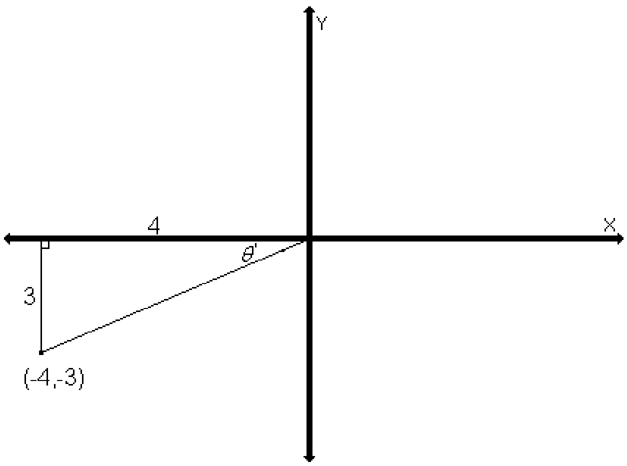
$$\tan\theta = -\frac{2}{3} \qquad \cot\theta = -\frac{3}{2}$$

Example

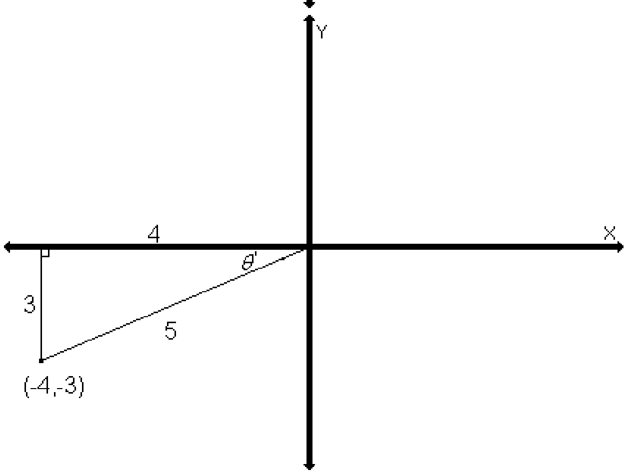
Evaluate the six trigonometric functions of the angle θ, in standard position, whose terminal side has an endpoint of (-4,-3).



Begin by drawing the angle θ in standard position whose terminal side has the endpoint of (-4,-3).



A right triangle is formed by drawing a line segment to the x axis. Now use the reference angle that is drawn in relation to the x axis to evaluate the six trigonometric functions.



Since this is obviously a 3-4-5 right triangle, there is no need to use the Pythagorean Theorem in this case.

$$\sin\theta = -\frac{3}{5} \qquad \csc\theta = -\frac{5}{3}$$

$$\cos\theta = -\frac{4}{5} \qquad \sec\theta = -\frac{5}{4}$$

$$\tan\theta = \frac{3}{4} \qquad \cot\theta = \frac{4}{3}$$

Since this angle resides in quadrant III, sine, cosine, cosecant and secant are negative values. Tangent is $\frac{y}{x}$ and cotangent is $\frac{x}{y}$. This means both tangent and cotangent will be positive values.

Evaluate the six trigonometric functions of the angle θ, in standard position, that has a terminal side with the following endpoints. ***(Remember, reference angles are always drawn in relation to the x axis.)***

A) $(3,5)$

$\sin\theta =$ $\csc\theta =$

$\cos\theta =$ $\sec\theta =$

$\tan\theta =$ $\cot\theta =$

B) $(2,-1)$

$\sin\theta =$ $\csc\theta =$

$\cos\theta =$ $\sec\theta =$

$\tan\theta =$ $\cot\theta =$

C) $(-4,2)$

$\sin\theta =$ $\csc\theta =$

$\cos\theta =$ $\sec\theta =$

$\tan\theta =$ $\cot\theta =$

D) $(-3,-5)$

$\sin\theta =$ $\csc\theta =$

$\cos\theta =$ $\sec\theta =$

$\tan\theta =$ $\cot\theta =$

E) $(1,-7)$

$\sin\theta =$ $\csc\theta =$

$\cos\theta =$ $\sec\theta =$

$\tan\theta =$ $\cot\theta =$

F) $(-6,1)$

$\sin\theta =$ $\csc\theta =$

$\cos\theta =$ $\sec\theta =$

$\tan\theta =$ $\cot\theta =$

G) $\left(\frac{1}{2},8\right)$

H) $\left(\frac{1}{4},-\frac{2}{5}\right)$

$\sin\theta =$ $\csc\theta =$

$\cos\theta =$ $\sec\theta =$

$\tan\theta =$ $\cot\theta =$

$\sin\theta =$ $\csc\theta =$

$\cos\theta =$ $\sec\theta =$

$\tan\theta =$ $\cot\theta =$

I) $(-2,-9)$

J) $(-1,6)$

$\sin\theta =$ $\csc\theta =$

$\cos\theta =$ $\sec\theta =$

$\tan\theta =$ $\cot\theta =$

$\sin\theta =$ $\csc\theta =$

$\cos\theta =$ $\sec\theta =$

$\tan\theta =$ $\cot\theta =$

K) $(4,-3)$

L) $\left(\frac{3}{4},\frac{4}{5}\right)$

$\sin\theta =$ $\csc\theta =$

$\cos\theta =$ $\sec\theta =$

$\tan\theta =$ $\cot\theta =$

$\sin\theta =$ $\csc\theta =$

$\cos\theta =$ $\sec\theta =$

$\tan\theta =$ $\cot\theta =$

Once again, think of the sine of an angle θ as yielding the y value, while the cosine yields the x value when the hypotenuse is 1. Since the tangent of an angle is y over x, as $\sin\theta = y$ and $\cos\theta = x$; it is a trigonometric identity that $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$.

If the hypotenuse is any other length, the following is true.

$$\sin\theta = \frac{y}{h} \qquad \csc\theta = \frac{h}{y}$$

$$\cos\theta = \frac{x}{h} \qquad \sec\theta = \frac{h}{x}$$

$$\tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{x}{y}$$

These are the actual equations used for evaluating the six trigonometric functions. The reason we think of sine being the y value, cosine being the x value, and tangent being sine divided by cosine is to determine whether the value of a trigonometric function is positive or negative. This of course all depends on where the terminal side of the angle lies.

The following questions will require evaluating the six trigonometric functions of an angle θ given different types of information. Understand that these are the same types of questions encountered on the previous pages, just asked in a different manner.

Evaluate the six trigonometric functions of an angle θ, *in standard position, where* $\sin\theta = \dfrac{2}{3}$ *and* $\cos\theta < 0$.

This question will be done shortly. Since the sine of an angle can be thought of as the y value there are two quadrants in which sine is positive. It is therefore necessary to have one more piece of information to answer the question. There is some vital information that is needed to answer this type of question.

The following guidelines will help determine in which quadrant an angle lies.

- *Sine is positive in quadrants I and II.*
- *Sine is negative in quadrants III and IV.*
- *Cosine is positive in quadrants I and IV.*
- *Cosine is negative in quadrants II and III.*
- *Tangent is positive in quadrants I and III.*
- *Tangent is negative in quadrants II and IV.*

For this particular problem, $\sin\theta > 0$ and $\tan\theta < 0$, this means the angle θ must reside in quadrant II. This information tells us where to construct our triangle.

Before we answer the sample question, here is some practice using this information.

Given the following information, determine the quadrant in which the angle θ resides.

A) $\cos\theta > 0$ and $\sin\theta < 0$

B) $\tan\theta > 0$ and $\cos\theta < 0$

C) $\sin\theta > 0$ and $\tan\theta < 0$

D) $\tan\theta < 0$ and $\cos\theta < 0$

E) $\sec\theta > 0$ and $\csc\theta > 0$

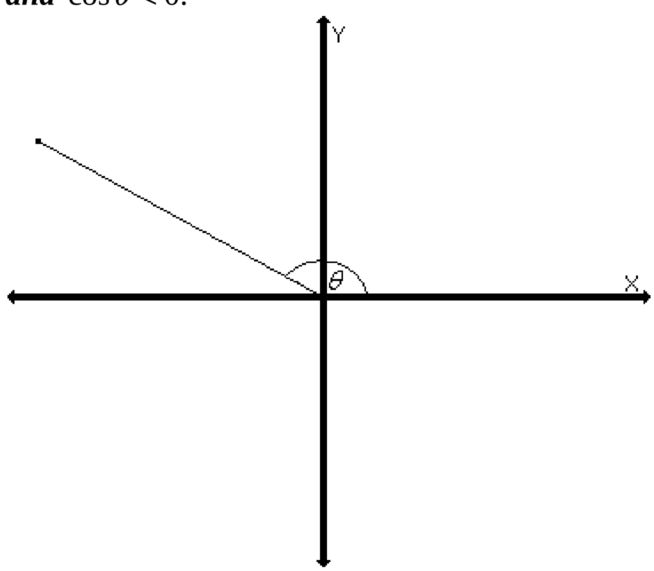
F) $\cot\theta > 0$ and $\sec\theta > 0$

G) $\csc\theta < 0$ and $\cos\theta < 0$

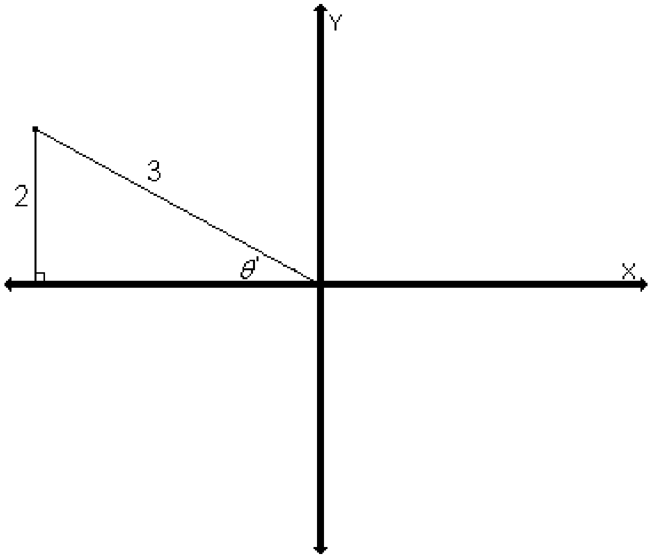
H) $\cot\theta > 0$ and $\sec\theta < 0$

