ALGEBRA II HONORS

WORKBOOK

Second Semester

SOLUTION MANUAL

MR. RAYA'S CLASS

ALGEBRA II HONORS

WORKBOOK

Second Semester

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LOGARITHMS

Objectives

The following is a list of objectives for this section of the workbook.

By the time the student is finished with this section of the workbook, he/she should be able to…

- ***Evaluate a simple logarithm without the aid of a calculator.***
- ***Express a logarithmic statement is exponential form.***
- ***Express a statement in exponential form in logarithmic form.***
- ***Expand a logarithmic expression as the sum or difference of logarithms using the properties of logs.***
- ***Condense the sum or difference of logarithms into a single logarithmic expression.***
- ***Evaluate logarithms using the base change formula.***
- ***Solve logarithmic equations.***
- ***Evaluate the solution to logarithmic equations to find extraneous roots.***
- ***Solve equations with variables in the exponents.***
- ***Find the range and domain of logarithmic functions.***
- ***Graph a logarithmic function using a table.***
- ***Find the inverse of a function.***
- ***Verify two functions are inverses of each other.***
- ***Identify a one-to-one function.***
- ***Use the compound interest formulas.***

Math Standards Addressed

The following state standards are addressed in this section of the workbook.

Algebra II

11.0 Students prove simple laws of logarithms.

> 11.1 Students understand the inverse relationship between exponents and logarithms and use this relationship to solve problems involving logarithms and exponents.
>
> 11.2 Students judge the validity of an argument according to whether the properties of real numbers, exponents, and logarithms have been applied correctly at each step.

13.0 Students use the definition of logarithms to translate between logarithms in any base.

14.0 Students understand and use the properties of logarithms to simplify logarithmic numeric expressions and to identify their approximate values.

15.0 Students determine whether a specific algebraic statement involving rational expressions, radical expressions, or logarithmic or exponential functions is sometimes true, always true, or never true.

24.0 Students solve problems involving functional concepts, such as composition, defining the inverse function and performing arithmetic operations on functions.

Logarithms are similar to radicals in that knowing what the question is asking makes the problem easier. Although this is a topic that is completely new to Algebra II students, Logarithms are simple. For example, the question $\log_3 27 =$ ***is asking "To what power do you raise 3 to get 27?" In this particular problem, 3 is the base of the logarithm. When reading the logarithm, it is read "Log base 3 of 27 is..."***

<u>Properties of Simple Logarithms</u>

$\log_a 1 = 0$

$\log_a a = 1$

$\log_a a^x = x \quad and \quad a^{\log_a x} = x \quad (inverse\ property)$

$If \quad \log_a x = \log_a y \quad then \quad x = y$

<u>Properties of Natural Logarithms</u>

$\ln 1 = 0$

$\ln e = 1$

$\ln e^x = x \qquad and \quad e^{\ln x} = x \qquad (inverse\ property)$

$If \quad \ln x = \ln y \quad then \quad x = y$

A standard logarithm can have any positive number as its base except 1, whereas a natural log is <u>always</u> base e***. Since the natural log is always base*** e***, it will be necessary to use a calculator to evaluate natural logs unless one of the first three examples of the properties of natural logs is used. For anything such as*** $\ln 2 =$***, a calculator must be used.***

When dealing with logarithms, switching between exponential and Logarithmic form is often necessary.

Logarithmic form	**Exponential Form**
$\log_a b = c$	$a^c = b$

Write each of the following in exponential form.

A) $\log_4 16 = 2$
$4^2 = 16$

B) $\log_9 3 = \frac{1}{2}$
$9^{1/2} = 3$

C) $\log_9 27 = \frac{3}{2}$
$9^{3/2} = 27$

E) $\log_4 \frac{1}{16} = -2$
$4^{-2} = 1/16$

Write each of the following in logarithmic form.

A) $3^4 = 81$
$\log_3 81 = 4$

B) $16^{1/4} = 2$
$\log_{16} 2 = 1/4$

C) $36^{-1/2} = \frac{1}{6}$
$\log_{36} 1/6 = -1/2$

D) $16^{5/4} = 32$
$\log_{16} 32 = 5/4$

Simplifying Logarithms

Evaluate each of the following logarithms without the use of a calculator.

A) $\log_3 81 = 4$

B) $\log_4 \frac{1}{2} = 1/2$

$4^x = 1/2$

$4^{2x} = 2^{-1}$

$2x = -1$

C) $\log_{12} 144 = 2$

D) $\log_6 \frac{1}{36} = -2$

E) $\log_{\frac{2}{3}} \frac{9}{4} = -2$

F) $\log_{0.25} 4 = -1$

$\log_{1/4} 4 = -1$

G) $\log_3 -3 =$

no solution

H) $\log_8 4 = 2/3$

$8^x = 4$

$2^{3x} = 2^2$

$3x = 2$

$x = 2/3$

I) $\log_{81} \frac{1}{27} = -3/4$

$81^x = 1/27$

$3^{4x} = 3^{-3}$

$4x = -3$

$x = -3/4$

J) $\log_{\frac{1}{16}} 32 = -5/4$

$1/16^x = 32$

$2^{-4x} = 2^5$

$-4x = 5$

$x = -5/4$

K) $\log_4 0 =$

no solution

L) $\log_{10} 1 = 0$

M) $\log_4 \frac{1}{8} =$

$4^x = 1/8$

$2^{2x} = 2^{-3}$

$2x = -3$

$x = -3/2$

N) $\log_{27} \frac{1}{3} =$

$27^x = 1/3$

$3^{3x} = 3^{-1}$

$3x = -1$

$x = -1/3$

O) $\log_9 3 = 1/2$

$9^x = 3$

$3^{2x} = 3$

$2x = 1$

$x = 1/2$

P) $\log_6 6^{3x} = 3x$

Q) $\log_{36} \frac{1}{6} = -1/2$

$36^x = 1/6$

$6^{2x} = 6^{-1}$

$2x = -1$

$x = -1/2$

R) $\log_{128} 2 = 1/7$

$128^x = 2$

$2^{7x} = 2$

$7x = 1$

$x = 1/7$

S) $\log_{\frac{1}{4}} 16 = -2$

$(\frac{1}{4})^x = 16$

$4^{-x} = 4^2$

$x = -2$

T) $\log_z z^{2x} = 2x$

U) $\ln e^{12} = 12$

V) $3^{\log_3 5} = 5$

W) $\ln 1 = 0$

X) $e^{\ln 4x} = 4x$

Y) $\log_2 16\sqrt{2} = 9/2$

$2^x = 16\sqrt{2}$

$2^x = 2^4 \cdot 2^{1/2}$

$2^x = 2^{9/2}$

$x = 9/2$

Z) $\log_3 \sqrt[5]{9} = 2/5$

$3^x = 9^{1/5}$

$3^x = 3^{2/5}$

$x = 2/5$

a) $\log_3 9\sqrt[3]{3} = 7/3$

$3^x = 9\sqrt[3]{3}$

$3^x = 3^2 \cdot 3^{1/3}$ $2^{1/3}$

$3^x = 3^{2+1/3}$

$3^x = 3^{7/3}$

$\boxed{x = 7/3}$

b) $\log_5 \frac{1}{\sqrt[3]{25}} = -2/3$

$5^x = (\sqrt[3]{25})^{-1}$

$5^x = (25^{1/3})^{-1}$

$5^x = (5^2)^{-1/3}$

$5^x = 5^{-2/3}$

$x = -2/3$

c) $\log_{\frac{5}{6}} \sqrt[3]{\frac{36}{25}} =$

$(5/6)^x = 36/25$

$(5/6)^x = (6/5)^{2/3}$

$(5/6)^x = (5/6)^{-2/3}$

$x = -2/3$

d) $e^{\ln 5x^2} = 5x^2$

Properties of Logarithms

The following properties serve to expand or condense a logarithm or logarithmic expression so it can be worked with.

Properties of logarithms	***Example***
$\log_a mn = \log_a m + \log_a n$	$\log_4 3x = \log_4 3 + \log_4 x$
$\log_a \frac{m}{n} = \log_a m - \log_a n$	$\log_2 \frac{x+1}{5} = \log_2 (x+1) - \log_2 5$
$\log_a m^n = n\log_a m$	$\log_3 (2x+1)^3 = 3\log_3 (2x+1)$

Properties of Natural Logarithms	***Example***
$\ln mn = \ln m + \ln n$	$\ln(x+1)(x-5) = \ln(x+1) + \ln(x-5)$
$\ln \frac{m}{n} = \ln m - \ln n$	$\ln \frac{x}{2} = \ln x - \ln 2$
$\ln m^n = n\ln m$	$\ln 7^3 = 3\ln 7$

These properties are used backwards and forwards in order to expand or condense a logarithmic expression. Therefore, these skills are needed in order to solve any equation involving logarithms. Logarithms will also be dealt with in Calculus. If a student has a firm grasp on these three simple properties, it will help greatly in Calculus.

Expanding Logarithmic Expressions

Write each of the following as the sum or difference of logarithms. In other words, expand each logarithmic expression.

A) $\log_2 \frac{3x^3y^2}{z^5}$

$\log_2 3x^3y^2 - \log_2 z^5$

$\log_2 3 + \log_2 x^3 + \log_2 y^2 - \log_2 z^5$

$\log_2 3 + 3\log_2 x + 2\log_2 y - 5\log_2 z$

B) $\log_3 5\sqrt[3]{xy^2}$

$\log_3 5 + \log_3 \sqrt[3]{xyz}$

$\log_3 5 + 1/3\log_3 xy^2$

$\log_3 5 + 1/3(\log_3 x + \log_3 y^2)$

$\log_3 5 + 1/3\log_3 x + 2/3\log_3 y$

C) $\log \sqrt[4]{(x+1)^3(x-2)^2}$

$1/4\log(x+1)^3(x-2)^2$

$1/4(\log(x+1)^3 + \log(x-2)^2$

$1/4(3\log(x+1) + 2\log(x-2))$

$3/4\log(x+1) + 1/2\log(x-2)$

D) $\log_5 \frac{6x^2}{11y^5z}$

$\log_5 6x^2 - \log_5 11y^5z$

$\log_5 6 + \log_5 x^2 - (\log_5 11 + \log_3 y^5 + \log_5 z)$

$\log_5 6 + 2\log_5 x - (\log_5 11 + 5\log_5 y + \log_5 z)$

$\log_5 6 + 2\log_5 x - \log_5 11 - 5\log_5 y - \log_5 z$

E) $\log_2 \frac{\sqrt[5]{3(x+2)^3}}{x-1}$

$\log_2 \sqrt[5]{3(x+2)^3} - \log_2(x-1)$

$1/5(\log_2 3(x+2)^3 - \log_2(x-1))$

$1/5(\log_2 3 + \log_2(x+2)^3) - \log_2(x-1)$

$1/5 \log_2 3 + 3/5 \log_2(x+2) - \log_2(x-1)$

F) $\log_{12} \frac{x-7}{x+2}$

$\log_{12}(x-7) - \log_{12}(x+2)$

G) $\log_a 12x^3\sqrt{y}$

$\log_a 12 + \log_a x^3 + \log_a y^{1/2}$

$\log_a 12 + 3\log_a x + 1/2 \log_a y$

H) $\log_3 \frac{\sqrt{5x^5y^3}}{\sqrt[3]{z^2}}$

$\log_3(5x^5y^3)^{1/2} - \log(z^2)^{1/3}$

$1/2(\log 5x^5y^3) - 1/3\log_3 z^2$

$1/2(\log_3 5 + \log_3 x^5 + \log_3 y^3) - 1/3 \log_3 z^2$

$1/2 \log_3 5 + 5/2 \log_3 x + 3/2 \log_3 y - 2/3 \log_3 z$

I) $\ln \frac{x^2-4}{x^3}$

$\ln(x^2-4) - \ln x^3$

$\ln(x^2-4) - 3\ln x$

OK

$\ln(x+2) + \ln(x-2) - 3\ln x$

J) $\ln 3x^5 y$

$\ln 3 + \ln x^6 + \ln y$

$\ln 3 + 5\ln x + \ln y$

K) $\ln \sqrt{x^3(x+4)}$

$1/2 \ln x^3(x+4)$

$1/2 (\ln x^3 + \ln(x+4)$

$3/2 \ln x + 1/2 \ln(x+4)$

L) $\ln \sqrt[3]{x^2(x-3)}$

$1/3 \ln x^2(x-3)$

$1/3[\ln x^2 + \ln(x-3)]$

$2/3 \ln x + 1/3 \ln(x-3)$

M) $\ln \sqrt{\frac{x^3 y}{z^5}}$

$1/2 \ln \frac{x^3y}{z^5}$

$1/2(\ln x^3 + \ln y - \ln z^5)$

$1/2(3\ln x + \ln y - 5\ln z)$

$3/2 \ln x + 1/2 \ln y - 5/2 \ln z$

N) $\ln \frac{x}{\sqrt{x-2}}$

$\ln x - \ln(x-2)^{1/2}$

$\ln x - 1/2 \ln(x-2$

Condensing Logarithmic Expressions

Rewrite each of the following logarithmic expressions using a single logarithm. Condense each of the following to a single expression. Do not multiply out complex polynomials. Just leave something like $(x+5)^3$ alone.

A) $3\log_4 x - 5\log_4 y + 2\log_4 z$

$\log_4 x^3 - \log_4 y^5 + \log_4 z^2$

$\log_4 \frac{x^3 z^2}{y^5}$

B) $2\log x + \frac{1}{2}\log y$

$\log x^2 + \log y^{1/2}$

$\log x^2\sqrt{y}$

C) $\frac{1}{3}\log 6 + \frac{1}{3}\log x + \frac{2}{3}\log y$

$1/3(\log 6 + \log x + 2\log y)$

$1/3 (\log 6xy^2)$

$\log \sqrt[3]{6xy^2}$

D) $\frac{3}{4}\log_3 16 - \frac{1}{3}\log_3 x^3 - 2\log_3 y$

$\log_3 16^{3/4} - \log_3 x^{3/3} - \log_3 y^2$

$\log_3 8 - \log_3 x - \log_3 y^2$

$\log_3 \frac{8}{xy^2}$

E) $3\log_5 x + 2\log_5 y + \log_5 z + 2$

$\log_5 x^3 + \log_5 y^2 + \log_z + 2(1)$

$\log_5 x^3y^2z + 2\log_5 5$

$\log_5 x^3 y^2z + \log_5 5^2$

$\log_5 25x^3y^2z$

F) $\frac{1}{3}\log_2 x + \frac{2}{3}\log_2 y - 3$

$1/3(\log_2 x + 2\log_2 y) - 3(1)$

$1/3 \log_2 x + 1/3\log_2 y^2 - 3\log_2 2$

$\log_2 \sqrt[3]{xy^2} - \log_2 2^3$

$\log_2 \frac{\sqrt[3]{xy^2}}{8}$

G) $\log_3(x+2) + \log_3(x-2) - \log_3(x+4)$

$\log_3 (x+2)(x-2) - \log_3(x+4)$

$\log_3 \frac{x^2-4}{x+4}$

H) $\frac{2}{3}\log(x+1) + \frac{1}{3}\log(x-2) - \frac{1}{3}\log(x+5)$

$1/3 (2\log(x+1) + \log(x-2) - \log(x+5))$

$1/3 \log \frac{(x+1)^2(x-2)}{x+5}$

$\log \sqrt[3]{\frac{(x+1)^2(x-2)}{x+5}}$

I) $\frac{1}{3}\left[2\ln(x+3) + \ln x\right] - \ln(2x-1)$

$1/3 \ln x(x+3)^2 - \ln(2x-1)$

$\ln \sqrt[3]{x(x+3)^2} - \ln(2x-1)$

$\ln \frac{\sqrt[3]{x(x+3)^2}}{2x-1}$

J) $\ln(x+3) - \ln(2x+5) + 2\ln(x-1)$

$\ln(x+3) - \ln(2x+5) + \ln(x-1)^2$

$\ln \frac{(x+3)(x-1)^2}{2x+5}$

K) $\frac{1}{2}\left[\ln(x+3)+2\ln(x-1)\right]-3\ln x$

$\frac{1}{2}[\ln(x+3)+\ln(x-1)^2]-\ln x^3$

$\frac{1}{2}\ln(x+3)(x-1)^2-\ln x^3$

$\ln\sqrt{(x+3)(x-1)^2}-\ln x^3$

$\ln\frac{\sqrt{(x+3)(x-1)^2}}{x^3}$

L) $2\ln 3+6\ln x-\frac{2}{3}\ln 27$

$\ln 3^2+\ln x^6-\ln 27^{2/3}$

$\ln 9+\ln x^6-\ln 4$

$\ln\frac{9x^6}{9}$

$\ln x^6$

M) $\frac{1+\log_3 x}{2}$

$\frac{1}{2}(1+\log_3 x)$

$\frac{1}{2}(\log_3 3+\log_3 x)$

$\frac{1}{2}\log_3 3x$

$\log_3\sqrt{3x}$

N) $\ln x-2\left[\ln(x+2)+\ln(x-2)\right]$

$\ln x-2\ln(x^2-4)$

$\ln x-\ln(x^2-4)^2$

$\ln\frac{x}{(x^2-4)}$

Practice Using Properties of Logarithms

Use the following information, to approximate the logarithm to 4 significant digits by using the properties of logarithms.