Example

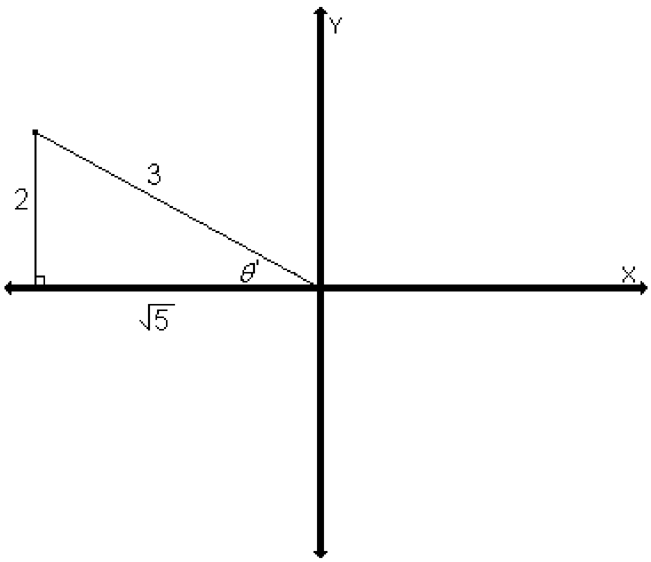
Evaluate the six trigonometric functions of an angle θ, *in standard position, where* $\sin\theta = \dfrac{2}{3}$ ***and*** $\cos\theta < 0$.



According to the information given, the sine of the angle is positive and cosine is negative. This means the terminal side of the angle to be evaluated must be in quadrant II.



From here, we will use the reference angle drawn in relation to the x axis. A right triangle is then constructed. Since the sine of an angle is opposite over hypotenuse, the 2 and the 3 can be placed on the appropriate sides of the triangle.



Using the Pythagorean Theorem, the adjacent side is found to be $\sqrt{5}$ ***units.***

$$x^2+2^2=3^2$$
$$x^2+4=9$$
$$x^2=5$$
$$x=\sqrt{5}$$

$\sin\theta = \frac{2}{3}$ $\quad\csc\theta = \frac{3}{2}$

$\cos\theta = -\frac{\sqrt{5}}{3}$ $\quad\sec\theta = -\frac{3}{\sqrt{5}} = -\frac{3\sqrt{5}}{5}$

$\tan\theta = -\frac{2}{\sqrt{5}} = -\frac{2\sqrt{5}}{5}$ $\quad\cot\theta = -\frac{\sqrt{5}}{2}$

At the beginning of the problem, the value of sine was given. Therefore, we can fill in the values of sine and cosecant right away. From that point, the other values can be found using:
Soh-Cah-Toa.

$\sin\theta = \frac{\text{opp}}{\text{hyp}} \quad \cos\theta = \frac{\text{adj}}{\text{hyp}} \quad \tan\theta = \frac{\text{opp}}{\text{adj}}$

$\sin\theta = \frac{2}{3}$ $\quad\csc\theta = \frac{3}{2}$

$\cos\theta = -\frac{\sqrt{5}}{3}$ $\quad\sec\theta = -\frac{3\sqrt{5}}{5}$

$\tan\theta = -\frac{2\sqrt{5}}{5}$ $\quad\cot\theta = -\frac{\sqrt{5}}{2}$

Here are the values of the six trigonometric functions of the angle θ***.***

Find the exact value of the six trigonometric functions of an angle θ, in standard position, given the following information.

A) $\cos\theta = \frac{1}{2}, \quad \sin\theta < 0$

B) $\sin\theta = -\frac{3}{4}, \quad \tan\theta > 0$

$\sin\theta =$	$\csc\theta =$
$\cos\theta =$	$\sec\theta =$
$\tan\theta =$	$\cot\theta =$

$\sin\theta =$	$\csc\theta =$
$\cos\theta =$	$\sec\theta =$
$\tan\theta =$	$\cot\theta =$

C) $\sin\theta = \frac{1}{4}, \quad \cos\theta > 0$

D) $\sec\theta = 3, \quad \csc\theta < 0$

$\sin\theta =$	$\csc\theta =$
$\cos\theta =$	$\sec\theta =$
$\tan\theta =$	$\cot\theta =$

$\sin\theta =$	$\csc\theta =$
$\cos\theta =$	$\sec\theta =$
$\tan\theta =$	$\cot\theta =$

E) $\tan\theta = -\frac{\sqrt{5}}{2}, \quad \sin\theta < 0$

F) $\cos\theta = \frac{\sqrt{3}}{2}, \quad \sin\theta < 0$

$\sin\theta =$	$\csc\theta =$
$\cos\theta =$	$\sec\theta =$
$\tan\theta =$	$\cot\theta =$

$\sin\theta =$	$\csc\theta =$
$\cos\theta =$	$\sec\theta =$
$\tan\theta =$	$\cot\theta =$

Continued

G) $\csc\theta = \frac{3}{2}, \quad \cos\theta < 0$

$\sin\theta =$ $\quad$ $\csc\theta =$

$\cos\theta =$ $\quad$ $\sec\theta =$

$\tan\theta =$ $\quad$ $\cot\theta =$

H) $\cot\theta = \sqrt{2}, \quad \cos\theta > 0$

$\sin\theta =$ $\quad$ $\csc\theta =$

$\cos\theta =$ $\quad$ $\sec\theta =$

$\tan\theta =$ $\quad$ $\cot\theta =$

I) $\sec\theta = -\frac{1}{5}, \quad \cot\theta < 0$

$\sin\theta =$ $\quad$ $\csc\theta =$

$\cos\theta =$ $\quad$ $\sec\theta =$

$\tan\theta =$ $\quad$ $\cot\theta =$

J) $\cot\theta = \frac{\sqrt{2}}{3}, \quad \sin\theta < 0$

$\sin\theta =$ $\quad$ $\csc\theta =$

$\cos\theta =$ $\quad$ $\sec\theta =$

$\tan\theta =$ $\quad$ $\cot\theta =$

K) $\csc\theta = \frac{2\sqrt{3}}{3}, \quad \cos\theta > 0$

$\sin\theta =$ $\quad$ $\csc\theta =$

$\cos\theta =$ $\quad$ $\sec\theta =$

$\tan\theta =$ $\quad$ $\cot\theta =$

L) $\sec\theta = -\frac{3\sqrt{5}}{5}, \quad \tan\theta < 0$

$\sin\theta =$ $\quad$ $\csc\theta =$

$\cos\theta =$ $\quad$ $\sec\theta =$

$\tan\theta =$ $\quad$ $\cot\theta =$

Answer the following questions, keeping the following information in mind.

$$\sin\theta = \frac{y}{h} \qquad \csc\theta = \frac{h}{y}$$

$$\cos\theta = \frac{x}{h} \qquad \sec\theta = \frac{h}{x}$$

$$\tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{x}{y}$$

Describe all angles that satisfy the questions below in the interval $0° \le \theta \le 360°$**.**

What angle would cause the value of $\tan\theta$ **to be undefined?**

What angle would cause the value of $\tan\theta$ **equal zero?**

What angle would cause $\cot\theta$ **to be undefined?**

What angle would cause the value of $\cot\theta$ **to be equal to zero?**

What angle would cause the value of $\sin\theta$ **to be equal to zero?**

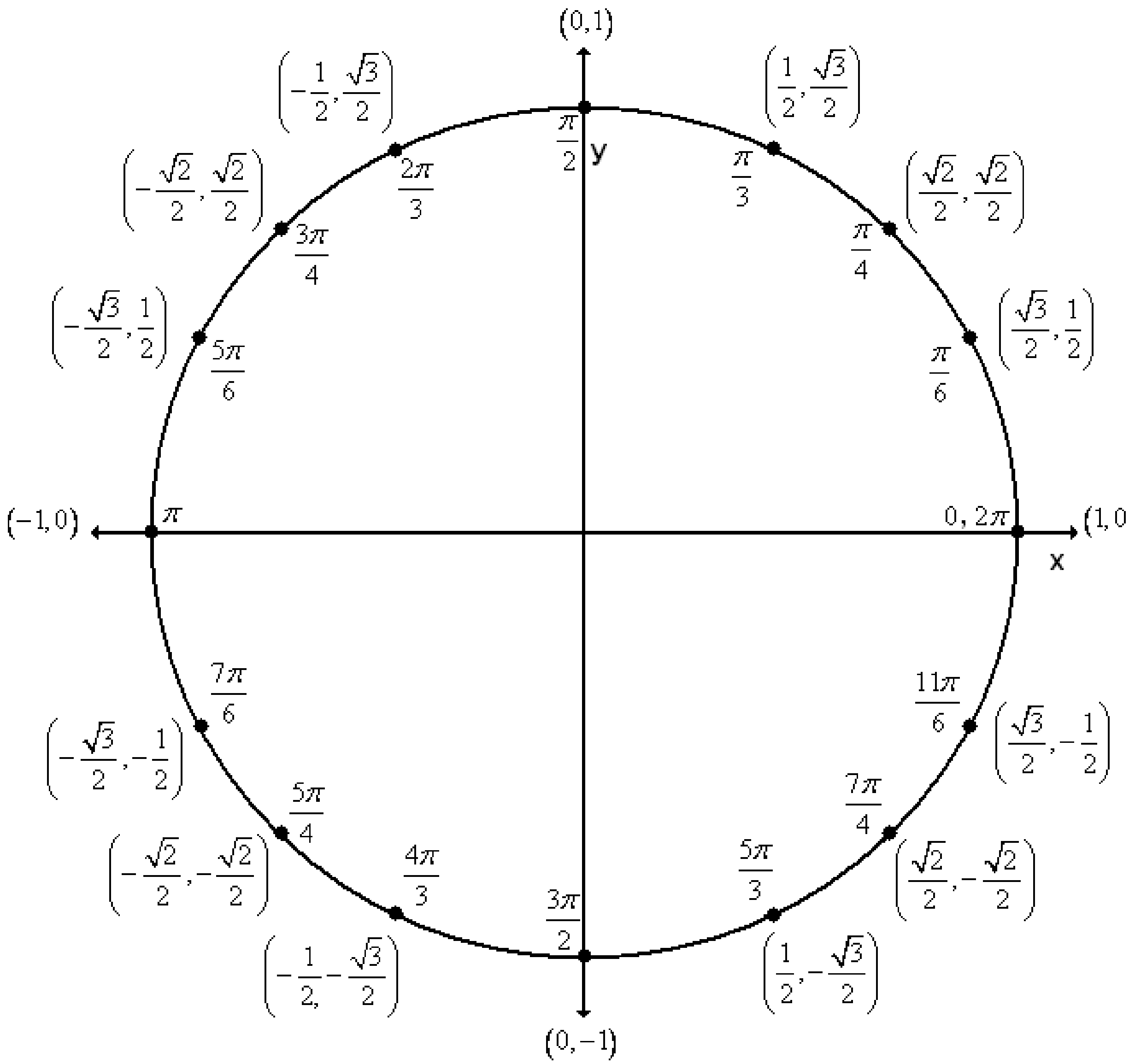
What angle would cause the value of $\csc\theta$ **to be undefined?**

What angle would cause the value of $\cos\theta$ **to be equal to zero.**

What angle would cause the value of $\sec\theta$ **to be undefined?**

The Unit Circle

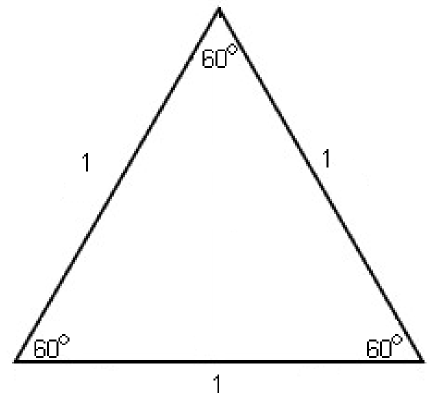
The unit circle is without a doubt the most critical topic a student must understand in trigonometry. The unit circle is the foundation on which trigonometry is based. If someone were to look at the unit circle and try to memorize it, they may find it difficult. In this section, we will discuss how to construct the unit circle, and exactly where those numbers on the unit circle come from.



This is called the unit circle, because the radius of the circle is exactly one unit. The numbers on the outside of the circle represent coordinates. These will be the x and y values with which various trigonometric functions can be evaluated. The numbers on the inside represent the radian measure of the angle. The construction of the unit circle entails the use of a conversion formula, and two different triangles. The two triangle used in the construction of a unit circle are a 30°-60°-90° right triangle, and a 45°-45°-90° right triangle. The lengths of the sides of the 30°-60°-90° triangle can be derived from a standard equilateral triangle.

The 30°-60°-90° triangle

To the right is a standard equilateral triangle. In this particular triangle, the length of each side is one unit.



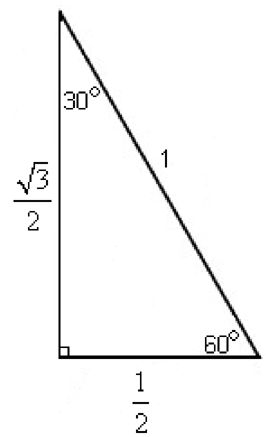
If the equilateral triangle is bisected, a 30°-60°-90° right triangle is formed. Since the triangle is bisected, the base is cut in half. The find the height of the triangle, use the Pythagorean Theorem.

$$x^2 + \left(\frac{1}{2}\right)^2 = 1^2$$

$$x^2 + \frac{1}{4} = 1$$

$$x^2 = \frac{3}{4}$$

$$x = \frac{\sqrt{3}}{2}$$



As a result, the side opposite the 60° angle has a length of $\frac{\sqrt{3}}{2}$ ***units, while the side opposite the 30° angle has a length of*** $\frac{1}{2}$ ***units. The hypotenuse was never touched, so the length of the hypotenuse remains 1 unit.***

The 45°-45°-90° triangle

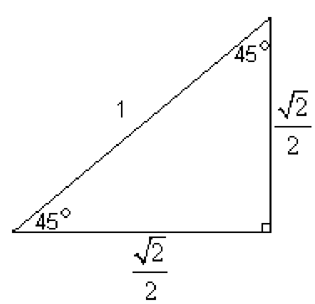
The lengths of the legs of a 45°-45°-90° triangle can be found using the Pythagorean Theorem. Since this is an isosceles triangle, the length of the two legs are equal to each other.

$$x^2 + x^2 = 1^2$$

$$2x^2 = 1$$

$$x^2 = \frac{1}{2}$$

$$x = \frac{\sqrt{2}}{2}$$



When dealing with a 45°-45°-90° triangle, the length of the sides opposite the 45° angles is $\frac{\sqrt{2}}{2}$

Building the Unit Circle

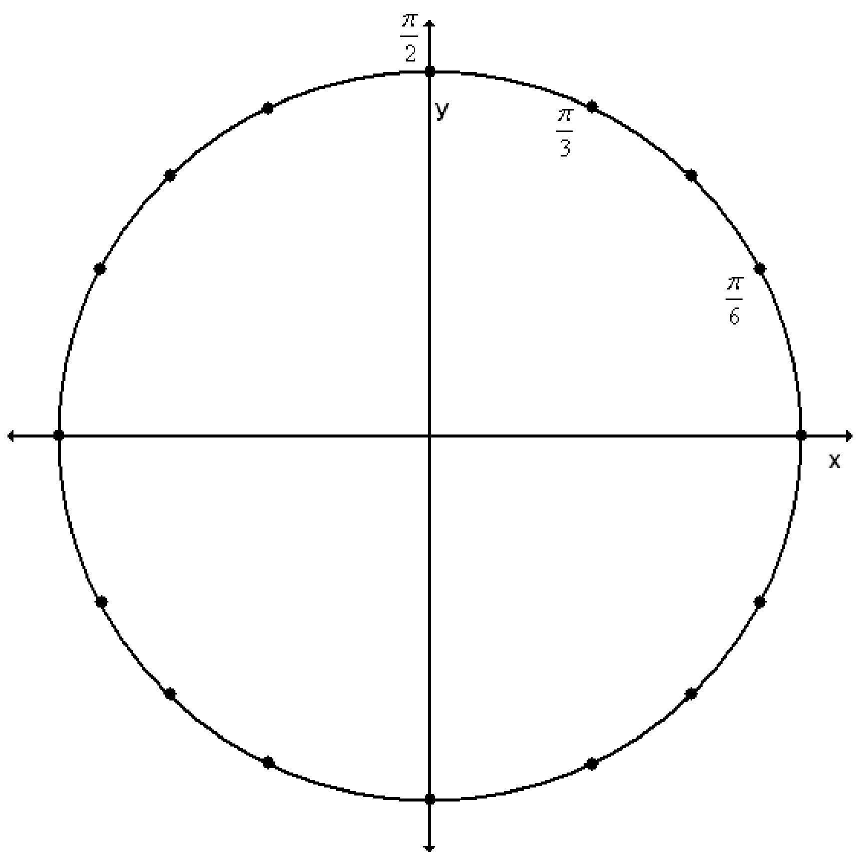
The first objective when building the unit circle is to use the conversion formula to find out the radian measures for a 30° angle, and a 45° angle. All of the angles used on the unit circle are multiples of the 30° angle and the 45° angle. Therefore, all that is needed it to add the required set. In other words, 120° is made up by 4 30° angles. A 30° angle is $\frac{\pi}{6}$ radians. Adding four of these together results in $\frac{4\pi}{6}$ radians which reduces to $\frac{2\pi}{3}$.

$$30° \cdot \frac{\pi}{180°} \qquad\qquad 45° \cdot \frac{\pi}{180°}$$

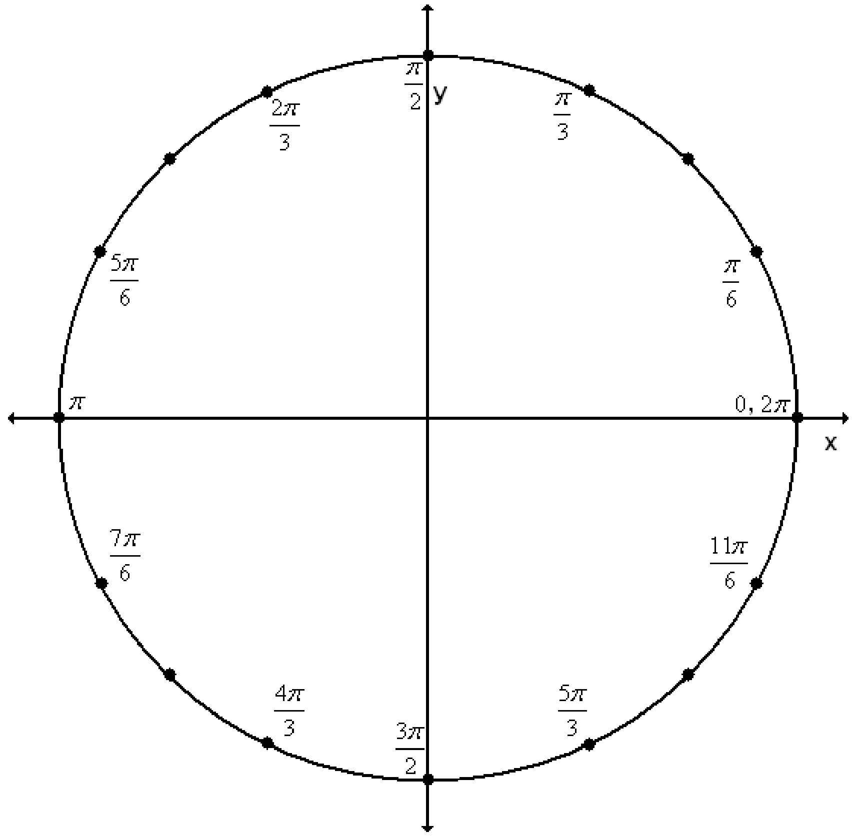
$$\frac{30\pi}{180} \qquad\qquad \frac{45\pi}{180}$$