$\log_a 2\approx 0.3562,\qquad \log_a 3\approx 0.5646,\qquad \textit{and}\qquad \log_a 5\approx 0.8271$

A) $\log_a\frac{6}{5}$

$\log_a 6-\log_a 5$

$\log_a 2+\log_a 3-\log_a 5$

$0.3562+0.5646-0.8271$

0.0937

B) $\log_a 18$

$\log_a(2-9)$

$\log_a 2+2\log 3$

$0.3562+2(5646)$

1.4854

C) $\log_a 100$

$\log_a(2^2+5^2)$

$2\log_a 2+2\log_a 5$

$2(0.3562)+2(.8271)$

2.3666

D) $\log_a 30$

$\log_a 2+\log_a 3+\log_a 5$

$0.3562+0.5646+0.8271$

1.7479

E) $\log_a\sqrt{3}$

$\frac{1}{2}\log_a 3$

$\frac{1}{2}(0.5646)$

0.2823

F) $\log_a\sqrt{75}$

$\frac{1}{2}\log_a 75$

$\frac{1}{2}(\log_a 25+\log_a 3)$

$\frac{1}{2}(\log_a 3+2\log 5)$

$\frac{1}{2}(\log_a 3+\log 5$

$\frac{1}{2}(0.5646)+0.8271$

1.1094

G) $\log_a\frac{4}{9}$

$\log_a\left(\frac{2}{3}\right)^2$

$2\log_a\frac{2}{3}$

$2(\log_a 2-\log_a 3)$

$2(0.3562-0.5646)$

-0.4168

H) $\log_a\sqrt[3]{15}$

$\frac{1}{3}\log_a 15$

$\frac{1}{3}(\log_a 3+\log_a 5)$

$\frac{1}{3}(.5646+.8271)$

$.4639$

I) $\log_a 54^2$

$2\log_a 54$

$2(\log_a 2+\log_a 27)$

$2\log_a 2+2\log_a 3^3$

$2\log_a 2+6\log_a 3$

2.4062

Use the following information for letters J – R.

$\log_{10} 2 \approx 0.3010$, $\log_{10} 3 \approx 0.4771$, *and* $\log_{10} 7 \approx 0.8451$

J) $\log_{10} 28$

$\log_{10} 4 + \log_{10} 7$

$2\log_{10} 2 + \log_{10} 7$

$2(.3010) + (.8457)$

1.4471

K) $\log_{10} 8$

$\log_{10} 2^3$

$3\log_{10} 2$

$3(.3010)$

0.9030

L) $\log_{10} 4.5$

$\log_{10} \frac{9}{2}$

$\log_{10} 9 - \log_{10} 2$

$2\log_{10} 3 - \log_{10} 2$

$2(0.4771) - 0.3010$

0.6532

M) $\log_{10} \sqrt{98}$

$\frac{1}{2}\log_{10} 98$

$\frac{1}{2}(\log_{10} 2 + \log_{10} 49)$

$\frac{1}{2}(\log_{10} 2 + 2\log_{10} 7)$

$\frac{1}{2}\log_{10} 2 + \log_{10} 7$

$\frac{1}{2}(0.3010) + 0.8461$

0.9956

N) $\log_{10} 20$

$\log_{10} 2 + \log_{10} 10$

$0.3010 + 1$

1.3010

O) $\log_{10} 210$

$\log_{10} 21 + \log_{10} 10$

$\log_{10} 3 + \log_{10} 7 + \log_{10} 10$

$0.4771 + 0.8451 + 1$

2.3222

P) $\log_{10} \frac{1}{300}$

$\log_{10} 1 - \log_{10} 300$

$\log_{10} 1 - (\log_{10} 3 + \log_{10} 100)$

$\log_{10} 1 - \log_{10} 3 - \log_{10} 100$

$0 - 0.4771 - 2$

-2.4771

Q) $\log_{10} \frac{1}{9}$

$\log_{10} 1 - \log_{10} 9$

$\log_{10} 1 - 2\log_{10} 3$

$0 - 2(0.4771)$

-0.9542

R) $\log_{10} \sqrt{42}$

$\frac{1}{2}\log_{10} 42$

$\frac{1}{2}(\log_{10} 6 + \log_{10} 7)$

$\frac{1}{2}(\log_{10} 3 + \log_{10} 2 + \log_{10} 7)$

$\frac{1}{2}\log_{10} 3 + \frac{1}{2}\log_{10} 7 + \frac{1}{2}\log_{10} 7$

$\frac{1}{2}(0.4771) + \frac{1}{2}(0.3010) + \frac{1}{2}(0.8401)$

0.8116

Base Change Formula

Up to this point calculators have not been used to evaluate logarithms. Remember, the logs on all calculators are base 10. If a calculator is used to evaluate a log with any other base, it will not give the correct estimate unless a specific formula is used. The following base change formula will be used to evaluate any log with a base other than 10.

$$\log_a b = \frac{\log b}{\log a}$$

The terms $\log a$ ***and*** $\log b$ ***will be evaluated with a base 10, which means a calculator can now be used. Of course natural logs are always base*** e***, so there is no need for a formula with natural logs. Each calculator is different. Be aware of what keystrokes are needed for your particular calculator, whether the log must be entered first, or the number first. Be very careful about grouping symbols when entering these types of problems into the calculator.***

Using a calculator, evaluate each of the following. Round all answers to three decimal places.

A) $\log_3 12$

$\frac{\log 12}{\log 3}$

2.262

B) $\log_6 17$

$\frac{\log 17}{\log 6}$

1.581

C) $\log_3 \frac{1}{5}$

$\frac{\log \frac{1}{5}}{\log 3}$

1.465

D) $\log_4 8$

$\frac{\log 8}{\log 4}$

1.500

E) $\log_6 \frac{1}{12}$

$\frac{\log \frac{1}{12}}{\log 6}$

-1.387

F) $\log_7(-35)$

no solution

G) $\ln 14$

2.639

H) $\ln 3.26$

1.182

I) $\ln\frac{1}{2}$

-0.693

J) $\ln 0$

no solution

K) $\ln e$

1

L) $\ln 6.2$

1.825

Why can't you take the log of a negative number?

Thinking of the question you ask yourself with $\log_a b$, you have "To what power do I raise a to get b?" As we saw with exponential functions, you cannot raise a number to a power and make it negative.

Solving Logarithmic Equations

Here we will solve some logarithmic equations. There are a couple of ways to solve a logarithmic equation. One method is to get all logs to the left side of the equal sign and condense the problem so there is only one log. The next step would be to write the problem in exponential form and solve.

Another method involves one of the properties of simple logarithms. In particular, the property that states: "If $\log_a x = \log_a y$ ***then*** $x = y$***." If the equation can be manipulated to resemble this property, take what is inside of each and set the two terms equal to each other. This follows the same principal as exponential equations where the bases are identical. If an exponential equation states*** $2^{3x+4} = 2^7$***; it stands to reason, because of the equal sign, that*** $3x + 4 = 7$***. The problem would then be solved from this point. With logarithmic equations, the problem*** $\log_4(8x-5) = \log_4 12$ ***can be written as*** $8x - 5 = 12$.

Always check for extraneous roots!!!

Solve each of the following logarithmic equations. **(Round any solutions with decimals to three decimal places)**

A) $\log_4 x + 6 = 8$

$\log_4 x = 2$

$4^2 = x$

$\boxed{x=16}$

ck

$\log_4 16 + 6 = 8$

$2+6=8$

$8=8$ ✓

B) $2\log_3 x - 4 = 1$

$2\log_3 x = 5$

$\log_3 x = \frac{5}{2}$

$3^{5/2} = x$

$\boxed{x \approx 15.588}$

ck

$2\log_3 15.588 - 4 = 1$

$.9999 \approx 1$ ✓

C) $5 + 2\log_4 x = 4$

$2\log_4 x = -1$

$\log_4 x = -\frac{1}{2}$

$4^{-1/2} = x$

$\boxed{x=\frac{1}{2}}$

ck

$5+2\log_4 \frac{1}{2} = 4$

$5+2(-\frac{1}{2}) = 4$

$5-1=4$

$4=4$ ✓

D) $\log_3 x - \log_3 2 = 5$

$\log_3 \frac{x}{2} = 5$

$3^5 = \frac{x}{2}$

$\frac{x}{2} = 243$

$\boxed{x=486}$

ck

$\log_3 486 - \log_3 2 = 5$

$\log_3 243 = 5$

$5=5$ ✓

E) $\log(2x-1) = 0$

$10^0 = 2x-1$

$1 = 2x-1$

$2x = 2$

$\boxed{x=1}$

ck/ $\log 1 = 0$

$0=0$ ✓

F) $\log(4x+8) = 2$

$10^2 = 4x+8$

$100 = 4x+8$

$-8 \quad -8$

$4x = 92$

$\boxed{x=23}$

ck/ $\log(4(23)+8) = 2$

$\log 100 = 2$

$2=2$ ✓

G) $\log_3(x+5) + \log_3(x+3) = \log_3 35$

$\log_3(x^2+8x+15) = \log_3 35$

$x^2+8x+15 = 35$

$-35 \quad -35$

$x^2+8x-20=0$

$(x-2)(x+10)=0$

$\boxed{x=2}$ $\cancel{x=-10}$

ck/ $\log_3 7 + \log_3 5 = \log_3 35$

$\log_3 35 = \log_3 35$

$\log_3(-5) + \log_3(-7) \neq \log_3 35$

H) $\log(x+3) + \log(x-4) = \log 30$

$\log(x^2-x-12) = \log 30$

$x^2-x-12 = 30$

$-30 \quad -30$

$x^2-x-42=0$

$(x-7)(x+6)=0$

$\boxed{x=7}$ $\cancel{x=-6}$

ck/ $\log 10 + \log 3 = \log 3$

$\log 30 = \log 30$

I) $\log_3 3x - 2 = 2$

$\log_3 3x = 4$

$3^4 = 3x$

$3x = 81$

$\boxed{x = 27}$

ck/ $\log_3 81 - 2 = 2$

$4 - 2 \overset{\checkmark}{=} 2$

J) $\log_5 x + \log_5 (x+4) = 4\log_5 2$

$\log_5(x^2+4x) = \log_5 2^4$

$x^2+4x=16$

$x^2+4x-16=0$

$x^2+4x+4 = 16+4$

$(\frac{b}{2})^2 = (\frac{4}{2})^2 = (2)^2 = 4$

$\sqrt{(x+2)^2} = \sqrt{20}$

$x+2 = \pm\sqrt{20}$

$x = -2 \pm \sqrt{20}$ $\boxed{x = -2 + \sqrt{20}}$

K) $\log_x 625 = 4$

$x^4 = 625$

$\sqrt[4]{x^4} = \sqrt[4]{625}$

$\boxed{x = 5}$

ck/ $\log_5 625 = 4$ $\checkmark$

$4 = 4$

L) $\log_2 (x-2) + \log_2 (x-6) = \log_2 13$

$\log_2(x^2-8x+12) = \log_2 13$

$\log_2 13 \; x^2-8x+12 = 13$

$x^2-8x-1=0$

$x^2-8x+16 = 1+16$

$(\frac{b}{2})^2 = (\frac{-8}{2})^2 = (-4)^2 = 16$

$\sqrt{(x-4)^2} = \sqrt{17}$

$x-4 = \pm\sqrt{17}$

$x = 4 \pm \sqrt{17}$

$\sqrt{17} \times 4$

$x \neq 4 - \sqrt{17}$

$\boxed{x = 4 + \sqrt{17}}$

M) $2\log_3 x - \log_3 (x-2) = 2$

$\log_3 \frac{x^2}{x-2} = 2$

$3^2 = \frac{x^2}{x-2}$ $x \neq 2$

$(x-2)\frac{x^2}{x-2} = 9(x-2)$

$x^2 = 9x - 18$

$x^2 - 9x + 18 = 0$

$(x-6)(x-3) = 0$

$x = 6$ $x = 3$

N) $\ln x - \ln 2 = 0$

$\ln \frac{x}{2} = 0$

$e^0 = \frac{x}{2}$

$(2)1 = \frac{x}{2}(2)$

$x = 2$

$\underline{ck}$

$\ln 2 - \ln 2 = 0$

$0 = 0$

O) $\ln(3x+5) = 8$

$e^8 = 3x+5$

$\frac{3x}{3} = e^8 - 5$

$x = \frac{e^8-5}{3}$

$x \approx 991.486$

$\underline{ck}$

$\ln(3(991.986)+5) = 8$

$8.00100 = 8$

P) $\ln(x-1) = 3.2$

$e^{3.2} = x-1$

$x = e^{3.2}+1$

$x \approx 25.533$

$\underline{ck}$

$\ln(25.533-1) = 3.2$

$3.2 = 3.2$

Q) $\log_3 (4x-2) = \log_3 (x+6)$

$4x-2 = x+6$

$-x+2 = -x+2$

$3x = 8$

$x = 8/3$

$\underline{ck}$

$\log_3(4(8/3)-2) = \log_3(8/3+6)$

$\log_3(\frac{32}{3} - \frac{6}{3}) = \log_3(\frac{8}{3} + \frac{18}{3})$

$\log_3 \frac{26}{3} = \log_3 \frac{26}{3}$

R) $\ln(3x-2) = 4$

$e^4 = 3x-2$

$\frac{e^4+2}{3} = \frac{3x}{3}$

$x = \frac{e^4+2}{3}$

$x \approx 18.866$

$\underline{ck}$

$\ln[3(18.866-2)] = 4$

$4 = 4$

S) $\ln e^{x+7} = 10$

$(x+7)\ln e = 10$

$x+7 = 10$

$x = 3$

$\underline{ck}$

$\ln e^{10} = 10$

$10 \ln e = 10$

$10 = 10$

T) $\log_a (x^2+7) = \frac{2}{3}\log_a 64$

$\log_a(x^2+7) = \log_a 64^{2/3}$

$\log_a(x^2+7) = \log_a 16$

$x^2+7 = 16$

$\sqrt{x^2} = \sqrt{9}$

$x = \pm 3$

ck/ $\log_a[(3)^2+7] = \log_a 64^{2/3}$

$\log_a(9+7) = \log_a 16$

$\log_a 16 = \log_a 16$

U) $\log_2 (x+3) + \log_2 (x-3) = 4$

$\log_2(x^2-9) = 4$

$2^4 = x^2-9$

$x^2-9 = 16$

$\sqrt{x^2} = \sqrt{25}$

$x = \pm 5$

$\underline{ck}$

$\log_2(8) + \log_2(2) = 4$

$\log_2 16 = 4$

$4 = 4$

Solving Exponential Equations

You will now be solving exponential equations. Logarithms allow you to solve equations with variables in the exponent. We have dealt with these types of problems before, however, in the previous examples the bases of the problem could be made to match making solving the problem as simple as setting the two exponents equal to each other and solving for the variable. When ever you come to a problem with the variable in the exponent, and you cannot get the bases to match, you will need to solve the problem using logarithms. You will be using all three properties of logarithms to simplify and eventually solve the problem. You can take either the log or natural log of both sides, it makes no difference unless e ***is in the problem. In that case, you must use natural logs to solve the equation.***

Solve each of the following exponential equations. Round solutions to three decimal places.

A) $3^{2x} = 5$

$\log 3^{2x} = \log 5$

$\frac{2x\log 3}{2\log 3} = \frac{\log 5}{2\log 3}$

$x = \frac{\log 5}{2\log 3}$

$x \approx 0.732$

B) $6^{4x} = 300$

$\log 6^{4x} = \log 300$

$\frac{4x\log 6}{4\log 6} = \frac{\log 300}{4\log 6}$

$x = \frac{\log 300}{4\log 6}$

$x = 0.796$

C) $12^{3x+1} = 7^2$

$\log 12^{3x+1} = \log 7^2$

$(3x+1)\log 12 = \log 49$

$3x\log 12 + \log 12 = \log 49$

$\frac{3x\log 12}{3\log 12} = \frac{\log 49 - \log 12}{3\log 12}$

$x \approx .189$

D) $5^{x+2} = 3^5$

$\log 5^{x+2} = \log 3^5$

$(x+2)\log 5 = 5\log 3$

$x\log 5 + 2\log 5 = 5\log 3$

$\frac{x\log 5}{\log 5} = \frac{5\log 3 - 2\log 5}{\log 5}$

$x \approx 1.413$

E) $4^{5x+8} = 8^{x-1}$

$(2^2)^{5x+8} = (2^3)^{x-1}$

$2^{10x+16} = 2^{3x-3}$

$10x+16 = 3x-3$

$\frac{7x = -19}{7}$

$x = \frac{-17}{7}$

F) $9^{3x+2} = 27^{x+8}$

$(3^2)^{3x+2} = (3^3)^{x+8}$

$3^{6x+4} = 3^{3x+24}$

$3x = 20$

$x = \frac{20}{3}$

G) $12^{3x-2} = 8^{5x+1}$

$\log 12^{3x-2} = \log 8^{5x+1}$

$(3x-2)\log 12 = (5x+1)\log 8$

$3x\log 12 - 2\log 12 = 5x\log 8 + \log 8$

$3x\log 12 - 5x\log 8 = \log 8 + 2\log 12$

$\frac{x(3\log 12 - 5\log 8)}{3\log 12 - 5\log 8} = \frac{\log 8 + 2\log 12}{3\log 12 - 5\log 8}$

$x \approx -2.346$

H) $5^{6x-5} = 6^{2x+1}$

$\log 5^{6x-5} = \log 6^{2x+1}$

$(6x-5)\log 5 = (2x+1)\log 6$

$6x\log 5 - 5\log 5 = 2x\log 6 + \log 6$

$6x\log 5 - 2x\log 6 = \log 6 + 5\log 5$

$\frac{x(6\log 5 - 2\log 6)}{6\log 5 - 2\log 6} = \frac{\log 6 + 5\log 5}{6\log 5 - 2\log 6}$

$x = \frac{\log 6 + 5\log 5}{6\log 5 - 2\log 6}$

$x \approx 1.620$

I) $4e^{3x}=40$

$e^{3x}=10$

$\ln e^{3x}=\ln 10$

$3x\ln e=\ln 10$

$3x=\ln 10$

$x=\ln\frac{10}{3}$

$x\approx 0.768$

J) $1200e^{x}=900$

$e^{x}=\frac{9}{12}$

$\ln e^{x}=\ln\frac{3}{4}$

$x\ln e=\ln\frac{3}{4}$

$x=\ln\frac{3}{4}$

$x=-0.288$

K) $3e^{4x}=12$

$e^{4x}=4$

$\ln e^{4x}=\ln 4$

$4x\ln e=\ln 4$

$4x=\ln 4$

$x=\frac{\ln 4}{4}$

$x=0.347$

L) $e^{2x}-3e^{x}+2=0$

$(e^x-2)(e^x-1)=0$

$e^x=2$ $\quad$ $e^x=1$

$\ln e^x=\ln 2$ $\quad$ $\ln e^x=\ln 1$

$x\ln e=\ln 2$ $\quad$ $x\ln e=\ln 1$

$x=\ln 2$ $\quad$ $x=\ln 1$

$x=0.693$ $\quad$ $x=0$

M) $e^{2x}-5e^{x}+6=0$

$(e^x-2)(e^x-3)=0$

$e^x=2$ $\quad$ $e^x=3$

$\ln e^x=\ln 2$ $\quad$ $\ln e^x=\ln 3$

$x\ln e=\ln 2$ $\quad$ $x\ln e=\ln 3$

$x=\ln 2$ $\quad$ $x=\ln 3$

$x\approx 0.693$ $\quad$ $x=1.099$

N) $e^{2x}+2e^{x}-24=0$

$(e^x+6)(e^x-4)=0$

$e^x=-6$ $\quad$ $*e^x=4$

$\ln e^x=\ln -6$ $\quad$ $\ln e^x=\ln 4$

$x\ln e=\ln -6$ $\quad$ $x\ln e=\ln 4$

$x\neq\ln -6$ $\quad$ $x=\ln 4$

no $\quad$ $x\approx 1.386$

solution.

O) $2e^{2x}+e^{x}-10=0$

$(2e^x+5)(e^x-2)=0$

$2e^x=-5x$ $\quad$ $e^x=2$

$e^x=-5/2$ $\quad$ $\ln e^x=\ln 2$

$\ln e^x=\ln -\frac{5}{2}$ $\quad$ $x\ln e=\ln 2$

$x\ln e=\ln -\frac{5}{2}$ $\quad$ $x=\ln 2$

$x=\ln -5/2$ $\quad$ $x=0.693$

no

solution

P) $5^{2x}-7\cdot 5^{x}+10=0$

$(5^x-2)(5^x-5)=0$

$5^x=2$ $\quad$ $5^x=5$

$\ln 5^x=\ln 2$

$x\ln 5=\ln 2$

$x=\frac{\ln 2}{\ln 5}$

$x=0.431$ $\quad$ $x=1$

Q) $3^{2x}+5\cdot 3^{x}-14=0$

$(3^x+7)(3^x-2)=0$

$3^x=-7$ $\quad$ $3^x=2$

$\ln 3^x=\ln 7$ $\quad$ $\ln 3^x=\ln 2$

$x\ln 3=\ln -7$ $\quad$ $x\ln 3=\ln 2$

$x\neq\ln -7$ $\quad$ $x=\frac{\ln 2}{\ln 3}$

no $\quad$ $x\approx 0.637$

solution

R) $\log_8(\log_4(\log_6 x))=\frac{1}{3}$

$\log_8 x=\frac{1}{3}$

$x=2$

$\log_4 x=2$

$x=16$

$\log_6 x=1$

S) $\log_2(\log_4 x)=1$

$\log_2 x=1$

$2^1=x$

$\log_4 x=2$

$4^2=x$

$\boxed{x=16}$

T) $\log_3(\log_{27} x)=-1$

$\log_3 x=-1$

$3^{-1}=x$

$x=1/3$

$\log_{27} x=1/3$

$27^{1/3}=x$

$\boxed{x=3}$

Finding the Domain of a Logarithmic Function

Since we can't take the log of zero or the log of a negative number, we can find the domain of a logarithmic function using a simple inequality. Set what is inside the log greater than zero and solve. The result will give you the domain of the function. Consider the following.

$$f_{(x)} = a\log_n(bx+c)+d \qquad f_{(x)} = a\ln(bx+c)+d$$

The variables a, b, c and d are used here to represent numbers that could possibly be used. To find the domain of either of these functions, evaluate the following inequality: $bx+c>0$.

Find the domain of each of the following logarithmic functions. Be sure to write the domain using interval notation.

A) $f_{(x)} = \log_5 x+2$

$x>0$

$(0,\infty)$

B) $f_{(x)} = \log_3(4x-1)-2$

$4x-1>0$

$4x>1$

$x>1/4$

$(1/4,\infty)$

C) $f_{(x)} = -\log_2 3x$

$3x>0$

$x>0$

$(0,\infty)$

D) $f_{(x)} = \log_2(5-x)$

$5-x>0$

$x<5$

$(-\infty,5)$

E) $f_{(x)} = \log_5(-x)+5$

$-x>0$

$x<0$

$(-\infty,0)$

F) $f_{(x)} = \ln x-4$

$x>0$

$(0,\infty)$

G) $f_{(x)} = \ln(2-3x)$

$2x-3x>0$

$2>3x$

$x<\frac{2}{3}$

$(-\infty,2/3)$

H) $f_{(x)} = \log_3(x+5)+1$

$x+5>0$

$x>-5$

$(-5,\infty)$

I) $f_{(x)} = \ln(x-3)$

$x-3>0$

$x>3$

$(3,\infty)$

J) $f_{(x)} = \log_2|x|$

$|x|>0$

$0>x>0$

$(-\infty,0)\cup(0,\infty)$

K) $f_{(x)} = |\log_3 x|$

$x>0$

$(0,\infty)$

L) $f_{(x)} = -\ln 5x+1$

$5x>0$

$x>0$

$(0,\infty)$

Explain the difference between the domains of letters J and K above. Although each problem contains an absolute value, the domains are different. Why is this?

For example J you can subtitute any value for x, as long as its not zero and evaluate the problem. To find the value of K you must use the standard method of $bx+c>0$

Finding the Vertical Asymptote of a Logarithmic Function

$f_{(x)} = a\log_n\left(bx+c\right)+d$ $\qquad$ $f_{(x)} = a\ln\left(bx+c\right)+d$

To find the vertical asymptote of a logarithmic function, set bx+c equal to zero and solve. This will yield the equation of a vertical line. In this case, that vertical line is the vertical asymptote.

Example

Find the vertical asymptote of the function $f_{(x)} = \log_3\left(4x-3\right)-2$.

$$4x-3=0$$
$$4x=3$$
$$x=\frac{3}{4}$$

Setting the bx+c term equal to zero results in this equation. Simply solve for x, and we now have the vertical asymptote of the function.

Find the vertical asymptote of each of the following logarithmic functions.

A) $f_{(x)} = \log_5 x+2$

$x=0$

B) $f_{(x)} = \log_3\left(4x-1\right)-2$

$4x-1=0$

$4x=1$

$x=\frac{1}{4}$

C) $f_{(x)} = -\log_2 3x$

$3x=0$

$x=0$

D) $f_{(x)} = \log_2\left(5-x\right)$

$5-x=0$

$+x \quad +x$

$x=5$

E) $f_{(x)} = \log_5\left(-x\right)+5$

$-x=0$

$x=0$

F) $f_{(x)} = \ln x-4$

$x=0$

G) $f_{(x)} = \ln\left(2-3x\right)$

$2-3x=0$

$\frac{2}{3}=\frac{3x}{3}$

$x=\frac{2}{3}$

H) $f_{(x)} = \log_3\left(x+5\right)+1$

$x+5=0$

$x=-5$

I) $f_{(x)} = \ln\left(x-3\right)$

$x-3=0$

$x=3$

J) $f_{(x)} = \log_2|x|$

$x=0$

K) $f_{(x)} = |\log_3 x|$

$x=0$

L) $f_{(x)} = -\ln 5x+1$

$5x=0$

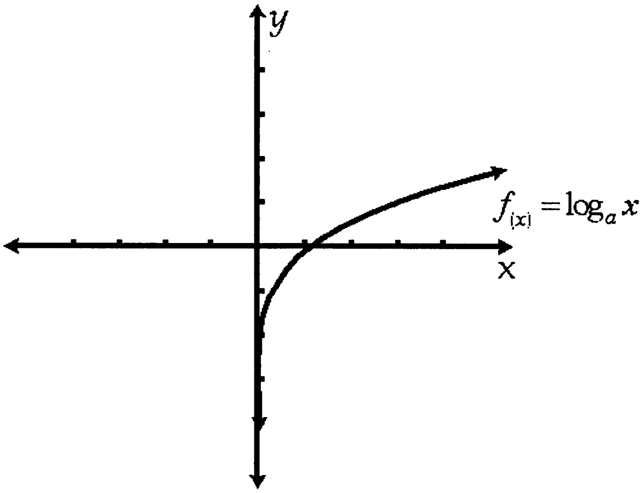
$x=0$

Graphing Logarithmic Functions

$$f_{(x)} = a\log_n\left(bx+c\right)+d \qquad\qquad f_{(x)} = a\ln\left(bx+c\right)+d$$

The parent functions for both logarithmic and natural logarithmic functions look almost identical. The following functions will be graphed by setting up a table. When faced with the following function: $f_{(x)} = \log_3 x$ ***, find the domain of the function first by evaluating*** $bx+c>0$. ***This is done to ensure that only appropriate values are substituted for x. The values that should be used for x are powers of the base of the logarithm such as*** $\frac{1}{9}, \frac{1}{3}, 1, 3, \text{ and } 9$. ***The values used to substitute for x are completely dependant on the base of the particular logarithm in the problem. When graphing a natural log with a table, any number in the domain of the function may be used. Using a calculator, get the decimal estimates for the y values of the function. DO NOT FORGET to plug in fractional powers for both logs and natural logs. The fractional powers will give the tail end of the function. Be sure to plug in the fractions that are right next to the vertical asymptote of the function. Note in the following picture the graph does not cross the y axis. That is because, in this particular example,*** $x=0$ ***is the vertical asymptote. By now, you should be aware that the equation*** $x=0$ ***is the equation for the y axis. Remember, always find the vertical asymptote of any logarithmic function when graphing. This must be done to ensure that it is not crossed. The range of any normal logarithmic function is all real numbers. The only time this will not be is the case is that of example K in the finding the domain section on the previous page.***

Remember, most people forget the tail end of the function. Do not forget it. Make sure you note the rise of the function. The function makes a very slow gradual rise. It is not steep.



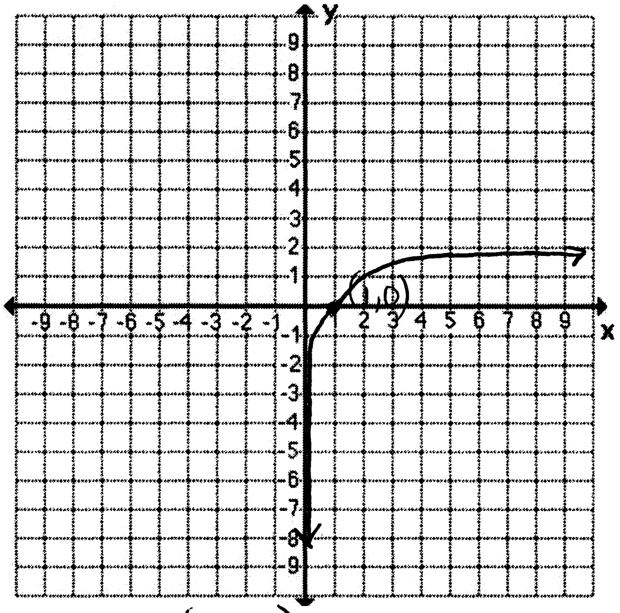
Normally, we will be graphing these functions by means of translations based on the parent function. This graphing portion is meant to give you practice at using your knowledge about logarithms and how to use powers to graph the functions. There will be an entire section devoted to nothing but functions later on in the workbook.

Graph each of the following by setting up a table. Identify the range and domain of each function. Label the x intercept of each function. ***We will deal with y intercepts later.***

A) $f_{(x)} = \log_3 x$

$\log_3 \frac{1}{9} = -2$
$\log_3 \frac{1}{3} = -1$
$\log_3 1 = 0$
$\log_3 3 = 1$
$\log_3 9 = 2$

x	$f_{(x)}$
$\frac{1}{9}$	-2
$\frac{1}{3}$	-1
1	0
3	1
9	2

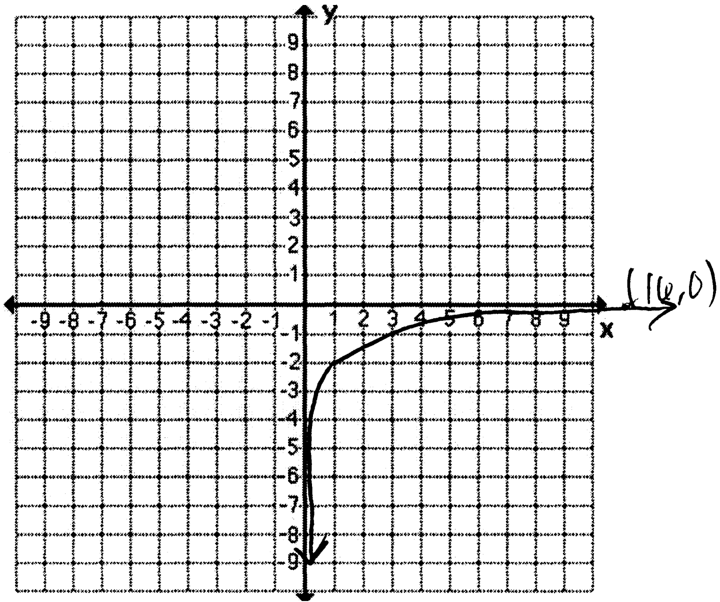


Domain: $(0, \infty)$
Range: $(-\infty, \infty)$

B) $f_{(x)} = \log_4 x - 2$

$\log_4 \frac{1}{16} - 2 = -2 - 2 = -4$
$\log_4 \frac{1}{4} - 2 = -1 - 2 = -3$
$\log_4 1 - 2 = 0 - 2 = -2$
$\log_4 4 - 2 = 1 - 2 = -1$
$\log_4 16 - 2 = 2 - 2 = 0$

x	$f_{(x)}$
$\frac{1}{16}$	-4
$\frac{1}{4}$	-3
1	-2
4	-1
16	0



Domain: $(0, \infty)$
Range: $(-\infty, \infty)$

C) $f_{(x)} = -\log_2 x$

$-\log_2 1/4 = -(-2) = 2$

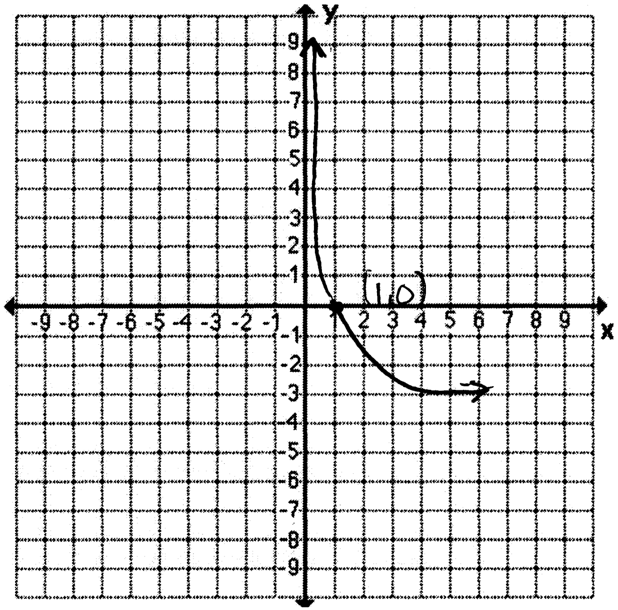
$-\log_2 1/2 = -(-1) = 1$

$-\log_2 1 = 0$

$-\log_2 2 = -(1) = -1$

$-\log_2 4 = -2$

x	$f_{(x)}$
$\frac{1}{4}$	2
$\frac{1}{2}$	1
1	0
2	-1
4	-2



Domain: $(0, \infty)$
Range: $(-\infty, \infty)$

D) $f_{(x)} = \log_3 (x+1)$

$x+1>0$
$x>-1$

$\log_3(-\frac{8}{9}+1) = \log_3 \frac{1}{9} = -2$

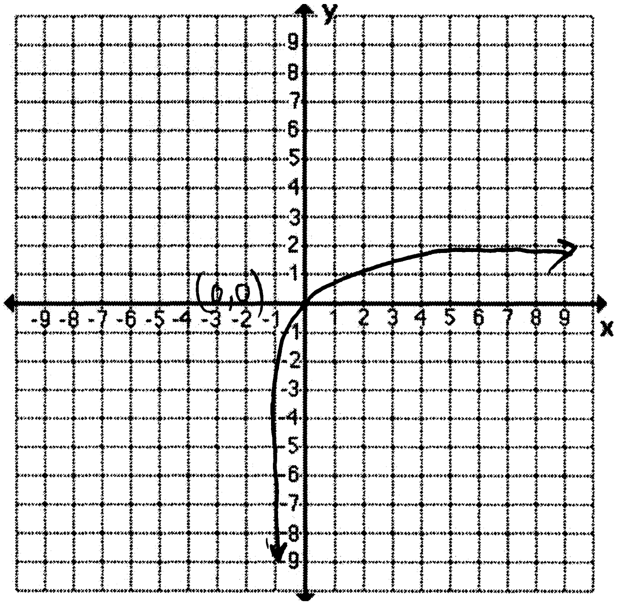
$\log_3(-\frac{2}{3}+1) = \log_3 \frac{1}{3} = -1$