$$\frac{\pi}{6} \qquad\qquad \frac{\pi}{4}$$

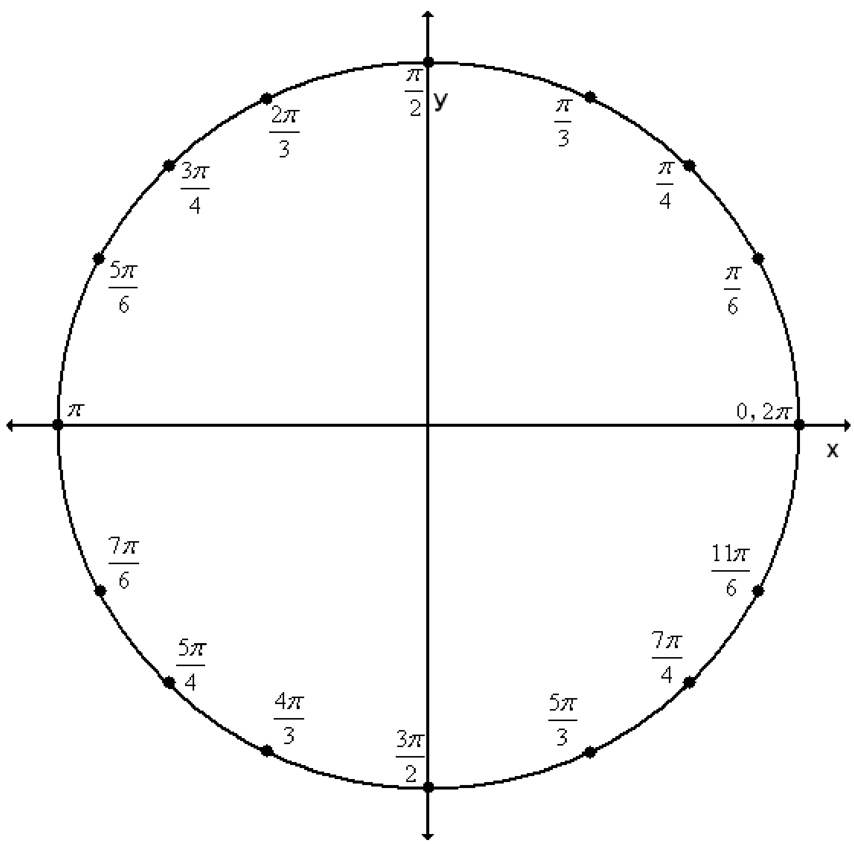
Using the conversion formula, a 30° angle is $\frac{\pi}{6}$ radians, and a 45° angle is $\frac{\pi}{4}$ radians.



Begin at the 30° angle. Place $\frac{\pi}{6}$ at that location and move around the circle in a counterclockwise direction adding by $\frac{\pi}{6}$ at every 30° increment. Make sure to reduce the totals when possible. For example, in the above diagram, to find the radian measure for 60°, add $\frac{\pi}{6}$ together twice. The results in $\frac{2\pi}{6}$ which is reduced to $\frac{\pi}{3}$.



In the diagram on the right, all multiples of the 30° angles have been completed.



The next step is to continue moving counterclockwise, inputting all 45° angles.

A 45° angle is $\frac{\pi}{4}$ ***radians.***

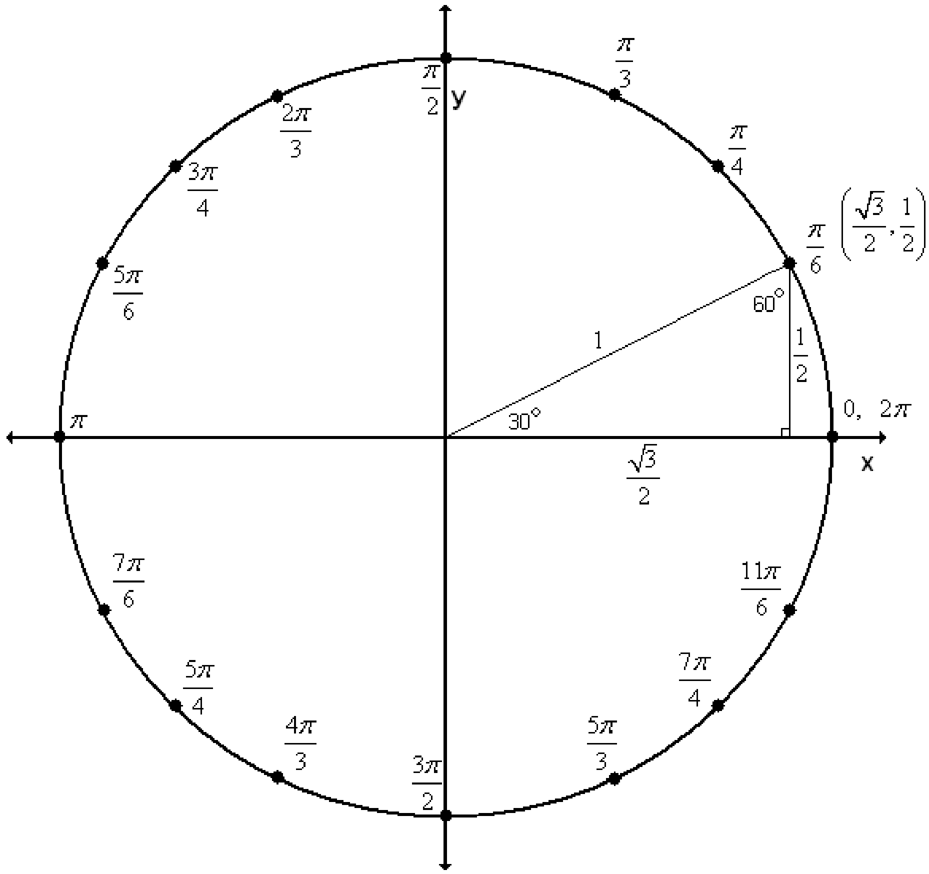
Remember to reduce any fractions if possible.

If a diagonal is drawn at the 30° angle, it will intercept the circle the first point. From there, a line is drawn to the x axis. This creates a 30°-60°-90° triangle. As demonstrated before:

The side opposite the 30° angle has a length of $\frac{1}{2}$.

The side opposite the 60° angle has a length of $\frac{\sqrt{3}}{2}$.

Since the length of the hypotenuse is 1 unit, we have found the x and y values at $\frac{\pi}{6}$. This yields to coordinates $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.

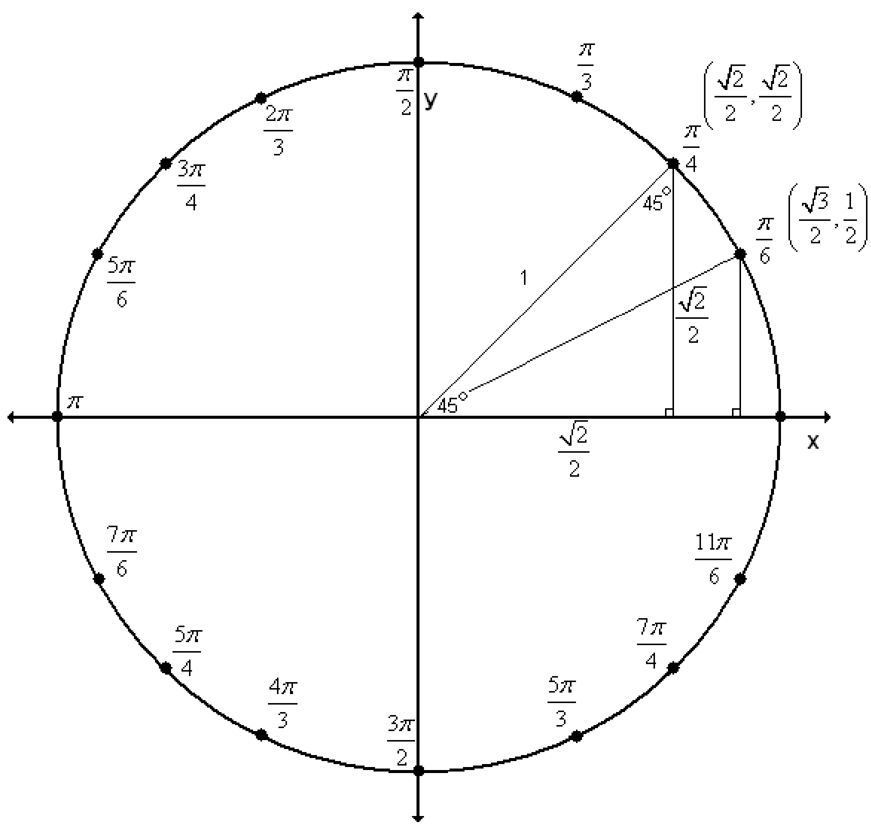


In this instance, a diagonal is drawn at the 45° angle to intercept the arc of the circle. The second line is then drawn to the x axis creating a 45°-45°-90° triangle.

The hypotenuse of the triangle has a length of 1 unit.

The length of the legs of the 45°-45°-90° triangle is $\frac{\sqrt{2}}{2}$ units.