$\log_3(0+1) = \log 1 = 0$

$\log_3(2+1) = \log_3 3 = 1$

$\log_3(8+1) = \log_3 9 = 2$

x	$f_{(x)}$
$-\frac{8}{9}$	-2
$-\frac{2}{3}$	-1
0	0
2	1
8	2



Domain: $(-1, \infty)$
Range: $(-\infty, \infty)$

Domain

$x+1>0$
$x>-1$

E) $f_{(x)} = \ln x + 3$

$\ln\frac{1}{10}+3 = .697 = .7$

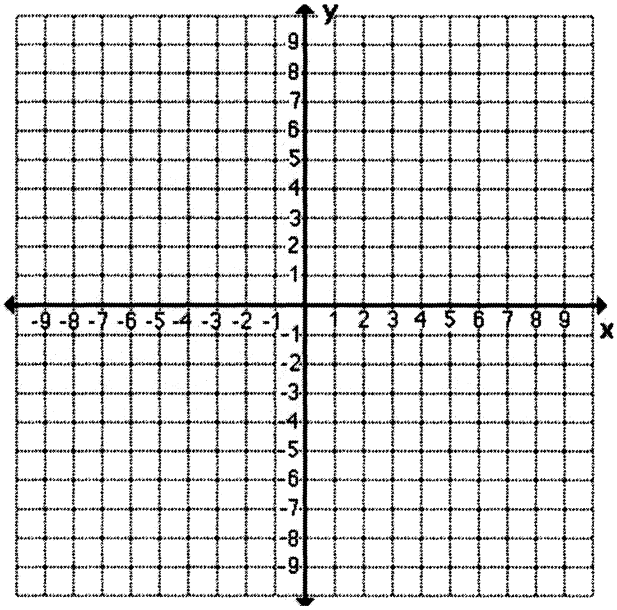
$\ln\frac{1}{5}+3 = 1.4$

$\ln 1+3 = 3$

$\ln 2+3 = 3.7$

$\ln 3+3 = 4.1$

x	$f_{(x)}$
$\frac{1}{10}$	.7
$\frac{1}{5}$	1.4
1	3
2	3.7
3	4.1



x-int

$\ln x + 3 = 0$

$\ln x = -3$

$x = e^{-3}$

$x = .05$

Domain: $(0, \infty)$

Range: $(-\infty, \infty)$

F) $f_{(x)} = \ln(x+2)$

$\ln(-1.8+2) = -1.6$

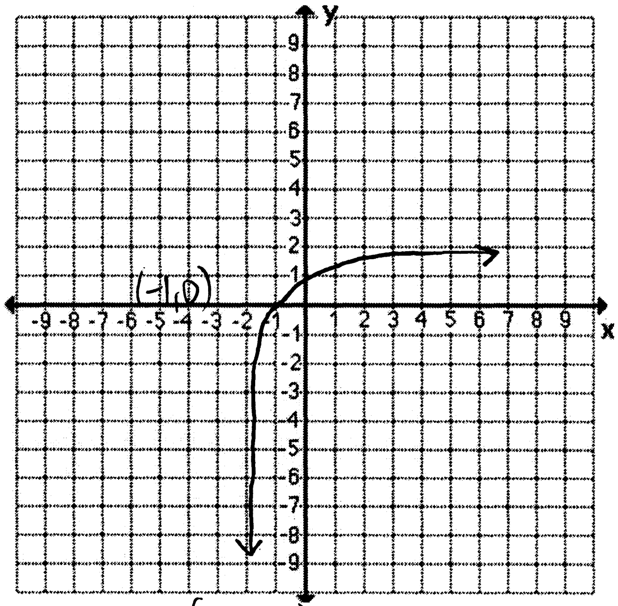
$\ln(-1.5+~~8~~2) = -0.7$

$\ln(-1+2) = 0$

$\ln(0+2) = 2$

$\ln(2+2) = 1.4$

x	$f_{(x)}$
-1.8	-1.6
-1.5	-0.7
-1	0
0	0.7
2	1.4



Domain:

$x+2>0$

$x>-2$

x-int:

$\ln(x+2) = 0$

$e^0 = x+2$

$1 = x+2$

$x = -1$

Domain: $(-2, \infty)$

Range: $(-\infty, \infty)$

G) $f_{(x)} = \log_3(-x)$

$\log_3(-(-1/9)) = -2$

$\log_3(-(-1/3)) = -1$

$\log_3(-(-1)) = 0$

$\log_3(-(-3)) = 1$

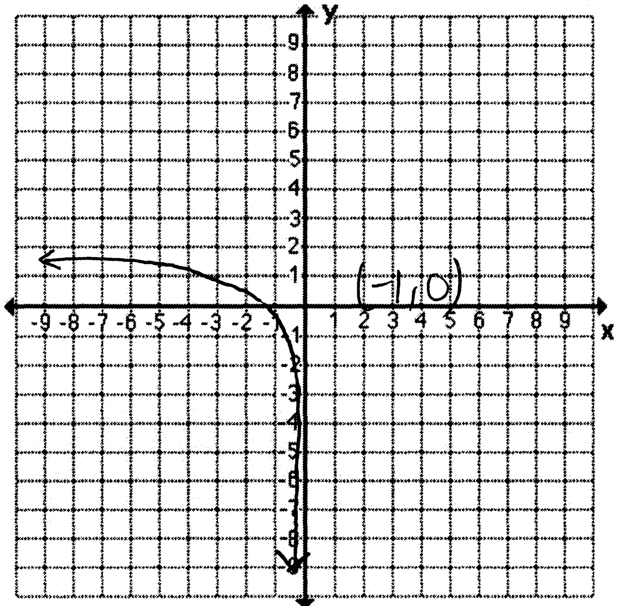
$\log_3(-(-9)) = 2$

Domain:

$-x > 0$

$x < 0$

x	$f_{(x)}$
$-\frac{1}{9}$	-2
$-\frac{1}{3}$	-1
-1	0
-3	1
-9	2



Domain: $(-\infty, 0)$

Range: $(-\infty, \infty)$

H) $f_{(x)} = |\log_2 x|$

$|\log_2 1/4| = |-2| = 2$

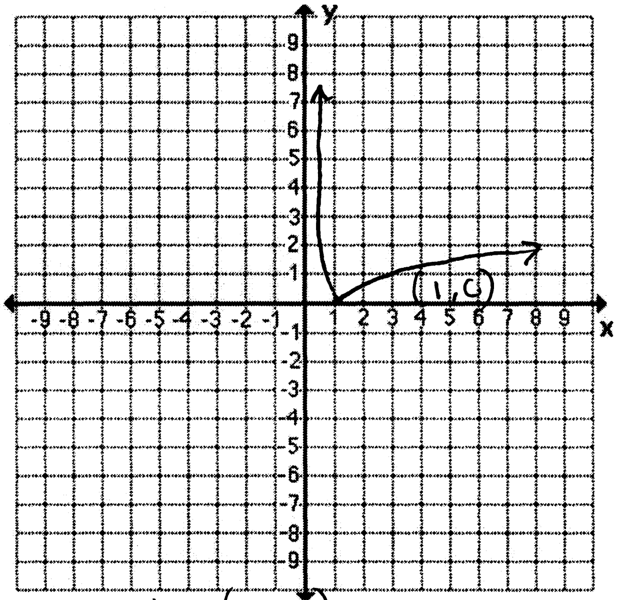
$|\log_2 \frac{1}{2}| = |-1| = 1$

$|\log_2 1| = |0| = 0$

$|\log_2 2| = |1| = 1$

$|\log_2 4| = |2| = 2$

x	$f_{(x)}$
$\frac{1}{4}$	2
$\frac{1}{2}$	1
1	0
2	1
4	2



Domain: $(0, \infty)$

Range: $[0, \infty)$

Finding the Inverse of a Function

In the past we worked on "function operations," in which we added, subtracted, multiplied, divided, and found composite functions (refer to page 68). The next function operation we will go over is finding the inverse of a function. If a function is denoted by $f_{(x)}$, its' inverse is denoted by $f^{-1}_{(x)}$. The "-1" is a superscript of the term, <u>not</u> an exponent.

<u>Finding the inverse of a function is a 4 step process.</u>

1. ***Replace $f_{(x)}$ with y***
2. ***Switch the x and y in the problem.***
3. ***Solve for y***
4. ***Replace y with $f^{-1}_{(x)}$***

A function only has an inverse if it is a one-to-one function. This means that for each y value, there exists only one corresponding x value. Simply put, a function is one-to-one if it passes a horizontal line test.

Proving whether or not two functions are inverses of each other is a different matter. Two functions $f_{(x)}$ and $g_{(x)}$ are inverses of each other if and only if the following is true:

$$f_{\left(g_{(x)}\right)} = x \text{ and } \quad g_{\left(f_{(x)}\right)} = x$$

If <u>both</u> composite functions $f_{\left(g_{(x)}\right)}$ and $g_{\left(f_{(x)}\right)}$ equal x, the functions are inverses of each other.

The official definition states: $f_{\left(g_{(x)}\right)} = x$ ***for all x in the domain of g.***

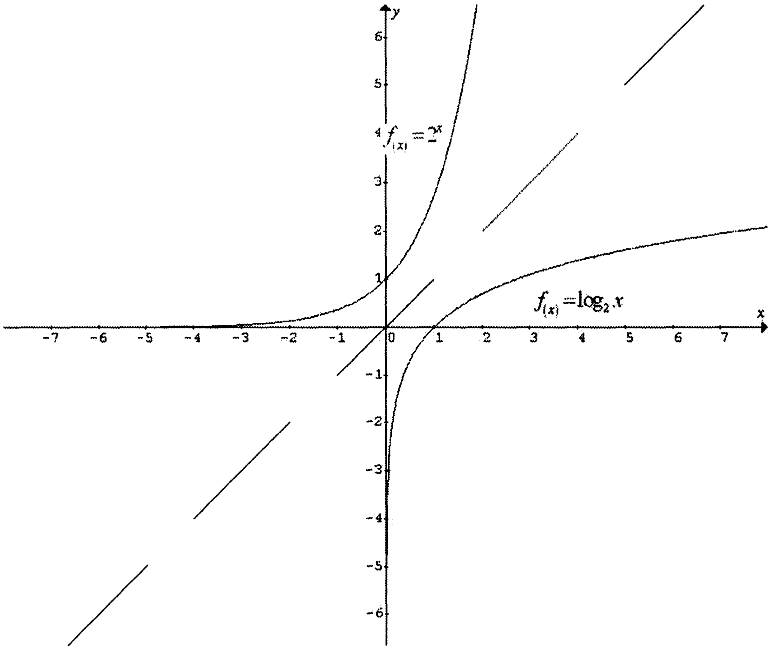
$g_{\left(f_{(x)}\right)} = x$ ***for all x in the domain of f.***

<u>Here is an example of finding the inverse of an exponential function.</u>

$f_{(x)} = 3^x - 1$	
$y = 3^x - 1$	***Replace $f_{(x)}$ with y.***
$x = 3^y - 1$	***Switch the x and y.***
$x + 1 = 3^y$	***Solve for y by adding 1 to both sides.***
$\log(x+1) = \log 3^y$	***Now you need to get to the y exponent, so take the log of both sides.***
$\log(x+1) = y \log 3$	***Pull out the exponent y using the properties of logarithms.***
$\dfrac{\log(x+1)}{\log 3} = y$	***Divide both sides now by*** $\log 3$.
$y = \log_3(x+1)$	***Use the symmetrical property to switch sides.*** ***Using the base change formula, you can rewrite the log with a base of 3.***
$f^{-1}_{(x)} = \log_3(x+1)$	***Replace y with $f^{-1}_{(x)}$***

As demonstrated in the previous example, the inverse of an exponential function is a logarithmic function and vice-versa.

Inverse functions are symmetrical to the $y = x$ axis. When an exponential function and its' logarithmic inverse are graphed on the same plane, the symmetry is apparent.



Drawing an imaginary diagonal using the line $y = x$, the functions could literally be folded along that axis, and the two functions would line up on top of each other.

In the following exercises you will be asked to find the inverse of a logarithmic function. It should be obvious that the inverse of a logarithmic function is exponential. In order to get to that point, the problem will need to be rewritten in exponential form.

Logarithmic form	**Exponential Form**
$\log_a b = c$	$a^c = b$

<u>Here is an example of finding the inverse of a logarithmic function.</u>

$f_{(x)} = \log_7(x-2)+3$	
$y = \log_7(x-2)+3$	***Replace $f_{(x)}$ with y.***
$x = \log_7(y-2)+3$	***Switch the x and y.***
$x-3 = \log_7(y-2)$	***Isolate the log by subtracting 3 to both sides.***
$\log_7(y-2) = x-3$	***Using the symmetrical property, the log is easier to see like this.***
$7^{x-3} = y-2$	***Rewrite the log in exponential form.***
$7^{x-3}+2 = y$	***Solve for y by adding 2 to both sides.***
$y = 7^{x-3}+2$	***Again rewrite using the symmetrical property.***
$f_{(x)}^{-1} = 7^{x-3}+2$	***Replace y with $f_{(x)}^{-1}$***

Find the inverse of each of the following functions if it exists.

A) $f_{(x)} = \frac{1}{2}x + 3$

$x = \frac{1}{2}y + 3$

$x - 3 = \frac{1}{2}y$

$\frac{1}{2}y = x - 3$

$y = 2x - 6$

$f^{-1}(x) = 2x - 6$

B) $f_{(x)} = 4x - 5$

$x = 4y - 5$

$x + 5 = 4y$

$4y = x + 5$

$y = \frac{x}{4} + \frac{5}{4}$

$f^{-1}(x) = \frac{1}{4}x + \frac{5}{4}$

C) $f_{(x)} = 4^{x+2}$

$x = 4^{x+2}$

$\log x = \log 4^{(x+2)}$

$\log x = (y+2)\log 4$

$(y+2)\log 4 = \log x$

$y\log 4 + 2\log 4 = \log x$

$y\log 4 = \log x - 2\log x$

$y\log 4 = \log\frac{x}{16}$ $\quad 4 = \frac{\log x}{\log 4} - \frac{2\log 4}{\log 4}$

$f^{-1}(x) = \log\frac{x}{16}$ or $f^{-1}(x) = \log_4 x - 2$

D) $f_{(x)} = 2^{x-3} + 1$

$x = 2^{y-3} + 1$

$x - 1 = 2^{y-3}$

$\log(x-1) = \log 2^{y-3}$

$\log(x-1) = (y-3)\log 2$

$\log(x-1) = y\log 2 - 3\log 2$

$y\log 2 - \log 8 = \log(x-1)$

$\frac{y\log 2}{\log 2} = \frac{\log(x-1)}{\log 2} + \frac{\log 8}{\log 2}$

$f^{-1}(x) = \log 2(x-1) + 3$

E) $f_{(x)} = \sqrt[3]{3x-2}$

$(x)^3 = (\sqrt[3]{3y-2})^3$

$x^3 = 3y - 2$

$x^3 + 2 = 3y$

$\frac{3y}{3} = \frac{x^3+2}{3}$

$y = \frac{1}{3}x^3 + \frac{2}{3}$

$f^{-1}(x) = \frac{x^3+2}{3}$

F) $f_{(x)} = 4(x+1)^2 - 3$

Does not exist

Not a one-to-one function.

G) $f_{(x)} = \log_6 x$

$x = \log_6 y$

$\log_6 y = x$

$6^x = y$

$y = 6^x$

$f^{-1}(x) = 6^x$

H) $f_{(x)} = \ln(x-2)$

$x = \ln(y-2)$

$\ln(y-2) = x$

$e^x = y - 2$

$e^x + 2 = y$

$y = e^x + 2$

$f^{-1}(x) = e^x + 2$

I) $f_{(x)} = 2\log_4 x - 5$

$x = 2\log_4 y - 5$

$2\log_4 y - 5 = x$

$\frac{2\log_4 y}{2} = \frac{x+5}{2}$

$\log_4 y = \frac{x+5}{2}$

$4^{\frac{x+5}{2}} = y$

$y = (4^{1/2})^{x+5}$

$f^{-1}(x) = 2^{x+5}$

J) $f_{(x)} = -\sqrt{x-4} + 6$

$x = -\sqrt{y-4} + 6$

$(\sqrt{y-4})^2 = (6-x)^2$ note $6-x$

$y - 4 = (x-6)^2$ $\quad -1(x-6)$

$y = (x-6)^2 + 4$ $\quad$ now square

$f^{-1}(x) = (x+6)^2 + 4$ $\quad [-1(x-6)]^2$

$(x-6)^2$

K) $f_{(x)} = (x+3)^3 - 2$

$x = (y+3)^3 - 2$

$x + 2 = (y+3)^3$

$\sqrt[3]{(y+3)^3} = \sqrt[3]{x+2}$

$y = \sqrt[3]{x+2} - 3$

$f^{-1}(x) = \sqrt[3]{x+2} - 3$

L) $f_{(x)} = 3^{x-20} + 1$

$x = 3^{y-20} + 1$

$x - 1 = 3^{y-20}$

$\log 3^{y-20} = \log(x-1)$

$(y-20)\log 3 = \log(x-1)$

$y\log 3 - 20\log 3 = \log(x-1)$

$$\frac{y\log 3 = \log(x-1) + 20\log 3}{\log 3}$$

$y = \frac{\log(x-1)}{\log 3} + 20$

$f^{-1}(x) = \log 3^{(x-1)} + 20$

Interest Formulas

There are three interest formulas commonly used in mathematics. The first is the standard interest formula. We will not be using that formula in this section. Instead, we will concentrate on the other two. We will be working with the compound interest formulas. There are two formulas for this. The first is a formula for interest being compounded a finite number of times per year. The second formula is for interest compounded continuously. For each of these functions, the variable time, t, should be in years. Which means 6 months is .5 years.

Standard Interest Formula	*Compound Interest Formula*	*Interest Compounded continuously*
$A = P + Prt$	$A = P\left(1+\frac{r}{n}\right)^{nt}$	$A = Pe^{rt}$

When using the compound interest formula, what does the variable n represent?