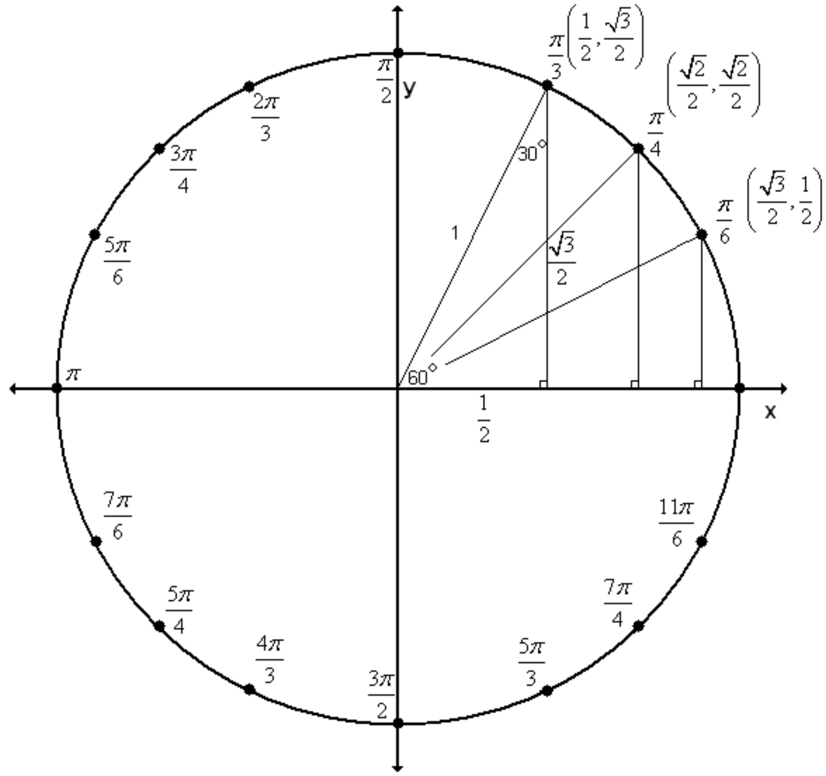
The x and y values here can be put together to find the coordinates of $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.



The same process is done for $\frac{\pi}{3}$.

Once again, a triangle is formed from which the lengths of the sides of the triangle are determined.

The coordinates here are $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.

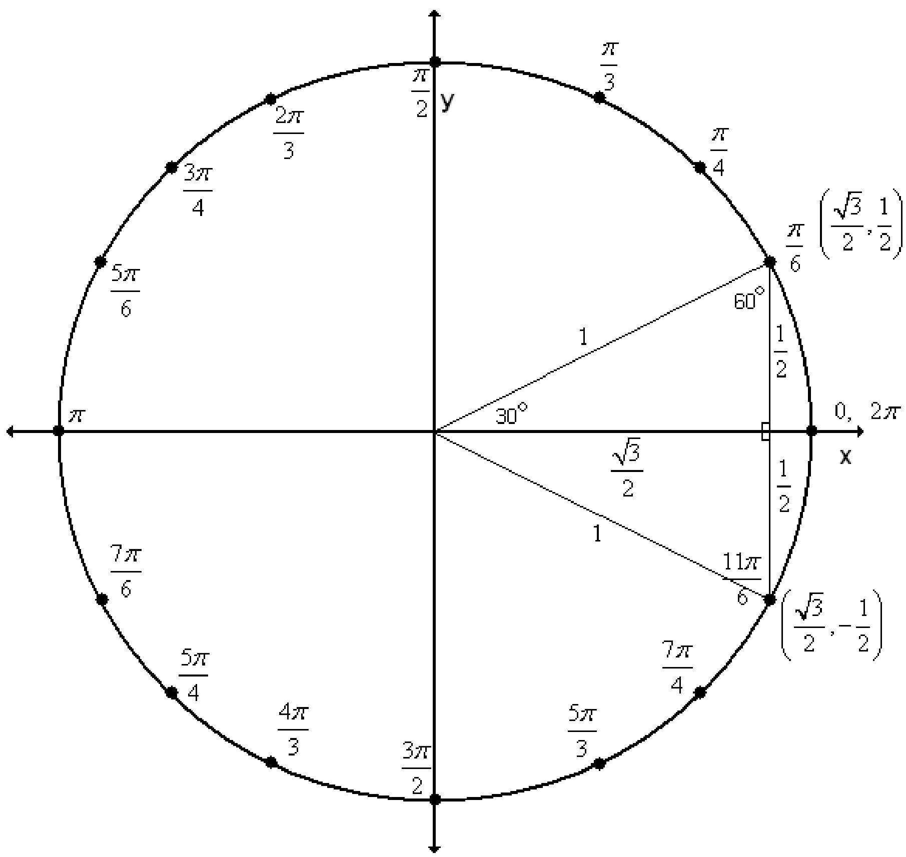


To fill in the remaining coordinates use reflections of the triangle. As illustrated here, the lengths of the sides of the triangle formed at $\frac{11\pi}{6}$ are the same as those for $\frac{\pi}{6}$.

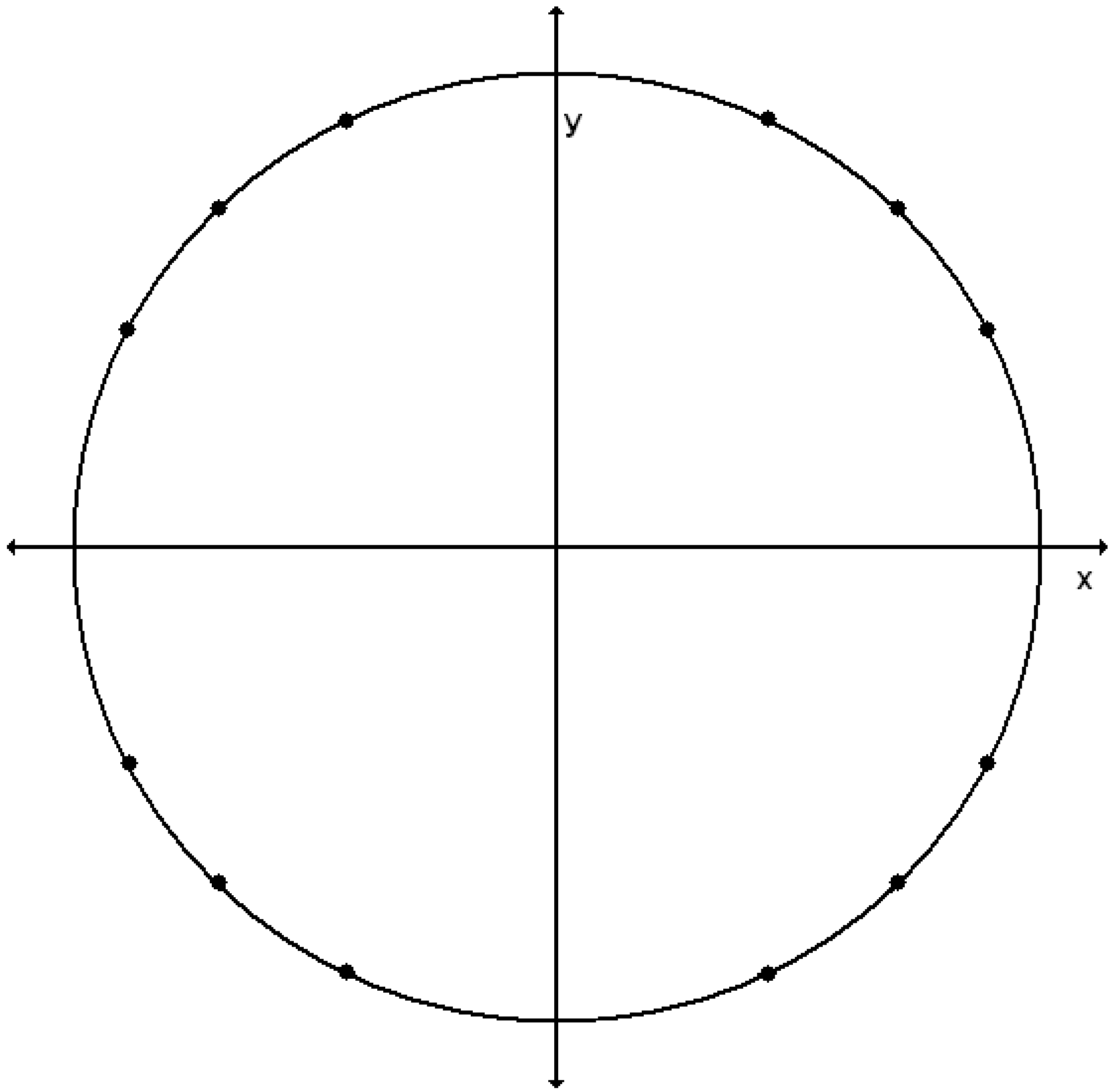
When labeling the coordinate here, however, the y value must be negative because the angle is in quadrant IV.

Once the coordinates are found in quadrant I, all others are reflections. Just take care with the sign being used.

Since the hypotenuse is always one, the coordinates on the axes are simple to find.



Complete the unit circle. Label all required radians and the coordinates for each.



<u>Using the Unit Circle</u>

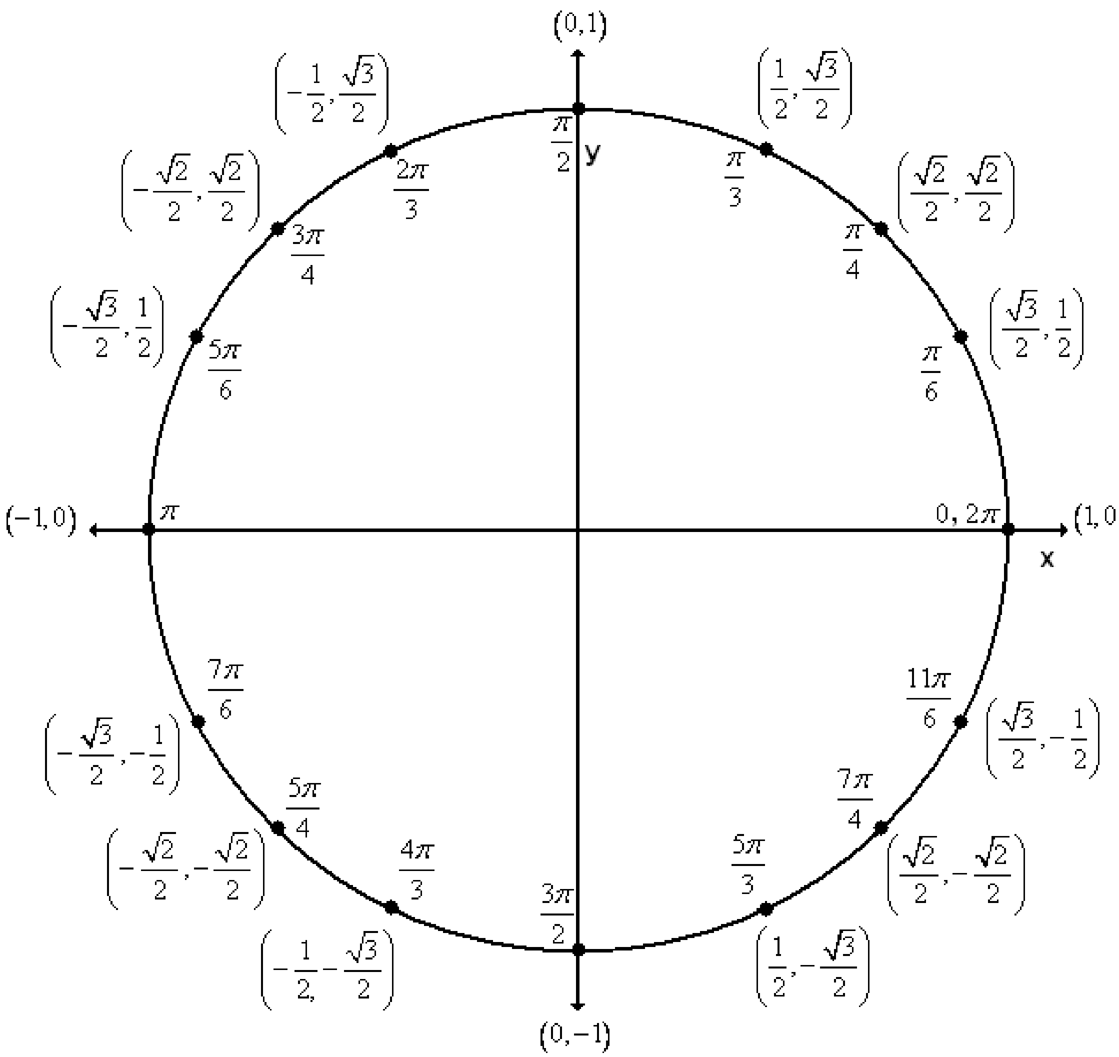
The hypotenuse of the unit circle has a length of one unit. Therefore, whenever any angle needs to be evaluated using any of the trigonometric functions, the following will be used.

$$\sin\theta = y \qquad \csc\theta = \frac{1}{y}$$

$$\cos\theta = x \qquad \sec\theta = \frac{1}{x}$$

$$\tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{x}{y}$$

Think of the sine of an angle being the y value of the coordinate, the cosine of an angle as being the x value of the coordinate, and the tangent of an angle being x over y. Then the reciprocals will be taken for the second set of functions.



When reading through the following examples, refer to the unit circle on the previous page.

Examples

Find the exact value of the six trigonometric functions for $\frac{4\pi}{3}$.

$\sin\frac{4\pi}{3} = -\frac{\sqrt{3}}{2}$ $\quad\csc\frac{4\pi}{3} = -\frac{2}{\sqrt{3}} = -\frac{2\sqrt{3}}{3}$

$\cos\frac{4\pi}{3} = -\frac{1}{2}$ $\quad\sec\frac{4\pi}{3} = -2$

$\tan\frac{4\pi}{3} = \left(-\frac{\sqrt{3}}{2} \div -\frac{1}{2}\right) = \left(-\frac{\sqrt{3}}{2} \cdot -2\right) = \sqrt{3}$ $\quad\cot\frac{4\pi}{3} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

$\sin\frac{4\pi}{3} = -\frac{\sqrt{3}}{2}$ $\quad\csc\frac{4\pi}{3} = -\frac{2\sqrt{3}}{3}$

$\cos\frac{4\pi}{3} = -\frac{1}{2}$ $\quad\sec\frac{4\pi}{3} = -2$

$\tan\frac{4\pi}{3} = \sqrt{3}$ $\quad\cot\frac{4\pi}{3} = \frac{\sqrt{3}}{3}$

Locate the coordinates at $\frac{4\pi}{3}$. The y value at $\frac{4\pi}{3}$ is $-\frac{\sqrt{3}}{2}$. The x value at $\frac{4\pi}{3}$ is $-\frac{1}{2}$.

Therefore, sin $\frac{4\pi}{3} = -\frac{\sqrt{3}}{2}$, and cos $\frac{4\pi}{3} = -\frac{1}{2}$.

Tangent is y over x, so the quotient of the two is found. The remaining three are evaluated using the reciprocal. All denominators <u>must</u> be rationalized. The exact value of the function means do not use decimal approximations..

Find the exact value of the six trigonometric functions for $-\frac{11\pi}{6}$.

$\sin -\frac{11\pi}{6} = \frac{1}{2}$ $\quad\csc -\frac{11\pi}{6} = 2$

$\cos -\frac{11\pi}{6} = \frac{\sqrt{3}}{2}$ $\quad\sec -\frac{11\pi}{6} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$

$\tan -\frac{11\pi}{6} = \left(\frac{1}{2} \div \frac{\sqrt{3}}{2}\right) = \left(\frac{1}{2} \cdot \frac{2}{\sqrt{3}}\right) = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$ $\quad\cot -\frac{11\pi}{6} = \sqrt{3}$

$\sin -\frac{11\pi}{6} = \frac{1}{2}$ $\quad\csc -\frac{11\pi}{6} = 2$

$\cos -\frac{11\pi}{6} = \frac{\sqrt{3}}{2}$ $\quad\sec -\frac{11\pi}{6} = \frac{2\sqrt{3}}{3}$

$\tan -\frac{11\pi}{6} = \frac{\sqrt{3}}{3}$ $\quad\cot -\frac{11\pi}{6} = \sqrt{3}$

In this case, $-\frac{11\pi}{6}$ is located in quadrant I. Moving in a clockwise direction, it is evident that $-\frac{11\pi}{6}$ is the same as $\frac{\pi}{6}$. This can also be found using coterminal angles. If we add 2π to $-\frac{11\pi}{6}$, the result is $\frac{\pi}{6}$. From this point, evaluate the six trigonometric functions.

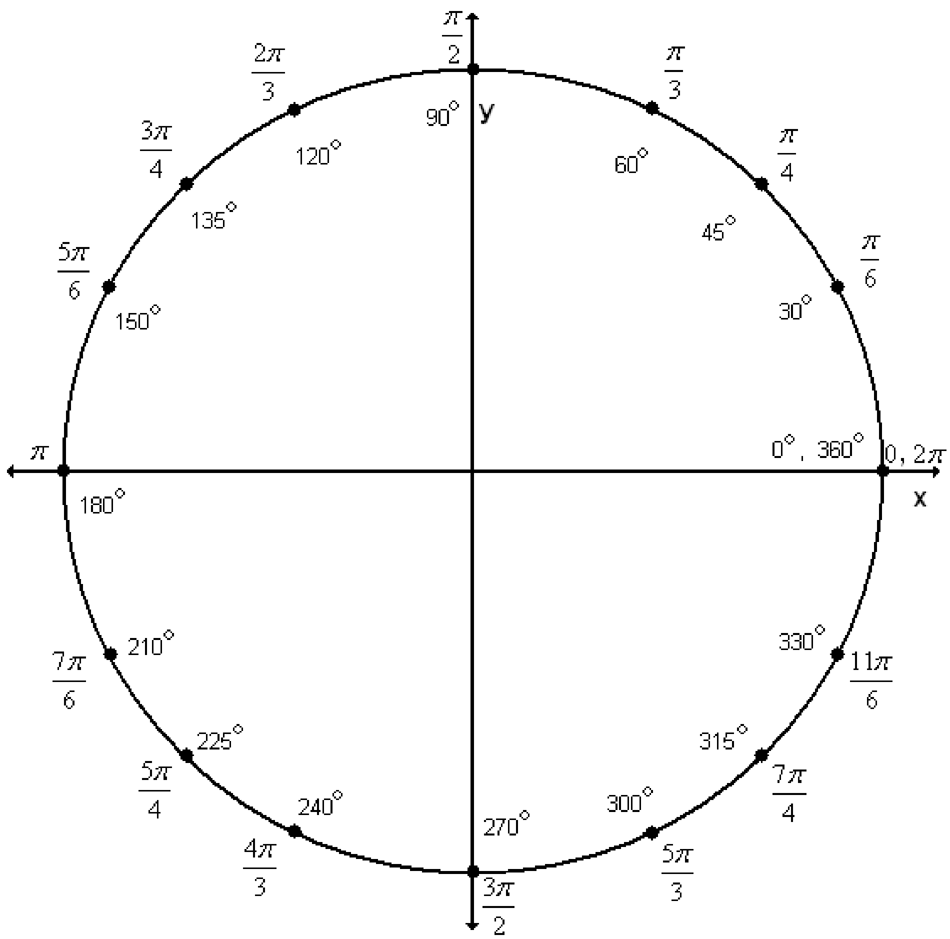
Find the exact value of the six trigonometric functions for $\frac{19\pi}{6}$

$\sin\frac{19\pi}{6} = -\frac{1}{2}$ $\quad\csc\frac{19\pi}{6} = -2$

$\cos\frac{19\pi}{6} = -\frac{\sqrt{3}}{2}$ $\quad\sec\frac{19\pi}{6} = -\frac{2\sqrt{3}}{3}$

$\tan\frac{19\pi}{6} = \frac{\sqrt{3}}{3}$ $\quad\cot\frac{19\pi}{6} = \sqrt{3}$

In this example, it is obvious that $\frac{19\pi}{6}$ is greater than 2π. This is called a periodic function. This means the angle makes at least one complete revolution before coming to rest. To find the angle that must be used, in this case, subtract 2π from $\frac{19\pi}{6}$. The result of this operation is $\frac{7\pi}{6}$. Therefore, in order to find the exact value of the six trigonometric functions of $\frac{19\pi}{6}$ use the angle $\frac{7\pi}{6}$.