The # of times compounded per year.

When dealing with the compound interest formula, your questions will always come in the form of word problems. Write down the word associated with the number of times interest is to be compounded per year when dealing with these word problems.

# of times compounded		*Word used in the problem*
once		annually
twice		semi-annually
four times		quarterly
twelve times		monthly
twenty-four times		semi - monthly

Explain, in your own words, what compounding interest means.

when calculating the balance of an interest, the previous interest earned is added to the balance and used to compute the current balance

If you had a choice to invest a sum of money into an account that yields 7% interest compounded quarterly verses one that compounds interest semi-annually, which would you choose? quarterly

Answer each of the following.

A) If you invest \$2500 in an account that pays 12% interest, compounded quarterly, how much would you have at the end of 17 years?

$A=P(1+\frac{r}{n})^{nt}$

$P=2500$

$r=.12$

$n=4$

$t=17$

$A=2500(1+\frac{.12}{4})^{4(17)}$

$A=2500(1.03)^{68}$

$A=\$18,658.27$

B) How much would you have to invest in an account that pays 6% interest, compounded monthly, to have a balance of \$30,000 at the end of 10 years?

$A=30000$

$P=$

$r=.06$

$n=12$

$t=10$

$30000=P(1+\frac{.06}{12})^{(12)(10)}$

$\frac{30000}{(1.005)^{120}}=\frac{P(1.005)^{120}}{(1.005)^{120}}$

$P=\$16,488.48$

C) How long will it take for an investment of \$2,000 in an account that pays $8\frac{1}{2}$% interest compounded quarterly to become \$15,000.

$A=15000$

$P=2000$

$r=.085$

$t=$

$\frac{15000}{2000}=\frac{2000}{2000}(1+\frac{.085}{4})^{4t}$

$7.5=(1.02125)^{4t}$

$\frac{4t\log(1.02125)}{4\log(1.02125)}=\frac{\log 7.5}{4\log(1.02125)}$

$t\approx 23.45$ yrs. or 24 yrs.

D) How long will it take for an amount of money to double if deposited in an account that pays 4.5% interest compounded monthly?

$r=.045$

$n=12$

$\frac{2P}{P}=\frac{P}{P}(1+\frac{.045}{12})^{12t}$

$2=(1+\frac{.045}{12})^{12t}$

$(1.00375)^{12t}=2$

$\frac{12t\log(1.00375)}{12\log(1.00375)}=\frac{\log 2}{12\log(1.00375)}$

$t\approx 15.4$ yrs

E) At what interest rate must you invest \$10,000 to have an ending balance of \$72,000 at the end of 14 years? (Assume interest is compounded quarterly.)

$A=P(1+\frac{r}{n})^{nt}$

$A=72000$

$P=10000$

$r=$

$n=4$

$t=14$

$\frac{72000}{10000}=\frac{10000}{10000}(1+\frac{r}{4})^{4(14)}$

$\sqrt[56]{7.2}=\sqrt[56]{(1+\frac{1}{4})^{56}}$

$1+\frac{r}{4}=\sqrt[56]{7.2}$

$\frac{r}{4}=\sqrt[56]{7.2-1}$

$r=4(\sqrt[56]{7.1}-1$

$r=.1435$

$r=14.35\%$

F) If you invest \$26,000 in an account that pays $3\frac{1}{2}$% interest compounded continuously, how much would your investment be worth in 18 years.

$A=Pe^{rt}$

$P=26000$

$r=.035$

$t=18$

$A=26000\,e^{(0.35)(18)}$

$A=26000\,e^{.63}$

$A=\$48,817.88$

G) How long will it take for an amount of money to double if invested in an account that pays 9% interest compounded continuously.

$A=Pe^{rt}$

$\frac{2P}{P}=\frac{Pe^{.096}}{P}$

$2=e^{.09t}$

$\ln 2=\ln e^{.09t}$

$\ln 2=.09t\ln e$

$\frac{.09t}{.09}=\frac{\ln 2}{.09}$

$t\approx 7.7$ yrs.

H) At what interest rate must you invest \$10,000 to have an ending balance of \$15,000 at the end of 9 years? (Assume interest is compounded continuously.)

$A=Pe^{rt}$

$\frac{15000}{10000}=\frac{10000}{10000}e^{9r}$

$e^{9r}=1.5$

$\ln e^{9r}=\ln 1.5$

$9r\ln e=\ln 1.5$

$\frac{9r}{9}=\frac{\ln 1.5}{9}$

$r\approx .445$

4.5%

I) How much must your principle investment be if you have \$30,000 after 16 years in an account that pays 7.5% interest compounded continuously.

$A=Pe^{rt}$

$30000=Pe^{(.075)(16)}$

$\frac{30000}{e^{1.2}}=\frac{Pe^{1.2}}{e^{1.2}}$

$P=\frac{30000}{e^{1.2}}$

$P=9,035.83$

J) Complete the following table to determine the value, A, of an initial investment, P, of \$15,000 into an account that pays 6.5% interest compounded n times for 10 years.

n	**1**	**2**	**4**	**12**	**365**
A	28157.06	28437.57	28583.38	28682.76	28731.45

$P=15000$ $r=.065$ $t=10$ $A=P(1+\frac{r}{n})^{nt}$

$n=1$

$A=15000(1+\frac{.065}{1})^{10}$

$A=15000(1.065)^{10}$

28,157.06

$n=2$

$A=15000(1+\frac{.065}{2})^{20}$

$A=15000(1.0325)^{20}$

28,437.57

$n=4$

$A=15000(1+\frac{.065}{4})^{40}$

$A=15000(1.01625)^{40}$

28,583.38

$n=12$

$A=15000(1+\frac{.065}{12})^{120}$

28,682.76

$n=365$

$A=15000(1+\frac{.065}{365})^{3650}$

28,731.45

K) Complete the table for the time necessary for an account to triple if interest is compounded continuously at rate r.

r	**2%**	**3%**	**4.5%**	**7%**	**12%**
t	54.9yrs	36.6yrs	24.4yrs	15.7yrs	9.2yrs

$A=Pe^{rt}$

$\frac{3P}{P}=\frac{P}{P}e^{.02t}$

$3=e^{.02t}$

$\ln 3=\ln e^{.02t}$

$\frac{\ln 3}{.02}=\frac{.02t}{.02}$

$t\approx 54.9$yrs

$3P=Pe^{.03t}$

$3=e^{.03t}$

$\ln 3=\ln e^{.03t}$

$\frac{\ln 3}{.03}=\frac{.03t}{.03}$

$t\approx 36.6$yrs

$3P=Pe^{.045t}$

$3=e^{.045t}$

$\ln 3=\ln e^{.045t}$

$\frac{\ln e}{.045}=\frac{.045t}{.045}$

$t\approx 24.4$yrs

$3P=Pe^{.07t}$

$3=e^{.07t}$

$\ln 3=\ln e^{.07t}$

$\frac{\ln 3}{.07}=\frac{.07t}{.07}$

$t\approx 15.7$yrs

$3P=Pe^{.12t}$

$3=e^{.12t}$

$\frac{\ln e}{.12}=\frac{.12t}{.12}$

$t\approx 9.2$yrs

Word Problems

A) The demand equation for a certain clock radio is given by $p = 400 - .06e^{0.003x}$. Find the demand, x, for the price of $p = \$99$.

$P = 400 - .06e^{0.003x}$

$99 = 400 - .06e^{0.003x}$

$-301 = -.06e^{.003x}$

$5016.66\overline{6} = e^{0.003x}$

$e^{0.003x} = \frac{301}{.06}$

$\ln e^{0.003x} = \ln \frac{301}{.06}$

$0.003x \ln e = \ln \frac{301}{.06}$

$\frac{0.003x}{0.003} = \frac{\ln \frac{301}{.06}}{0.003}$

$x = \frac{\ln \frac{3.01}{.06}}{0.003}$

$x \approx 2840.2$

B) The population, P, where P is measured in thousands, of one city is given by $P = 30e^{kt}$. In this particular model, $t = 0$ represents the year 2000. In 1990, the population was 52,000. Find the value of k and use the result to estimate the population of the city in the year 2012.

$P = 30e^{kt}$

$\frac{52}{30} = \frac{30e^{-10k}}{30}$

$e^{-10k} = \frac{52}{30}$

$\ln e^{-10k} = \ln \frac{52}{30}$

$-10k \ln e = \ln \frac{52}{30}$

$\frac{-10k}{-10} = \frac{\ln \frac{52}{30}}{-10}$

$k \approx -.055$

check

$P = 30e^{-10(-.055)}$

$P = 51.947$

$\times 1000$

51447

in t 2012

$t = 12.0$

$P = 30e^{(-.055)(12)}$

$P = 15.505$

population in 2013

is approximately 15,505

On the Richter scale, the magnitude R of an earthquake with intensity I is measured by

$$R = \log_{10} \frac{I}{I_0}$$

Where $I_0 = 1$ is the minimum intensity used for comparison.

C) Find the intensity of an earthquake that measures 6.5 on the Richter scale.

$R = \log_{10} \frac{I}{I_0}$

$6.5 = \log_{10} \frac{I}{1}$

$6.5 = \log_{10} I$

$\log I = 6.5$

$10^{6.5} = I$

$I = 3{,}162{,}277.7$

D) Find the intensity of an earthquake the measures 3.2 on the Richter scale.

$R = \log_{10} \frac{I}{I_0}$

$3.2 = \log_{10} I$

$\log_{10} I = 3.2$

$\log 3.2 = I$

$I \approx 1584.9$

E) Find the magnitude of an earthquake that has an intensity of 325,000.

$R = \log_{10} \frac{I}{I_0}$

$R = \log_{10} \frac{325000}{1}$

$R = \log_{10} 325000$

$R = 5.5$

Checking Progress

You have now completed the "Logarithms" section of the workbook. The following is a checklist so that you may check your progress. Check off each of the objectives you have accomplished.

The student should be able to…

- ☐ ***Evaluate a simple logarithm without the aid of a calculator.***
- ☐ ***Express a logarithmic statement is exponential form.***
- ☐ ***Express a statement in exponential form in logarithmic form***
- ☐ ***Expand a logarithmic expression as the sum or difference of logarithms using the properties of logs.***
- ☐ ***Condense the sum or difference of logarithms into a single logarithmic expression.***
- ☐ ***Evaluate logarithms using the base change formula.***
- ☐ ***Solve logarithmic equations.***
- ☐ ***Evaluate the solution to logarithmic equations to find extraneous roots.***
- ☐ ***Solve equations with variables in the exponents.***
- ☐ ***Find the range and domain of logarithmic functions.***
- ☐ ***Graph a logarithmic function using a table.***
- ☐ ***Find the inverse of a function.***
- ☐ ***Verify two functions are inverses of each other.***
- ☐ ***Identify a one-to-one function.***
- ☐ ***Use the compound interest formulas.***

QUADRATIC EQUATIONS

Objectives

The following is a list of objectives for this section of the workbook.

By the time the student is finished with this section of the workbook, he/she should be able to…

- ***Solve quadratic equations by completing the square.***
- ***Derive the quadratic formula.***
- ***Solve quadratic equations by using the quadratic formula.***
- ***Determine the nature of the roots of a quadratic using the discriminant.***
- ***Put a quadratic function in standard form.***
- ***Determine how the values of a, h, and k affect the graph of a function in*** $y=a(x-h)^2+k$ ***form.***
- ***Find the vertex of a quadratic function.***
- ***Find the x and y intercepts of a quadratic function.***
- ***Find the range and domain of a quadratic function without graphing it.***
- ***Graph a quadratic function.***
- ***Determine the maximum or minimum value of a quadratic function.***
- ***Determine the interval in which the value of a function is increasing, decreasing or constant.***
- ***Determine the interval in which the value of a function is positive or negative.***
- ***Find the equation of a parabola given the vertex and a point on the parabola.***
- ***Find the equation of a parabola given the x intercepts of the graph of the function.***
- ***Find the equation of a parabola given three points on the curve.***

Math Standards Addressed

The following state standards are addressed in this section of the workbook.

Algebra II

8.0 Students solve and graph quadratic equations by factoring, completing the square, or using the quadratic formula. Students apply these techniques in solving word problems. They also solve quadratic equations in the complex number system.

9.0 Students demonstrate and explain the effect that changing a coefficient has on the graph of quadratic functions; that is, students can determine how the graph of a parabola changes as *a, b,* and *c* vary in the equation $y = a(x-b)^2 + c$.

10.0 Students graph quadratic functions and determine the maxima, minima, and zeros of the function.

The first thing we are going to do in this section is solve quadratic equations. In the "Polynomials" section of this workbook we went over solving quadratic equations by factoring. Since not all quadratic equations are factorable, we must have alternative methods for solving these problems. In this section we will be solving quadratic equations by completing the square, and using the quadratic formula. We will begin with completing the square.

Completing the Square

When solving a quadratic equation by completing the square, the goal is to create a perfect square binomial on the left side of the equal sign. A perfect square binomial resembles the following.

x^2+6x+9

You can see that x^2+6x+9 ***is really*** $(x+3)^2$***, therefore, it is a perfect square; just as 4 is the perfect square of 2.***

****The most important rule when completing the square is "You can only complete the square when the leading coefficient is one." If the leading coefficient is any other number, you will need to multiply the entire equation by its' reciprocal. This will yield a leading coefficient of one. Observe the following example.***

$x^2+8x-4=0$	***Begin with the quadratic equation in standard form*** $ax^2+bx+c=0$
$x^2+8x\quad=4$	***Begin by adding 4 to both sides of the equation***
$\left(\frac{b}{2}\right)^2=\left(\frac{8}{2}\right)^2=(4)^2=16$	***Now we need to create a perfect square binomial. We need to find the missing number after the 8x, so we need to evaluate*** $\left(\frac{b}{2}\right)^2$.
$x^2+8x+16=4+16$	***Once we evaluate b over 2 squared, we add the result to both sides of the equation.***
$(x+4)^2=20$	***Now, on the left side of the equal sign is the perfect square in factored form. When you evaluate b over 2, that number, in this case 4, is what goes in the factor.***
$\sqrt{(x+4)^2}=\pm\sqrt{20}$	***Now take the square root of both sides of the equation.*** ***Do not forget to use*** ± ***on the right side.***
$(x+4)=\pm2\sqrt{5}$	***Simplify the radical if possible.***
$x=-4\pm2\sqrt{5}$	***Now subtract 4 to both sides. Since we do not have two separate rational solutions, the answers will be written as algebraic expressions. For now, the solution may be left like this, however in the future, it will be necessary to use each separately.***

Here is a more complicated example.

$$2x^2 - 7x + 3 = 0$$

Here is the quadratic equation in standard form
$ax^2 + bx + c = 0$

$$x^2 - \frac{7}{2}x + \frac{3}{2} = 0$$

Begin by multiplying the entire equation by ½.

$$x^2 - \frac{7}{2}x \quad = -\frac{3}{2}$$

Subtract $\frac{3}{2}$ ***to both sides.***

$$\left(\frac{b}{2}\right)^2 = \left(-\frac{7}{2} \div 2\right)^2 = \left(-\frac{7}{2} \cdot \frac{1}{2}\right)^2 = \left(-\frac{7}{4}\right)^2 = \frac{49}{16}$$

Now we need to create a perfect square binomial. We need to find the missing number, so we need to evaluate $\left(\frac{b}{2}\right)^2$.

$$x^2 - \frac{7}{2}x + \frac{49}{16} = -\frac{3}{2} + \frac{49}{16}$$

Once we evaluate b over 2 squared, we add the result to both sides of the equation.

$$\left(x - \frac{7}{4}\right)^2 = \frac{25}{16}$$

Now, on the left side of the equal sign is the perfect square in factored form. When b over 2 is evaluated, the result is -7/4. That is the number that goes in the binomial on the left.

$$\sqrt{\left(x - \frac{7}{4}\right)^2} = \pm\sqrt{\frac{25}{16}}$$

Now take the square root of both sides of the equation. Do not forget to use ± on the right side.

$$x - \frac{7}{4} = \pm\frac{5}{4}$$

Simplify the radical if possible.

$$x = \frac{7}{4} \pm \frac{5}{4}$$

$$x = \frac{7}{4} + \frac{5}{4} \quad \text{and} \quad x = \frac{7}{4} - \frac{5}{4}$$

$$x = \frac{12}{4} \qquad x = \frac{2}{4}$$

$$x = 3 \quad \text{and} \quad x = \frac{1}{2}$$

Now add 7/4 to both sides. And simplify the solutions if possible.

In this example, once everything is divided by 2, the leading coefficient is one. This means the "completing the square" method can now be used. This is a complicated problem because the b term is a fraction. Which means all those rules regarding fractions will be coming into play. Be mindful of such problems in the future.

Being able to complete the square is VITAL as it will be used many times in the future!!!

For following problems, you are to find the number needed to create a perfect square binomial.

A) $x^2-6x+\underline{9}$

$(\frac{b}{2})^2=(\frac{-6}{2})^2=(-3)^2=9$

B) $x^2-3x+\underline{9/4}$

$(\frac{b}{2})^2=(-\frac{3}{2})^2=\frac{9}{4}$

C) $x^2-7x+\underline{\frac{49}{4}}$

$(\frac{b}{2})^2=(-\frac{7}{2})^2=\frac{49}{4}$

D) $x^2+\frac{1}{2}x+\underline{\frac{1}{16}}$

$(\frac{b}{2})^2=(\frac{1}{2}\div 2)^2=(\frac{1}{2}\times\frac{1}{2})^2=(\frac{1}{4})^2=\frac{1}{16}$

E) $x^2-4x+\underline{4}$

$(\frac{b}{2})^2=(-\frac{4}{2})^2=(-2)^2=4$

F) $x^2-\frac{2}{3}x+\underline{\frac{1}{9}}$

$(\frac{b}{2})^2=(-\frac{2}{3}\div 2)^2=(-\frac{x}{3}\cdot\frac{1}{x})^2=$

$(-\frac{1}{3})^2=\frac{1}{9}$

Find all solutions to the following quadratic equations by completing the square.

A) $(x-3)^2=12$

$\sqrt{(x-3)^2}=\sqrt{12}$

$x-3=\pm2\sqrt{3}$

$x=3\pm2\sqrt{3}$

B) $(x+5)^2=17$

$\sqrt{(x+5)^2}=\sqrt{17}$

$x+5=\pm\sqrt{17}$

$x=-5\pm\sqrt{17}$

C) $x^2-4x-6=0$

$(-\frac{4}{2})^2=(-2)^2=4$ $x^2-4x+4=6+4$

$\sqrt{(x-2)^2}=\sqrt{10}$

D) $x^2+5x-3=0$

$x^2+5x+\frac{25}{4}=3+\frac{25}{4}$

$(\frac{5}{2})^2=\frac{25}{4}$

$(x+\frac{5}{2})^2=\frac{12}{4}+\frac{25}{4}$

$\sqrt{2x+5/2^2}=\sqrt{\frac{37}{4}}$ $x=\frac{-5\pm\sqrt{37}}{2}$

$x+\frac{5}{2}=\pm\sqrt{\frac{37}{2}}$

E) $x^2+6x+18=0$

$x^2+6x+9=-18+9$

$(\frac{6}{2})^2=(3)^2=9$

$\sqrt{(x+3)^2}=\sqrt{-9}$

$x+3=\pm3i$

$x=-3\pm3i$

F) $\frac{1}{2}x^2+3x-1=0$

$x^2+6x+9=2+9$

$(\frac{6}{2})^2=(3)^2=9$

$\sqrt{(x+3)^2}=\sqrt{11}$

$x+3=\pm\sqrt{11}$

$x=-3\pm\sqrt{11}$

G) $3x^2+12x-6=0$

$x^2+4x-2=0$

$x^2+4x+4=2+4$

$(\frac{4}{2})^2=(2)^2=4$

$\sqrt{(x+2)^2}=\sqrt{6}$

$x+2=\pm\sqrt{6}$

$x=-2\pm\sqrt{6}$

H) $x^2-6x-7=0$

$x^2-6x+9=7+9$

$(-\frac{6}{2})^2=(-3)^2=9$

$\sqrt{(x-3)^2}=\sqrt{16}$

$x-3=\pm4$

$x=3\pm4$

$x=\{-1,7\}$

I) $9x^2+12x-12=0$

$x^2+\frac{4}{3}x+\frac{4}{9}=\frac{4}{3}+\frac{4}{9}$

$(\frac{4}{3}\cdot\frac{1}{2})^2=(\frac{2}{3})^2=\frac{4}{9}$

$\sqrt{(x+\frac{2}{3})^2}=\sqrt{\frac{16}{9}}$

$x+\frac{2}{3}=\pm\frac{4}{3}$

$x=\{-2,\frac{2}{3}\}$ $x=-\frac{2}{3}\pm\frac{4}{3}$

J) $16x^2+8x-63=0$

$x^2+\frac{1}{2}x+\frac{1}{16}=\frac{63}{16}+\frac{1}{16}$

$(\frac{1}{2}\div2)^2=(\frac{1}{2}\cdot\frac{1}{2})^2=(\frac{1}{4})^2=\frac{1}{16}$

$\sqrt{(x+1/4)^2}=\sqrt{\frac{64}{16}}$

$x+\frac{1}{4}=\pm\frac{8}{4}$

$x=-\frac{1}{4}\pm\frac{8}{4}$

$x=\{-\frac{9}{4},\frac{7}{4}\}$

K) $9x^2+6x-15=0$

$x^2+\frac{2}{3}x+\frac{1}{9}=\frac{5}{3}+\frac{1}{9}$

$(\frac{2}{3}\cdot\frac{1}{2})^2=(\frac{1}{3})^2=\frac{1}{9}$

$\sqrt{(x+\frac{1}{3})^2}=\sqrt{\frac{16}{9}}$

$x+\frac{1}{3}=\pm\frac{4}{3}$

$x=\{-\frac{5}{3},1\}$

L) $\frac{1}{3}x^2-12x+1=0$

$x^2-36x+324=-3+324$

$(-\frac{36}{2})^2=(-15)^2=324$

$\sqrt{(x-18)^2}=\sqrt{321}$

$x-18=\pm\sqrt{321}$

$x=18\pm\sqrt{321}$

M) $\frac{1}{2}x^2 + x + 6 = 0$

$x^2 + 2x + 1 = -12 + 1$

$(\frac{2}{2})^2 = (1)^2 = 1$

$\sqrt{(x+1)^2} = \sqrt{-11}$

$x + 1 = \pm\sqrt{11}\, i$

$x = -1 \pm \sqrt{11}\, i$

N) $x^2 - 6x + 13 = 0$

$x^2 - 6x + 9 = 13 + 9$

$(-\frac{6}{2})^2 = (-3)^2 = 9$

$\sqrt{(x-3)^2} = \sqrt{-4}$

$x - 3 = \pm 2i$

$x = 3 \pm 2i$

O) $x^2 + 20x + 200 = 0$

$x^2 + 20x + 100 = -200 + 100$

$(\frac{20}{2})^2 = (10)^2 = 100$

$\sqrt{(x+10)^2} = \sqrt{-100}$

$x + 10 = \pm 10i$

$x = -10 \pm 10i$

Do not forget to check for extraneous roots for the following problems.

P) $0.01x^2 + .12x - .06 = 0$

$x^2 + 12x - 6 = 0$

$x^2 + 12x + 36 = 6 + 36$

$(\frac{12}{2})^2 = (6)^2 = 36$

$\sqrt{(x+6)^2} = \sqrt{42}$

$x + 6 = \pm\sqrt{42}$

$x = -6 \pm \sqrt{42}$

Q) $\frac{a^2}{4} + \frac{a}{2} - \frac{1}{2} = 0$

$a^2 + 2a - 2 = 0$

$a^2 + 2a + 1 = 2 + 1$

$(\frac{2}{2})^2 = (1)^2 = 1$

$\sqrt{(a+1)^2} = \sqrt{3}$

$a + 1 = \pm\sqrt{3}$

$a = 1 \pm \sqrt{3}$

R) $\frac{3}{x+1} + \frac{1}{x-1} = 1$

$3(x-1) + (x+1) = x^2 - 1$

$3x - 3 + x + 1 = x^2 - 1$

$4x - 2 = x^2 - 1$

$x^2 + 4x + 1 = 0$

$x^2 - 4x + 4 = -1 + 4$

$(-\frac{4}{20})^2 = (-2)^2 = 4$

$\sqrt{(x-2)^2} = \sqrt{3}$

$x - 2 = \pm\sqrt{3}$

$x = 2 \pm \sqrt{3}$

S) $\frac{5}{x+3} - \frac{2}{x+2} = 1$

$5(x+2) - 2(x+3) = x^2 + 5x + 6$

$5x + 10 - 2x - 6 = x^2 + 5x + 6$

$3x + 4 = x^2 + 5x + 6$

$x^2 + 2x + 2 = 0$

$x^2 + 2x + 1 = -2 + 1$

$(\frac{2}{2})^2 = (1)^2 = 1$

$\sqrt{(x+1)^2} = \sqrt{-1}$

$x + 1 = \pm i$

$x = 1 \pm i$

T) $\sqrt{x+2} - \frac{6}{\sqrt{x+2}} = 1$

$\sqrt{x+2}(\sqrt{x+2}) - (\sqrt{x+2})(\frac{6}{\sqrt{x+2}}) = 1(\sqrt{x+2})$

$x + 2 - 6 = \sqrt{x+2}$

$(x-4)^2 = (\sqrt{x+2})^2$

$x^2 - 8x + 16 = x + 2$

$-x \quad -2 \quad -x \quad -2$

$x^2 - 9x + 14 = 0$

$x^2 - 9x + \frac{81}{4} = \frac{81}{4} - 14$

$(\frac{b}{2})^2 = (-\frac{9}{2})^2 = \frac{81}{4}$ $\quad \frac{81}{4} - \frac{56}{4}$

$\sqrt{(x - \frac{9}{2})^2} = \sqrt{25}$

$x - \frac{9}{2} = \pm\frac{5}{2}$

$x = \frac{9}{2} \pm \frac{5}{2}$

$\boxed{x = 7}$ $\quad x \neq 2$

U) $2x = \sqrt{x+5}$

$(2x)^2 = (\sqrt{x+5})^2$

$4x^2 = x + 5$

$4x^2 - x = 5$

$x^2 - \frac{1}{4}x + \frac{1}{64} = \frac{5}{4} + \frac{1}{64}$

$(-\frac{1}{4} \cdot \frac{1}{2})^2 = (\frac{1}{8})^2 = \frac{1}{64}$

$\sqrt{(x - \frac{1}{8})^2} = \sqrt{\frac{81}{64}}$

$x - \frac{1}{8} = \pm\frac{9}{8}$

$x = \frac{1}{8} \pm \frac{9}{8}$

$x \neq -1$ $\quad \boxed{x = 5/4}$

The Quadratic Formula

The solutions to any quadratic equation in the form $ax^2+bx+c=0$, **where** $a \neq 0$, ***can be found by using the quadratic formula.***

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

The quadratic formula can be derived when solving the general quadratic equation, $ax^2+bx+c=0$, **by completing the square.**

Beginning with the general quadratic equation $ax^2+bx+c=0$, **derive the quadratic formula.**

$$\frac{ax^2}{a}+\frac{bx}{a}+\frac{c}{a}=0$$

$$x^2+\frac{b}{a}x=-\frac{c}{a}$$

$$\left(\frac{b}{a}\div 2\right)^2=\left(\frac{b}{a}\cdot\frac{1}{2}\right)^2=\left(\frac{b}{2a}\right)^2=\frac{b^2}{4a^2}$$

$$x^2+\frac{b}{a}x+\frac{b^2}{4a^2}=\frac{b^2}{4a^2}-\frac{c}{a}\left(\frac{4a}{4a}\right)$$

$$\left(x+\frac{b}{2a}\right)^2=\sqrt{\frac{b^2-4ac}{4a}}$$

$$x+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{2a}$$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

There are times when completing the square can be complicated. This happens in particular, when the leading coefficient is a number other than 1 and the b term is odd. If the b term is odd it will be a bit more complicated to evaluate $\left(\frac{b}{2}\right)^2$. ***In these cases, it is easier to just use the quadratic formula. Since it has been proven that the quadratic formula is actually derived by completing the square, the solutions will be identical.***

Solve each of the following quadratic equations by using the quadratic formula.

A) $2x^2+3x-4=0$

$x=\frac{-3\pm\sqrt{(3)^2-4(2)(-4)}}{2(2)}$

$x=\frac{-3\pm\sqrt{9+32}}{4}$

$x=\frac{-3\pm\sqrt{41}}{4}$

B) $6x^2+7x-20=0$

$x=\frac{-7\pm\sqrt{(7)^2-4(6)(-20)}}{2(16)}$

$x=\frac{-7\pm\sqrt{529}}{12}$

$x=\frac{-7\pm 23}{12}$

C) $9x^2+12x+4=0$

$x=\frac{-12\pm\sqrt{(12)^2-4(9)(4)}}{2(9)}$

$x=\frac{-12\pm\sqrt{0}}{18}$

$x=-\frac{12}{18}$

$x=-\frac{2}{3}$

D) $x^2-4x-7=0$

$x=\frac{4\pm\sqrt{(-4)^2-4(1)(7)}}{2}$

$x=4\pm\sqrt{44}$

$x=4\pm 2\sqrt{11}$

$x=2\pm\sqrt{11}$

E) $4x^2-2x-3=0$

$x=\frac{2\pm\sqrt{(-2)^2-4(4)(-3)}}{2(4)}$

$x=\frac{2\pm\sqrt{52}}{8}$

$x=\frac{2\pm 2\sqrt{13}}{8}$

$x=\frac{1\pm\sqrt{13}}{4}$

F) $12a^2+5a-2=0$

$x=\frac{-5\pm\sqrt{(5)^2-4(12)(-2)}}{2(12)}$

$x=\frac{-5\pm\sqrt{121}}{24}$

$x=\frac{-5\pm 11}{24}$

$x=-\frac{16}{24}, \frac{6}{24}$

$x=\left\{-\frac{2}{3}, \frac{1}{4}\right\}$

G) $15t^2+2t-1=0$

$x=\frac{-2\pm\sqrt{(2)^2-4(15)(-1)}}{2(15)}$

$x=\frac{-2\pm\sqrt{64}}{30}$

$x=\frac{-2\pm 8}{30}$

$x=-\frac{10}{30}, \frac{6}{30}$

$x=\left\{-\frac{1}{3}, \frac{1}{5}\right\}$

H) $x^2-4x+29=0$

$x=\frac{4\pm\sqrt{(-4)^2-4(1)(29)}}{2}$

$x=\frac{4\pm\sqrt{100}}{2}$

$x=\frac{4\pm 10i}{2}$

$x=2\pm 5i$

I) $a^2+2a+2=0$

$x=\frac{-2\pm\sqrt{(2)^2-4(1)(2)}}{2}$

$x=\frac{-2\pm\sqrt{-4}}{2}$

$x=\frac{-2\pm 2i}{2}$

$x=-1\pm i$

J) $(2x+1)(2x-1)=4x$

$4x^2-1=4x$

$4x^2-4x-1=0$

$x=\frac{4\pm\sqrt{(-4)^2-4(4)(-1)}}{2(8)}$

$x=\frac{4\pm\sqrt{32}}{8}$

$x=\frac{4\pm 4\sqrt{2}}{8}$

$x=\frac{1\pm\sqrt{2}}{2}$

K) $\frac{2x^2+3}{5}=2x$

$2x^2+3=10x$

$2x^2-10x+3=0$

$x=\frac{10\pm\sqrt{(-10)^2-4(2)(3)}}{2(2)}$

$x=\frac{10\pm\sqrt{76}}{4}$

$x=\frac{10\pm 2\sqrt{19}}{4}$

$x=\frac{5\pm\sqrt{19}}{2}$

L) $\frac{3x^2-1}{4}=x$

$3x^2-1=4x$

$3x^2-4x-1=0$

$x=\frac{4\pm\sqrt{(-4)^2-4(3)(1)}}{2(3)}$

$x=\frac{4\pm\sqrt{28}}{6}$

$x=\frac{4\pm 2\sqrt{7}}{6}$

$x=\frac{2\pm\sqrt{7}}{38}$

M) $2x(x+3)=5$

$$2x^2+6x=5$$
$$2x^2+6x-5=0$$
$$x=\frac{-6\pm\sqrt{(6)^2-4(2)(-5)}}{2(2)}$$
$$x=\frac{-6\pm\sqrt{76}}{4}$$
$$x=\frac{-6\pm2\sqrt{14}}{4}$$
$$x=\frac{-3\pm\sqrt{14}}{2}$$

N) $x^2-x-\frac{15}{4}=0$

$$4x^2-4x-15=0$$
$$x=\frac{4\pm\sqrt{(-4)^2-4(4)(-15)}}{2(4)}$$
$$x=\frac{4\pm\sqrt{256}}{8}$$
$$x=\frac{4\pm16}{8}$$
$$x=\frac{20}{8},\ \frac{-12}{8}$$
$$x=\left\{-\frac{3}{2},\ \frac{5}{2}\right\}$$

O) $x^2+2\sqrt{3}x-3=0$

$$x=\frac{-2\sqrt{3}\pm\sqrt{(2\sqrt{3})^2-4(1)(-3)}}{2(1)}$$
$$x=-2\sqrt{3}\pm\sqrt{24}$$
$$x=\frac{-2\sqrt{3}\pm2\sqrt{6}}{2}$$
$$x=-\sqrt{3}+\sqrt{6}$$

P) $x^2-2\sqrt{2}x+1=0$

$$x=\frac{2\sqrt{2}\pm\sqrt{(-2\sqrt{2})^2-4(1)(1)}}{2(1)}$$
$$x=\frac{2\sqrt{2}\pm\sqrt{4}}{2}$$
$$x=\frac{2\sqrt{2}\pm2}{2}$$
$$x\sqrt{2}\pm1$$

Q) $\sqrt{x+2}-\frac{6}{\sqrt{x+2}}=1$

$$\sqrt{x+2}(\sqrt{x+2})-(\cancel{\sqrt{x+2}})\left(\frac{6}{\cancel{\sqrt{x+2}}}\right)=\sqrt{(x+2)}$$
$$x+2-6=\sqrt{x+2}$$
$$(x-4)^2=(\sqrt{x+2})$$
$$x^2-8x+16=x+2$$
$$x^2-9x+14=0$$
$$x=\frac{9\pm\sqrt{81-4(1)(4)}}{2}$$
$$x=\frac{9\pm\sqrt{25}}{2}$$
$$x=\frac{9\pm5}{2}\quad\boxed{x=7}$$

$x\neq2$

R) $\frac{3}{x+4}-\frac{2}{2x+1}=1$

$$3(2x+1)-2(x+4)=(2x+1)(x+4)$$
$$6x+3-2x-8=2x^2+9x+4$$
$$4x-5=2x^2+9x+4$$
$$2x^2+5x+9=0$$
$$x=\frac{-5\pm\sqrt{(5)^2-4(2)(9)}}{2(2)}$$
$$x=\frac{-5\pm\sqrt{-47}}{4}$$
$$x=\frac{-5}{4}\pm\frac{\sqrt{47}}{4}i$$

S) $\sqrt{x+2}+\frac{1}{\sqrt{x+2}}=2$

$$\sqrt{x+2}\left[\sqrt{x+2}+\frac{1}{\sqrt{x+2}}\right]=2\left[\sqrt{x+2}\right]$$
$$x+2+1=2\sqrt{x+2}$$
$$(x+3)^2=(2\sqrt{x+2})^2$$
$$x^2+6x+9=4(x+2)$$
$$x^2+6x+9=4x+8$$
$$x^2+2x+1=0$$
$$x=\frac{-2\pm\sqrt{(2)^2-4(1)(1)}}{2(1)}$$
$$x=\frac{-2\pm\sqrt{0}}{2}$$
$$x=-1$$

T) $3x^2-\sqrt{5}x+4=0$

$$x=\frac{\sqrt{5}\pm\sqrt{(-\sqrt{5})^2-4(3)(4)}}{2(3)}$$
$$x=\frac{\sqrt{5}\pm\sqrt{-43}}{6}$$
$$x=\frac{\sqrt{5}\pm\sqrt{43}i}{6}$$
$$x=\frac{\sqrt{5}}{6}\pm\frac{\sqrt{43}}{6}i$$

U) $\frac{m^2}{4}+\frac{m}{2}-\frac{1}{2}=0$

$$4\left(\frac{m^2}{4}+\frac{m}{2}-\frac{1}{2}\right)=0(4)$$
$$m^2+2m-2=0$$
$$m=\frac{-2\pm\sqrt{(2)^2-4(1)(-2)}}{2}$$
$$m=\frac{-2\pm\sqrt{4+8}}{2}$$
$$m=\frac{-2\pm\sqrt{12}}{2}$$
$$m=\frac{-2\pm2\sqrt{3}}{2}$$
$$m=1\pm\sqrt{3}$$

The Discriminant

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Within the quadratic formula lies the discriminant. The discriminant, $b^2 - 4ac$, tells you about your solutions.