Use the unit circle above to find the exact value of the six trigonometric functions for each of the following angles.

A) $\frac{3\pi}{4}$

$\sin\theta =$ $\csc\theta =$

$\cos\theta =$ $\sec\theta =$

$\tan\theta =$ $\cot\theta =$

B) $300°$

$\sin\theta =$ $\csc\theta =$

$\cos\theta =$ $\sec\theta =$

$\tan\theta =$ $\cot\theta =$

C) $-\frac{5\pi}{6}$

$\sin\theta =$ $\csc\theta =$

$\cos\theta =$ $\sec\theta =$

$\tan\theta =$ $\cot\theta =$

D) $\frac{2\pi}{3}$

$\sin\theta =$ $\csc\theta =$

$\cos\theta =$ $\sec\theta =$

$\tan\theta =$ $\cot\theta =$

E) $\frac{13\pi}{3}$

$\sin\theta =$ $\csc\theta =$

$\cos\theta =$ $\sec\theta =$

$\tan\theta =$ $\cot\theta =$

F) $-240°$

$\sin\theta =$ $\csc\theta =$

$\cos\theta =$ $\sec\theta =$

$\tan\theta =$ $\cot\theta =$

G) $-\frac{7\pi}{2}$

$\sin\theta =$ $\csc\theta =$

$\cos\theta =$ $\sec\theta =$

$\tan\theta =$ $\cot\theta =$

H) $135°$

$\sin\theta =$ $\csc\theta =$

$\cos\theta =$ $\sec\theta =$

$\tan\theta =$ $\cot\theta =$

I) $\frac{13\pi}{6}$

$\sin\theta =$ $\csc\theta =$

$\cos\theta =$ $\sec\theta =$

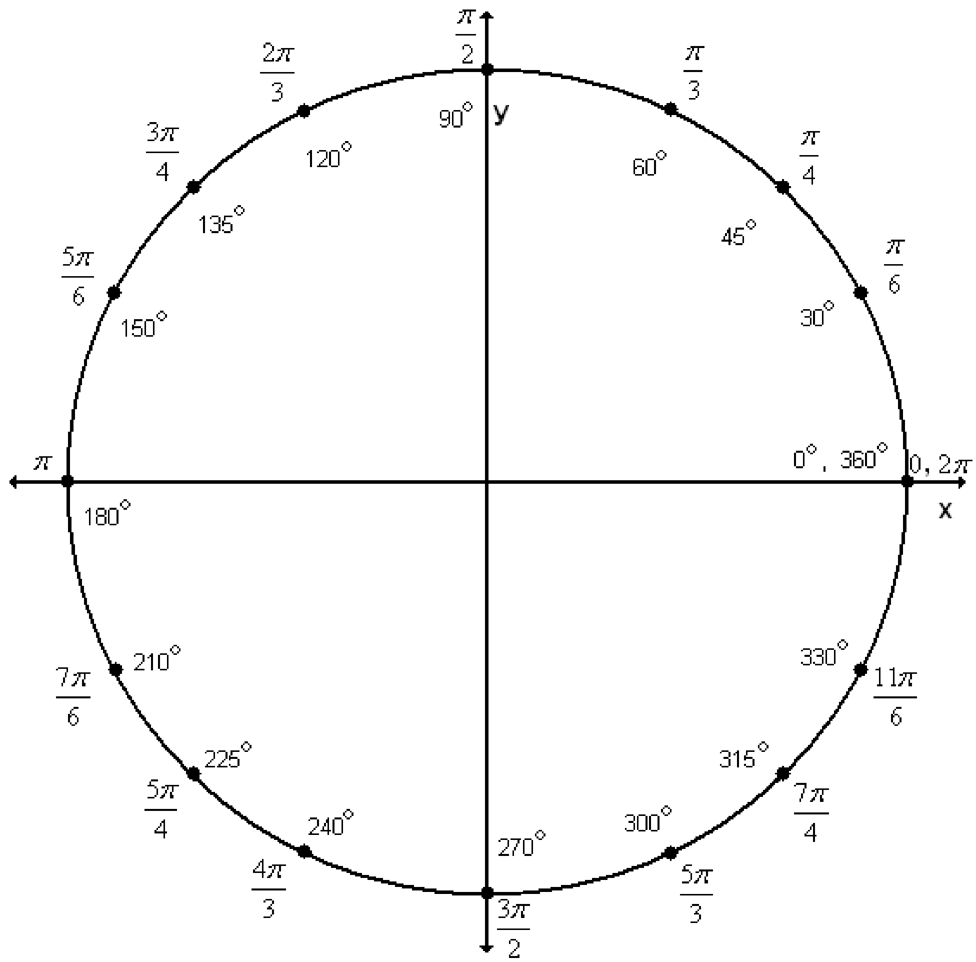
$\tan\theta =$ $\cot\theta =$

J) $-\frac{2\pi}{3}$

$\sin\theta =$ $\csc\theta =$

$\cos\theta =$ $\sec\theta =$

$\tan\theta =$ $\cot\theta =$



Use the unit circle above to find the exact value of each of the following.

A) $\tan\frac{\pi}{4} =$ **B)** $\cos\frac{2\pi}{3} =$ **C)** $\cos\pi =$

D) $\sin\frac{11\pi}{6} =$ **E)** $\tan\left(-\frac{2\pi}{3}\right) =$ **F)** $\csc\frac{\pi}{3} =$

G) $\sec\frac{4\pi}{3} =$ **H)** $\cos\left(-\frac{11\pi}{6}\right) =$ **I)** $\sin\frac{13\pi}{4} =$

J) $\csc\left(-\frac{5\pi}{6}\right) =$

K) $\tan\left(-\frac{\pi}{6}\right) =$

L) $\cot\frac{2\pi}{3} =$

M) $\sec\left(-\frac{19\pi}{3}\right) =$

N) $\cot\frac{\pi}{4} =$

O) $\cot\frac{11\pi}{6} =$

P) $\cos\left(-\frac{9\pi}{2}\right) =$

Q) $\sin\frac{21\pi}{4} =$

R) $\cot\frac{7\pi}{4} =$

S) $\sin\left(-\frac{7\pi}{6}\right) =$

T) $\cot\frac{26\pi}{3} =$

U) $\cos\frac{\pi}{3} =$

V) $\sec\left(-\frac{11\pi}{6}\right) =$

W) $\sin(-7\pi) =$

X) $\cot 2\pi =$

Y) $\cot\left(-\frac{2\pi}{3}\right) =$

Z) $\csc\frac{5\pi}{3} =$

a) $\sec\left(-\frac{3\pi}{2}\right) =$

b) $\sec\left(-\frac{23\pi}{6}\right) =$

c) $\tan\frac{3\pi}{4} =$

d) $\csc\frac{7\pi}{6} =$

e) $\sin\frac{14\pi}{3} =$

f) $\cos\left(-\frac{17\pi}{6}\right) =$

g) $\cot\frac{3\pi}{2} =$

It is also possible to use the unit circle going backwards. The previous exercises require a student to evaluate the trigonometric function of an angle using the unit circle. The samples below require the student to work backwards. Given the value of the trigonometric function of an angle θ, refer to the unit circle, and find the angle θ that makes the statement true. Given a statement such as $\sin\theta = \frac{1}{2}$, we will work backwards to try to determine the angle θ that would make the statement true.

<u>For Example</u>

Find all values of θ in the interval $(0, 2\pi]$ that make the statement $\cos\theta = \frac{1}{2}$ true.

Referring to the unit circle, look for x coordinates of $\frac{1}{2}$. This happens in two places, $\frac{\pi}{3}$ and $\frac{5\pi}{3}$.

As a result, $\theta = \frac{\pi}{3}, \frac{5\pi}{3}$

Find all values of θ in the interval $(0, 2\pi]$ that make the statement $\tan\theta = -\sqrt{3}$ true.

The only coordinate that has $\sqrt{3}$ in it is $\frac{\sqrt{3}}{2}$. That rules out any and all of the 45° angles or multiples thereof.

It would therefore follow, that the $\sqrt{3}$ is a result of either $\left(\frac{\sqrt{3}}{2} \div \frac{1}{2}\right)$ or $\left(\frac{1}{2} \div \frac{\sqrt{3}}{2}\right)$.

Working backwards will reveal what the result is.

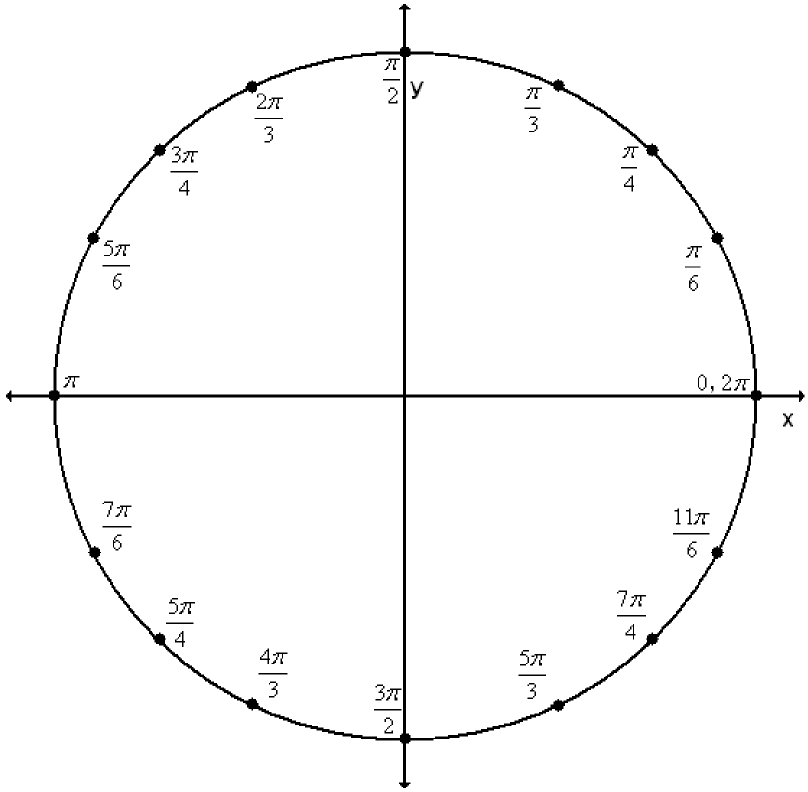
$$\left(\frac{\sqrt{3}}{2} \div \frac{1}{2}\right) = \left(\frac{\sqrt{3}}{2} \cdot \frac{2}{1}\right) = \sqrt{3} \qquad \left(\frac{1}{2} \div \frac{\sqrt{3}}{2}\right) = \left(\frac{1}{2} \div \frac{2}{\sqrt{3}}\right) = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

According to the work above a y value of $\frac{\sqrt{3}}{2}$ divided by an x value of $\frac{1}{2}$ would yield a result of $\sqrt{3}$.