If $b^2 - 4ac > 0$ you will have two real solutions
If $b^2 - 4ac = 0$ you will have one real solution
If $b^2 - 4ac < 0$ you will have no real solutions

Think about what this means in terms of the graph of the function. The quadratic formula allows you to find roots of a quadratic equation. If the quadratic is given to you as a function, then you are finding the zeros of the function. The only difference is an equation verses a function. In other words, one of them has $f_{(x)}$ in front of it. In either case, whether you are being asked to find the roots of the equation, or the zeros of the function, you are solving for x. So, finding the roots of an equation involve the same procedures as finding the zeros of a function. As we have discussed in class, the zeros of a function are the x intercepts of the graph of that function.

Therefore, as an extension, you can conclude the following.

If $b^2 - 4ac > 0$ you will have two x intercepts
If $b^2 - 4ac = 0$ you will have one x intercept
If $b^2 - 4ac < 0$ you will have no x intercepts

If the graph of a quadratic function is a parabola, how is it possible to have only one x intercept?

If the vertex of the parabola is on the x-axis there will be only one x-intercept.

How can you have a parabola with no x intercepts?

If the vertex of a parabola that opens up is above the x-axis, or the vertex of a parabola that open down is below the x-axis, the graph of the function will not cross the x-axis. Thus, no x-intercepts exist.

Determine the nature of the roots of each of the following quadratic equations without solving. Tell whether you have two or one real solutions, or no real solutions.

A) $2x^2-4x-2=0$

b^2-4ac

$(-4)^2-4(2)(-2)$

$16+16$

32

2 real solutions

B) $-2\left(x^2-5\right)=x+12$

$-2x^2+10=x+12$

$-2x^2-x-2=0$

b^2-4ac

$(-1)^2-4(-2)(-2)$

$1-16$

-15 no real solutions

C) $-x^2+3x+7=0$

$(3)^2-4(-1)(4)$

$9+28$

39

2 real solutions

D) $3x^2-\sqrt{2}x+4=0$

$(-\sqrt{2})^2-4(3)(4)$

$2-48$

-46

no real solutions

E) $6x^2-\sqrt{6}x+2=0$

$(-\sqrt{6})^2-4(6)(2)$

$6-48$

-42

no real solutions

F) $\frac{1}{2}x^2+\frac{2}{3}x+\frac{4}{5}=0$

$(\frac{2}{3})^2-4(\frac{1}{2})(\frac{4}{5})$

$\frac{4}{9}-\frac{8}{5}$

$\frac{20}{46}-\frac{72}{45}$ no real solutions

$-\frac{52}{45}$

G) $0.4x^2-0.5x-0.3=0$

$4x^2-5x-3=0$

$(-5)^2-4(4)(-3)$

$25+48$

73

2 real solutions

H) $\left(3+\sqrt{2}\right)x^2-\left(2-\sqrt{2}\right)x+3=0$

$[-(2-\sqrt{2})]^2-4(3+\sqrt{2})(3)$

$4-4\sqrt{2}+2-12(3+\sqrt{2})$

$6-4\sqrt{2}-36-12\sqrt{2}$

$-30-16\sqrt{2}$

no real solutions

I) $-\frac{1}{3}x^2+4x-\frac{5}{8}=0$

$(4)^2-4(-\frac{1}{3})(-\frac{5}{8})$

$16\cancel{(4)^2}-(-\frac{1}{3})(-\frac{5}{2})$

$16-\frac{5}{6}$

2 real solutions

J) $3x^2+5x+4=0$

$(5)^2-4(3)(4)$

$25-48$

no real solutions

K) $-x^2+3x-\frac{1}{\ }=0$

$(3)^2-4(-1)(^2\frac{1}{2})$

$9-2$

7

2 real solutions.

L) $x^2-8x+16=0$

$(8)^2-4(1)(16)$

$64-64$

0

1 real solution.

M) $4x^2-12x+9=0$

$(-12)^2-4(4)(9)$

$144-144$

0

1 real solution

N) $2x^2-6x-5=0$

$(-6)^2-4(2)(-5)$

$36+40$

76

2 real solutions

O) $\frac{1}{4}x^2-\frac{1}{3}x+\frac{1}{9}=0$

$(-1/3)^2-4(\frac{1}{4})(\frac{1}{9})$

$\frac{1}{9}-\frac{1}{9}$

0

1 real solution

What do you notice about the quadratic equations that have one real solution?

They are perfect square trinomials.

Equations in Quadratic Form

There are many examples of equations that can be rewritten into the form of a quadratic equation. The following are examples of such. A process of substitution will be used to create a quadratic equation.

For example:

$x+6\sqrt{x}+5=0$ ***This equation can be written in quadratic form because the problem can be seen as*** $\left(\sqrt{x}\right)^2+6\sqrt{x}+5=0$

$let \quad A=\sqrt{x}$ ***The variable*** A ***will be substituted for*** $\sqrt{x}$***. Bear in mind that you can usually find what you need to substitute for in the middle term of the equation.***

$A^2+6A+5=0$ ***This problem is now recognizable in quadratic form. One of the previous methods of solving quadratic equations may now be used to solve the problem.***

It should be understood that this really wasn't necessary. This particular problem could have been solved as a radical equation by isolating $\sqrt{x}$***, and squaring both sides. However, with different variations of the problem such as*** $\frac{1}{x}+\frac{6}{\sqrt{x}}+5=0$***, it may not be so obvious what to do. This method makes solving these problems easier. It puts them in a form that is not only immediately recognizable, but will save time. Focus on the middle term to find out what to substitute. Do not forget to check your answers for extraneous roots by substituting into the original problem. All of the skills introduced here will show up at some time in your future coursework. The key is to recognize them when you encounter them. Look beyond the problem, and into the interior of the equation at hand. Look for patterns or something recognizable, much in the same way complex polynomials are factored. This type of problem will show up with trigonometric equations as well; such as the problem*** $16\sin^4 x-16\sin^2 x+3=0$***. When substituting, do not pick variables that are in the original problem, this could lead to confusion.***

Solve each of the following equations.

A) $(x-2)^2+4(x-2)-21=0$

let $a=x-2$

$a^2-4a-21=0$

$a^2-4a+4=21+4$

$(\frac{b}{2})^2=(-\frac{4}{2})^2=(-2)^2=4$

$\sqrt{(a-2)^2}=\sqrt{25}$

$a-2=\pm5$

$a=2\pm5$

$x-2=-3$ $\quad$ $x-2=7$ $\quad$ $x=\{-1,9\}$

$x=-1$ $\quad$ $x=9$

B) $\left(\frac{1}{2x}\right)^2-\frac{7}{2x}+12=0$

let $a=\frac{1}{2x}$ $\quad$ $a^2-7a+12=0$

$a=\frac{7\pm\sqrt{(-7)^2-4(1)(12)}}{2}$

$a=\frac{7\pm\sqrt{1}}{2}$ $\quad$ $x=\{\frac{1}{8},\frac{1}{6}\}$

$a=\frac{7\pm1}{2}$

$\frac{1}{2x}=4$ $\quad$ $\frac{1}{2x}=3$

$2x=1/4$ $\quad$ $2x=\frac{1}{3}$

$x=1/8$ $\quad$ $x=1/6$

C) $y^4-12y^2+36=0$

let $a=y^2$ $\quad$ $a^2-12a+36=0$

$\sqrt{(a-6)^2}=\sqrt{0}$

$a-6=0$

$a=6$

$y^2=6$

$y=\pm\sqrt{6}$

D) $(1-3a)^2-13(1-3a)+36=0$

let $a=1-3a$ $\quad$ $a^2-13a+36=0$

$(a-4)(a-9)=0$

$a=4$ $\quad$ $a=9$

$1-3a=4$ $\quad$ $1-3a=9$

$-3a=3$ $\quad$ $-3a=8$

$a=-1$ $\quad$ $a=-8/3$

$a=\{-8/3,-1\}$

E) $(3x+2)^2+4(3x+2)-21=0$

let $a=3x+2$ $\quad$ $a^2+4a-21=0$

$(a+7)(a-3)=0$

$a=-7$ $\quad$ $a=3$

$3x+2=-7$ $\quad$ $3x+2=3$

$3x=-9$ $\quad$ $3x=1$

$x=-3$ $\quad$ $x=1/3$

$x=\{-3,1/3\}$

F) $x^{-1}-5x^{-1/2}+6=0$

let $a=x^{-1/2}$ $\quad$ $a^2-5a+6=0$

$(a-2)(a-3)=0$

$a=2$ $\quad$ $x^{-1/2}=3$

$x^{-1/2}=2$ $\quad$ $\frac{1}{\sqrt{x}}=3$

$\frac{1}{\sqrt{x}}=2$ $\quad$ $(\sqrt{x})^2=(\frac{1}{3})^2$

$\frac{1}{\sqrt{x}}=2$ $\quad$ $x=1/9$

$(\sqrt{x})^2=(\frac{1}{2})^2$ $\quad$ $x=\{\frac{1}{4},\frac{1}{9}\}$

$x=1/4$

G) $x^{-4}-13x^{-2}+36=0$

let $a=x^{-2}$ $\quad$ $a^2-13a+36=0$

$(a-4)(a-9)=0$

$a=4$ $\quad$ $a=9$

$x^{-2}=4$ $\quad$ $\sqrt{\frac{1}{x^2}}=\sqrt{9}$

$\sqrt{\frac{1}{x^2}}=\sqrt{4}$ $\quad$ $\frac{1}{x}=\pm3$

$\frac{1}{x}=\pm2$ $\quad$ $x=\pm\frac{1}{3}$

$x=\pm\frac{1}{2}$ $\quad$ $x=\{\pm\frac{1}{2},\pm\frac{1}{3}\}$

H) $(a^2-3)^2-4(a^2-3)-12=0$

let $b=a^2-3$ $\quad$ $b^2-4b-12=0$

$(b-6)(b+2)=0$

$b=6$ $\quad$ $b=-2$

$a^2-3=6$ $\quad$ $a^2-3=-2$

$\sqrt{a^2}=\sqrt{9}$ $\quad$ $\sqrt{a^2}=\sqrt{1}$

$a=\pm3$ $\quad$ $a=\pm1$

$a=\{\pm1,\pm3\}$

The Standard Form of a Parabola

In this class we will be graphing quadratic functions using the standard form of a parabola.

$$y = a(x-h)^2 + k$$

This is the standard form of a parabola, where a, h and k represent real number constants.

In Algebra I, students are taught to graph a parabola in the form $y = ax^2 + bx + c$, ***by first finding the vertex. This meant the student had to first evaluate*** $-\frac{b}{2a}$ ***to find the x value of the vertex. The student would then proceed to evaluate*** $f_{\left(-\frac{b}{2a}\right)}$ ***to find the y value of the vertex. In other words, first find the x value of the vertex, then plug it back into the problem to find the y value. From here the student would set up a table to find any additional coordinates. The results would then be graphed. From this point forward, you will be given the function in the form*** $y = ax^2 + bx + c$, ***and rewrite it in the standard form,*** $y = a(x-h)^2 + k$. ***Once in this form, it is easier to identify the vertex, x-intercepts and any other information you need to graph the function.***

Once the function is in standard form, the vertex is given by (h,k). ***The*** a ***value determines if the function opens up or down, and gives some indication as to how wide or narrow the graph will be. The*** h ***value dictates how the function shifts horizontally. Remember, in the standard form of a parabola you see*** $-h$, ***but in the vertex you see*** h. ***This means the x value of the vertex is the opposite of what you see when it is standard form. Remember P.L.N.R., Positive Left Negative Right, and it will help you with the shift. The*** k ***value of the function is the y value of the vertex, thereby making the graph of the function shift vertically.***

As stated earlier, the process of completing the square is vital in mathematics. This is the method used to put the equation of a parabola in standard form. As you complete the square of the function $y = ax^2 + bx + c$, ***the dependant variable,*** y, ***<u>must</u> be kept in the problem. The*** y ***variable will keep track of the*** a ***term for you since the completing the square method cannot be used if the leading coefficient is any number other that one. The*** y ***variable holds on to the number for you, then gives it back at the end. The following example shows this happening.***

Example

$y=-2x^2+8x-14$ ***Remember to keep y in the problem.***

$\frac{y}{-2}=\frac{-2x^2}{-2}+\frac{8x}{-2}-\frac{14}{-2}$ ***To put into standard form, we will need to complete the square, which means the coefficient for the*** x^2 ***term must be one, so divide the whole equation by -2.***

$\frac{y}{-2}=x^2-4x+7$ ***Division by -2 yields this equation***

$\frac{y}{-2}-7 \quad =x^2-4x$ ***Subtract 7 to both sides of the equation***

$\left(\frac{b}{2}\right)^2=\left(\frac{-4}{2}\right)^2=(-2)^2=4$ ***To find the missing term evaluate*** $\left(\frac{b}{2}\right)^2$ ***to create a perfect square binomial on the right.***

$\frac{y}{-2}-7+4=x^2-4x+4$ ***Add the result, 4, to both sides of the equation***

$\frac{y}{-2}-3=(x-2)^2$ ***Simplify***

$\frac{y}{-2}=(x-2)^2+3$ ***Now we need to begin isolating the y, so add 3 to both sides of the equation***

$-2\left(\frac{y}{-2}\right)=-2\left[(x-2)^2+3\right]$ ***To isolate the y, multiply the entire equation by -2.***

$y=-2(x-2)^2-6$ ***Notice the y was just holding on to -2 throughout the entire process. The equation for this parabola is now in standard form.***

As previously stated, this skill is vital. In the future, it will be necessary to complete the square twice within the same problem. This is the procedure used when putting the equation of a circle, ellipse, or hyperbola into standard form. In each case, you will be required to complete the square for two different variables in order to write the equations in standard form.

Write each of the following quadratic functions in standard form.