This occurs at $\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}$ and $\frac{5\pi}{3}$.

Since tangent is y divided by x, and in this case, the tangent of θ is negative, it would stand to reason that one of the coordinates used will be a negative, while the other is a positive. As discussed earlier, the tangent of θ will be negative in quadrants II and IV.

The solution to this problem is: $\theta = \frac{2\pi}{3}, \frac{5\pi}{3}$



For each of the following, find all values of θ in the interval $(0, 2\pi]$ that make the following statements true.

A) $\sin\theta = \frac{1}{2}$ **B)** $\cos\theta = -\frac{\sqrt{3}}{2}$ **C)** $\cos\theta = -1$

D) $\sin\theta = -\frac{\sqrt{3}}{2}$ **E)** $\csc\theta = 2$ **F)** $\cot\theta = -1$

G) $\tan\theta = \textit{undefined}$ **H)** $\csc\theta = -\frac{2\sqrt{3}}{3}$ **I)** $\tan\theta = -1$

J) $\tan\theta = \frac{\sqrt{3}}{3}$ **K)** $\csc\theta = \textit{undefined}$ **L)** $\sin\theta = \pm\frac{1}{2}$

Trigonometric Equations

Many of the skills used for solving algebraic equations will be used to solve trigonometric equations. Trigonometric equations are solved using inverse operations. The ultimate objective of solving trigonometric equations is to find the angle that makes the statement true.

Examples

Solve the following equations in the interval $(0, 2\pi]$.

$$2\sin\theta + 1 = 0$$

$$2\sin\theta + 1 = 0$$
$$2\sin\theta = -1$$
$$\sin\theta = -\frac{1}{2}$$

Begin solving a trigonometric equation by isolating the trigonometric function involved.

$$\theta = \frac{7\pi}{6},\ \frac{11\pi}{6}$$

At this point, find all angles in the interval $(0, 2\pi]$ which make the equation true.

Solve the following equations in the interval $(0, 2\pi]$.

$$\sin^2\theta - 1 = 0$$

When dealing with trigonometric equations, $\sin^2\theta$ is the same type of thing as x^2

In other words, $(\sin 30°)^2$ is written as $\sin^2 30°$.

$$\sin^2\theta - 1 = 0$$
$$\sin^2\theta = 1$$
$$\sqrt{\sin^2\theta} = \pm\sqrt{1}$$
$$\sin\theta = \pm 1$$

Solve the equation by isolating the trigonometric function, then taking the square root of both sides. Do not forget to use $\pm$ when finding the solution.

$$\theta = \frac{\pi}{2},\ \frac{3\pi}{2}$$

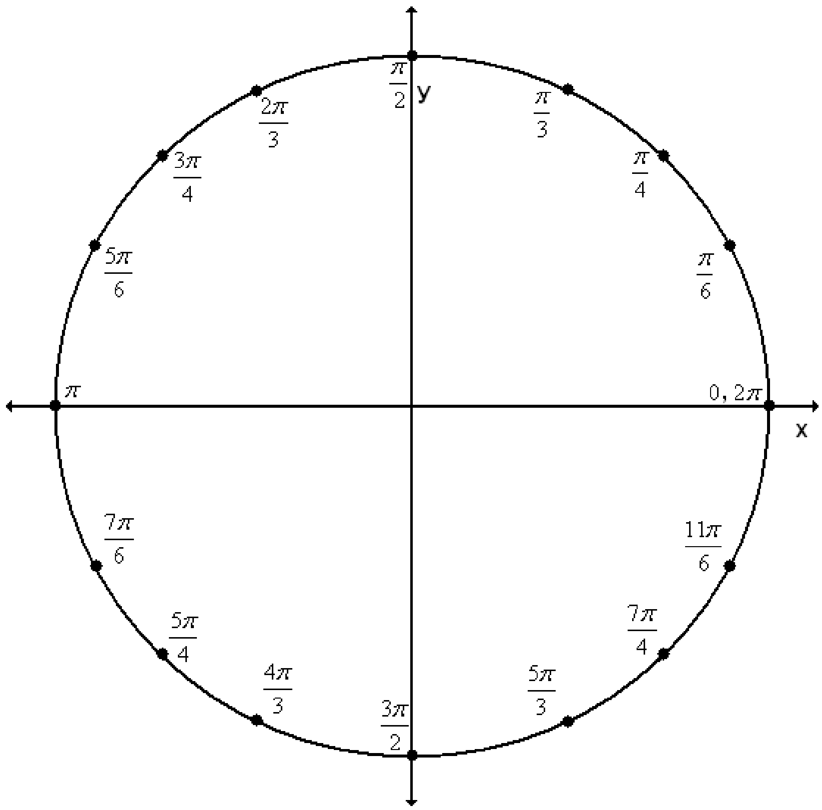
Here are the angles in the interval $(0, 2\pi]$ that satisfy the equation.

Every algebraic equation that was solved previously will play into your ability to solve some of the trigonometric equations you may face.

For example, the trigonometric equation $\csc^2\theta + 3\csc\theta + 2 = 0$ is very similar to the quadratic equation $x^2 + 3x + 2 = 0$. This trigonometric equation would be solved in the same manner as the algebraic. Factor the equation out to $(\csc\theta + 1)(\csc\theta + 2) = 0$. Then, proceed to set each factor equal to zero and solve.

Here are a couple of examples of trigonometric equations involving the reciprocal functions.

Solve the following equations in the interval $(0, 2\pi]$**.** $\sec\theta + 2 = 0$	
$\sec\theta + 2 = 0$ $\sec\theta = -2$ $\frac{1}{\cos\theta} = -2$ $\cos\theta = -\frac{1}{2}$	***Begin by isolating the trigonometric function. Once that is done, raise both sides of the equation to the negative first power essentially taking the reciprocal of both sides. This yields one of the basic three trigonometric functions.***
$\theta = \frac{2\pi}{3},\ \frac{4\pi}{3}$	***The angles in the interval*** $(0, 2\pi]$ ***that satisfy the equation are here.***
Solve the following equations in the interval $(0, 2\pi]$**.** $3\csc^2\theta + 6 = 10$	
$3\csc^2\theta + 6 = 10$ $3\csc^2\theta = 4$ $\csc^2\theta = \frac{4}{3}$ $\sqrt{\csc^2\theta} = \pm\sqrt{\frac{4}{3}}$ $\csc\theta = \pm\frac{2}{\sqrt{3}}$ $\frac{1}{\sin\theta} = \pm\frac{2}{\sqrt{3}}$ $\sin\theta = \pm\frac{\sqrt{3}}{2}$	***Once again, the trigonometric function is isolated.*** ***Taking the square root of both sides always results in*** $\pm$ ***answers.*** ***Since the cosecant function is really the reciprocal of the sine function, both sides are flipped over.***
$\theta = \frac{\pi}{3},\ \frac{2\pi}{3},\ \frac{4\pi}{3},\ \frac{5\pi}{3}$	***Here are the angles in the interval*** $(0, 2\pi]$ ***that satisfy the equation.***



Solve each of the following trigonometric equations in the interval $(0, 2\pi]$.

A) $2\sin\theta - 1 = 0$

B) $\cos\theta + 1 = 0$

C) $\tan\theta + 1 = 0$

D) $4\cos^2\theta - 3 = 0$

E) $5\tan\theta + 4 = 4$

F) $\csc\theta + 2 = 0$

G) $4\sin^2\theta - 2 = 0$

H) $3\csc^2\theta - 4 = 0$

I) $\sqrt{3}\cot\theta + 1 = 0$

H) $2\sin^3\theta + \sin^2\theta - 2\sin\theta - 1 = 0$

J) $\sin^2\theta + \sin\theta = 0$

K) $\sec\theta - 1 = 0$

L) $\sec^2\theta - 3\sec\theta + 2 = 0$

M) $\csc^2\theta + 3\csc\theta + 2 = 0$

N) $\tan^2\theta - 1 = 0$

O) $4\cos^3\theta - 2\cos\theta = 0$

P) $\csc^2\theta - 2 = 0$

Q) $\cot^2\theta - 3 = 0$

R) $4\sin^2\theta - 4\sin\theta + 1 = 0$

S) $2\sin\theta\cos\theta - \cos\theta = 0$

T) $9\sec^2\theta - 12 = 0$

U) $4\sin\theta\cos\theta + 2\cos\theta - 2\sin\theta - 1 = 0$

V) $\tan^2\theta - \left(1+\sqrt{3}\right)\tan\theta + \sqrt{3} = 0$

Checking Progress

You have now completed the "Introduction to Trigonometry" section of the workbook. The following is a checklist so that you may check your progress. Check off each of the objectives you have accomplished.

The student should be able to…

- [ ] ***Give a graphical representation of any angle***
- [ ] ***Find positive and negative coterminal angles***
- [ ] ***Convert an angle measured in degrees to radians***
- [ ] ***Convert an angle measured in radians to degrees***
- [ ] ***Evaluate the basic trigonometric functions***
- [ ] ***Use reference angles to evaluate the basic trigonometric functions***
- [ ] ***Construct a unit circle***
- [ ] ***Use the unit circle to evaluate basic trigonometric functions***
- [ ] ***Use the unit circle to solve trigonometric equations***
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