A) $y=x^2-4x+3$

$y-3+4=x^2-4x+4$

$(\frac{b}{2})^3=(-\frac{4}{2})^2=(-2)^2=4$

$y+1=(x-2)^2$

$y=(x-2)^2-1$

B) $y=x^2+6x-2$

$y+2+9=x^2+6x+9$

$(\frac{b}{2})^2=(\frac{b}{2})^2=(3)^2=9$

$y+11=(x+3)^2$

$y=(x+3)^2-11$

C) $y=-2x^2+8x-6$

$\frac{y}{2}+\frac{6}{2}=\frac{-2x^2}{-2}+\frac{8x}{-2}$

$-\frac{4}{2}-3+4=x^2-4x+4$

$(\frac{b}{2})^2=(\frac{4}{2})^2=(-2)^2=4$

$-\frac{y}{2}+1\ (x-2)^2$

$\frac{y}{-2}\ 2(x-2)^2-1$

D) $y=x^2+5x-4$

$y+4+\frac{25}{4}=x^2+5x+\frac{25}{4}$

$(\frac{b}{2})^2=(\frac{5}{2})^2=\frac{25}{4}$

$y+\frac{16}{4}+\frac{25}{4}=x^2+5x+\frac{25}{4}$

$y+\frac{41}{4}=x^2+5x+\frac{25}{4}$

$y=(x+\frac{5}{2})^2=\frac{41}{4}$

E) $\frac{y}{3}=\frac{3x^2}{3}-\frac{9x}{3}+\frac{12}{3}$

$\frac{y}{3}=x^2-3x+4$

$\frac{y}{3}-4+\frac{1}{4}=x^2-3x+\frac{1}{4}$

$(\frac{b}{2})^2=(-\frac{3}{2})^2=\frac{1}{4}$

$\frac{y}{3}-\frac{16}{4}+\frac{9}{4}=x^2-3x+{}^{-1}/4$

$\frac{y}{3}-\frac{7}{4}=(x-\frac{3}{2})^2$

$y/3=(x-3/2)^2+7/4$ $\quad y=3(x-\frac{3}{2})^2+\frac{21}{4}$

F) $y=-x^2-6x-9$

$-y=x^2+6x+9$

$-y-9+9=x^2+6x+9$

$(\frac{b}{2})^2=(\frac{6}{2})^2=(3)^2=9$

$-y=(x+3)^2$

$y=-(x+3)^2$

G) $y=\frac{1}{2}x^2-5x-6$

$2y=x^2-10x-12$

$2y+12+25=x^2-10x+25$

$(\frac{b}{2})^2=(\frac{10}{2})^2=25$

$2y+37=(x-5)^2$

$2y=(x-5)^2-37$

$y=\frac{1}{2}(x-5)^2-\frac{37}{2}$

H) $\frac{y}{4}=\frac{4x^2}{4}-\frac{6x}{4}+\frac{5}{4}$

$\frac{y}{4}-\frac{5}{4}+\frac{9}{16}=x^2-\frac{3}{2}x+\frac{9}{16}$

$(\frac{b}{2})^2=(-\frac{3}{2}\cdot\frac{1}{2})^2=(-\frac{3}{4})^2=\frac{9}{16}$

$\frac{y}{4}-\frac{11}{16}=(x-\frac{3}{4})^2$

$\frac{y}{4}=(x-\frac{3}{4})^2+\frac{11}{16}$

$y=4(x-\frac{3}{4})^2+\frac{11}{4}$

I) $y=\frac{1}{3}x^2-x+2$

$3y=x^2-3x+6$

$3y-6+\frac{9}{4}=x^2-3x+\frac{9}{4}$

$(\frac{b}{2})^2=(-\frac{3}{2})^2=\frac{9}{4}$

$3y-\frac{24}{4}+\frac{9}{4}=x^2-3x+\frac{9}{4}$

$3y-\frac{15}{4}=(x-\frac{3}{2})^2$

$3y=(x-3/2)^2+\frac{15}{4}$

$y=\frac{1}{3}(x-3/2)^2+\frac{5}{4}$

J) $y=-\frac{1}{2}x^2+x-1$

$-2y=x^2-2x+2$

$-2y-2+1=x^2-2x+1$

$(\frac{b}{2})^2=(-\frac{2}{2})^2=(-1)^2=1$

$-2y-1=(x-1)^2$

$-2y=(x-2)^2+1$

$y=-1/2(x-1)^2-\frac{1}{2}$

K) $y=-x^2-5x+2$

$-y+2+\frac{25}{4}=x^2+5x+\frac{25}{4}$

$(\frac{b}{2})^2=(\frac{5}{2})^2=\frac{25}{4}$

$-y+\frac{8}{4}+\frac{25}{4}=(x+\frac{5}{2})^2$

$-y+\frac{33}{4}=(x+\frac{5}{2})^2$

$y=-(x+\frac{5}{2})^2+\frac{33}{4}$

L) $y=3x^2-2$

M) $y=-\frac{1}{3}x^2-6x+4$

$-3y=x^2+18x-12$

$-3y+12+81=x^2+18x+81$

$-3y+93=(x+9)^2$

$-3y=(x+9)^2-93$

$y=\frac{1}{3}(x-9)^2+\frac{93}{3}$

$y=-\frac{1}{3}(x-9)^2+31$

N) $y=3x^2-2x+5$

$\frac{y}{3}-\frac{5}{3}+\frac{1}{9}=x^2-\frac{2}{3}x+\frac{1}{9}$

$(-\frac{2}{3}\cdot\frac{1}{2})^2=(\frac{1}{3})^2=\frac{1}{9}$

$\frac{y}{3}-\frac{15}{9}+\frac{1}{9}=x^2-\frac{2}{3}+\frac{1}{9}$

$\frac{y}{3}-\frac{14}{9}=(x-\frac{1}{3})$

$\frac{y}{3}=(x-\frac{1}{3})^2+\frac{14}{9}$

$y=3(x-\frac{1}{3})^2+\frac{14}{4}$

O) $y=2x^2-\frac{1}{2}x+\frac{1}{3}$

$\frac{y}{2}-\frac{1}{6}+\frac{1}{64}=x^2-\frac{1}{4}x+\frac{1}{64}$

$(-\frac{1}{4}\cdot\frac{1}{2})^2=(-\frac{1}{8})^2=\frac{1}{64}$

$\frac{y}{2}-\frac{29}{192}=(x-\frac{1}{8})^2$

$\frac{y}{2}=(x-\frac{1}{8})^2+\frac{29}{192}$

$y=2(x-\frac{1}{8})^2+\frac{29}{96}$

Working with the Standard Form of a Quadratic Function

Answer each of the following questions regarding the standard form of a parabola.

$$y = a(x-h)^2 + k$$

Where a, h and k represent constants.

Which constant is responsible for a vertical shift of the function?

k

Which constant is responsible for a horizontal shift of the function?

H

Which constant will cause the function to open up or down?

a

What is the most notable difference between the graphs of the following functions?

$$y = 4(x-2)^2 + 3 \quad \textbf{and} \quad y = \frac{1}{4}(x-2)^2 + 3$$

The first function is more narrow while the second is wider.

Once the quadratic function is in $y = a(x-h)^2 + k$ form, what makes up the vertex of the parabola? (h, k)

The highest point of a parabola that opens down is the vertex.

The lowest point of a parabola that opens up is the vertex.

What is the domain of any quadratic function?

$(-\infty, \infty)$

What would a parabola that has only one x-intercept look like?

The vertex would be on the x-axis.

Referring to a parabola in $y = a(x-h)^2 + k$ form, answer the following.

If $a < 0$, the range of the function is the interval $(-\infty, k]$.

If $a > 0$, the range of the function is the interval $[k, \infty)$.

Explain how it is possible to have a parabola with no x-intercepts.

If the parabola opens up & its vertex is above the x-axis or if the parabola opens down and the vertex is below the x-axis; the graph will have no x-intercepts.

Is it possible to have a parabola with no y-intercepts? Why or why not?

No, because the domain of the function is all real numbers.

A standard parabola can have two x intercepts. Can a parabola have two y intercepts as well? Why or why not.

No it cannot. If a parabola had two y-int. it would fail the vertical line test. That means it is not a function.

Explain, in your own words, what an axis of symmetry is?

the vertical line that crosses through the vertex of the parabola. The graph of the parabola is symmetrical to its axis of symmetry this means if you were to fold the graph, along the axis of symmetry, they would match.

What are the two things you need to know to find the range of a quadratic function?

In order to find the range of a quadratic function you need to know how it opens, and the y value of the vertex. In other words, the values of a and k.

Find the vertex of each of the following. These are the same problems you wrote in standard form earlier on page 208, so it is not necessary to go through that process again. However, be sure to write the standard form of the quadratic function below each one.

A) $y = x^2 - 4x + 3$

$y = (x-2)^2 - 1$

$(2, -1)$

B) $y = x^2 + 6x - 2$

$y = (x+3)^2 - 11$

$(-3, -11)$

C) $y = -2x^2 + 8x - 6$

$y = -2(x-2)^2 + 2$

$(2, 2)$

D) $y = x^2 + 5x - 4$

$y = (x + \frac{5}{2})^2 - \frac{41}{4}$

$(-\frac{5}{2}, \frac{-41}{4})$

E) $y = 3x^2 - 9x + 12$

$y = 3(x - \frac{3}{2})^2 + \frac{21}{4}$

$(\frac{3}{2}, \frac{21}{4})$

F) $y = -x^2 - 6x - 9$

$y = -(x+3)^2$

$(-3, 0)$

G) $y = \frac{1}{2}x^2 - 5x - 6$

$y = \frac{1}{2}(x-5)^2 - \frac{37}{2}$

$(5, -\frac{37}{2})$

H) $y = 4x^2 - 6x + 5$

$y = 4(x - \frac{3}{4})^2 + \frac{11}{4}$

$(\frac{3}{4}, \frac{11}{4})$

I) $y = \frac{1}{3}x^2 - x + 2$

$y = \frac{1}{3}(x - \frac{3}{2})^2 + \frac{5}{4}$

$(\frac{3}{2}, \frac{5}{4})$

J) $y = -\frac{1}{2}x^2 + x - 1$

$y = -\frac{1}{2}(x-1)^2 - \frac{1}{2}$

$(1, -\frac{1}{2})$

K) $y = -x^2 - 5x + 2$

$y = -(x + \frac{5}{2})^2 + \frac{33}{4}$

$(-\frac{5}{2}, \frac{33}{4})$

L) $y = 3x^2 - 2$

$y = 3x^2 - 2$

$(0, -2)$

M) $y = -\frac{1}{3}x^2 - 6x + 4$

$y = -\frac{1}{3}(x+9)^2 + 31$

$(-9, 31)$

N) $y = 3x^2 - 2x + 5$

$y = 3(x - \frac{1}{3})^2 + \frac{14}{3}$

$(\frac{1}{3}, \frac{14}{5})$

O) $y = 2x^2 - \frac{1}{2}x + \frac{1}{3}$

$y = 2(x - \frac{1}{8})$

$(\frac{1}{8}, \frac{29}{96})$

Maximum and Minimum Values

In some problems you will be asked about the maximum or minimum value of a function. Whenever you are asked about the value of a function, we are always referring to the y value. In the case of a parabola, the maximum or minimum value will be the y value of the vertex. The x value of the vertex is where you get that maximum or minimum. For example, let's look at a business; where the profit margin is given by the formula $P = -(x-32)^2 + 8000$***, where x is the number of units sold, and P is the profit. According to the equation for the graph of the function, the parabola will open down, and the vertex will be located at*** $(32, 8000)$***. This means the maximum profit is $8000. The business will achieve maximum profitability if 32 units are sold. So the x value is where the max occurred, and the y value is simply what the maximum or minimum is depending on the case and function.***

Therefore, it can be concluded…

If the parabola opens down, meaning $a < 0$***, the function will have a maximum.***
If the parabola opens up, meaning $a > 0$***, the function will have a minimum.***

In either case, the maximum or minimum value of the function is the y value of the vertex.

When asked to find the maximum or minimum value of a function, or where this occurs, it is not necessary to actually write the function in standard form to find the vertex. As previously stated, working from the general form of a quadratic function, $y = ax^2 + bx + c$***, the vertex can be found. Evaluating for*** $-\frac{b}{2a}$***, will reveal the x value of the vertex. This tells you where the maximum or minimum value occurs. Keeping our previous example in mind, you could simply be asked how many units need to be sold to maximize profits. If you are being asked were the maximum or minimum occur,*** $-\frac{b}{2a}$ ***will give you the solution to the problem. If you are asked to take it a step further and find the maximum or minimum value, plug*** $-\frac{b}{2a}$***, in this case 32, back into the problem to find the y value of the vertex. Why is this important? Consider the following example.***

Find the maximum value of the function $y = -0.25x^2 + 5.62x + 2$.
You can see putting this in standard form will be relatively complicated. Even more so if the leading coefficient is a decimal value not easily converted to a fraction. Following this procedure, the maximum value of the function is found.

$$-\frac{b}{2a} = -\frac{(5.62)}{2(-0.25)} = 11.24$$

now substitute 11.24 *for* x,

$$y = -0.25(11.24)^2 + 5.62(11.24) + 2$$

$$y = 33.5844$$

The maximum value of this function is 33.5844

Answer each of the following.

Find the maximum value of the function: $f_{(x)} = -2x^2 + 6x + 12$

$y = -2x^2 + 6x + 12$

$-\frac{y}{2} = x^2 - 3x - 6$

$-\frac{y}{2} + 6 + \frac{9}{4} = x - 3x + \frac{9}{4}$

$(-\frac{3}{2})^2 = \frac{9}{4}$

$-\frac{y}{2} + \frac{24}{4} + \frac{9}{4} = x - 3x + \frac{9}{4}$

$-\frac{y}{2} + \frac{33}{4} = (x - \frac{3}{2})$

$-\frac{y}{2} = (x - \frac{3}{2})^2 - \frac{33}{4}$

$y = -2(x - \frac{3}{2})^2 + \frac{33}{2}$

$\boxed{\frac{33}{2}}$

Find the maximum value of the function: $f_{(x)} = -5x^2 + 30x - 200$

$-\frac{b}{2a} = \frac{-30}{2(-5)} = \frac{-30}{-10} = 3$

$f(3) = -5(3)^2 + 30(3) - 200$

$f(3) = -45 + 90 - 200$

$f(3) = -155$

$\boxed{-155}$

For what value of x does the function $f_{(x)} = -5x^2 + 200x + 2300$ **achieve its maximum value?**

$-\frac{b}{2a} = \frac{-200}{2(-5)} = \frac{200}{-100} = 20$

$\boxed{20}$

Find the minimum value of the function: $f_{(x)} = \frac{1}{4}x^2 - 10x + 800$

$-\frac{b}{2a} = \frac{-10}{2(\frac{1}{4})} = \frac{10}{1/2} = 10 \div \frac{1}{2} = 10 \cdot 2 = 20$

$f(20) = \frac{1}{4}(20)^2 - 10(20) + 800$

$f(20) = 100 - 200 + 800 = 700$

$\boxed{700}$

Find the minimum value of the function: $f_{(x)} = 3x^2 + 4x + 3$

$-\frac{b}{2a} = \frac{4}{2(3)} = -\frac{2}{3}$

$f(-\frac{2}{3}) = 3(-\frac{2}{3})^2 + 4(-\frac{2}{3}) + 3$

$= 3(\frac{4}{9}) + \frac{8}{3} + 3$

$= \frac{4}{3} - \frac{8}{3} + \frac{9}{3}$

$= \boxed{\frac{5}{3}}$

Finding X and Y intercepts of Parabolas

You will need to be able to find the x and y intercepts of each function in order to accurately graph it.

We have already gone over the procedures for finding the x intercepts of a quadratic function.

- ***First replace $f_{(x)}$ or y, whichever happens to be the case, with zero. This has the effect of setting the quadratic equation equal to zero. Then find all zeros of the function by factoring, completing the square, or using the quadratic formula. Since the zeros of a function are really x-intercepts, your results will be the x-intercepts of the function. It doesn't matter what form of the function you are looking at, whether it be the general form of a quadratic function, $y = ax^2 + bx + c$, or the standard form of the quadratic $y = a(x-h)^2 + k$. the procedure is the same.***

Finding the y intercept depends on what form of the quadratic function you are looking at.

- ***Given a quadratic function in $y = ax^2 + bx + c$ form, substitute zero for x, and solve for y. In this case, the result is simply $(0, c)$. Since both a and b have the variable x next to them, the only thing left is the constant c. Remember, if the constant c is not there, that means it is zero, and the y intercept is at the origin.***

- ***Given a quadratic function in $y = a(x-h)^2 + k$ form, substitute zero for x and solve for y. Be careful! Don't assume the constant k is the y-intercept. That is the y value of the vertex. You must find the y-intercept arithmetically. There is no shortcut as with a quadratic in general form.***

Find the x and y intercepts of the following functions.

A) $f_{(x)} = -x^2 + 2x + 5$

$-x^2+2x+5=0$

$x^2-2x-5=0$

$x^2-2x+1=5+1$

$(\frac{b}{2})^2=(-\frac{2}{2})^2=(-1)^2=1$

$\sqrt{(x-1)^2}=\sqrt{6}$

$x-1=\pm\sqrt{6}$

$x=1\pm\sqrt{6}$

$(1+\sqrt{6},0)(1-\sqrt{6},0)(0,5)$

B) $f_{(x)} = x^2 + x - 6$

$x^2+x-6=0$

$x=\frac{-1\pm\sqrt{(1)^2-4(1)(-6)}}{2(1)}$

$x=\frac{-1\pm\sqrt{25}}{2}$

$x=\frac{-1}{2}$

$x=-3,2$

$(-3,0)(2,0)(0,-6)$

C) $f_{(x)} = -x^2 + 10x - 24$

$-x^2+10x-24=0$

$x^2-10x+24=0$

$x^2-10x+25=-24+25$

$(\frac{b}{2})^2=(\frac{-10}{2})^2=(-5)^2=25$

$\sqrt{(x-5)^2}=\sqrt{1}$

$x-5=\pm1$

$x=5\pm1$

$(4,0)(6,0)(0,-24)$

D) $f_{(x)} = 3x^2 + 12x + 12$

$\frac{y}{3}=x^2+4x+4$

$\frac{y}{3}=(x+2)^2$

$y\sqrt{(x+2)^2}=\sqrt{0}$

$x+2=0$

$x=-2$

$(-2,0)(0,12)$

E) $f_{(x)} = 2x^2 - 3x + 4$

$2x^2-3x+4=0$

$x=\frac{3\pm\sqrt{(-3)^2-4(2)(4)}}{2(2)}$

$x=\frac{3\pm\sqrt{23}}{4}$

y-int (0,4)

no x-int.

F) $f_{(x)} = x^2 - 6x + 7$

$x^2-6x+7=0$

$x^2-6x+9=-7+9$

$(\frac{b}{2})^2=(\frac{-6}{2})^2=(-3)^2=9$

$\sqrt{(x-3)^2}=\sqrt{2}$

$x-3=\pm\sqrt{2}$

$x=3\pm\sqrt{2}$

Finding the Range and Domain

We know that the domain for any quadratic function is all real numbers, so that won't be a problem. The difficulty can arise from finding the range of the function. To find the range of a quadratic function, we need to know how it opens, and the y value of the vertex. As it was discussed earlier, the y value of the vertex is the maximum or minimum value of a parabola, depending on how it opens; so it is the absolute highest or lowest point on the graph of the function.

The following is true for a quadratic function in standard form $y = a(x-h)^2 + k$.

If $a > 0$, ***the range is*** $[k, \infty)$.

If $a < 0$, ***the range is*** $(-\infty, k]$.

State the range and domain for each of the following.

A) $y = 4x^2 - 6x + 2$

$\frac{-b}{2a} = -\left(\frac{-6}{2(4)}\right) = \frac{3}{4}$

$y = 4\left(\frac{3}{4}\right)^2 - 6\left(\frac{3}{4}\right) + 2$

$y = 4\left(\frac{9}{16}\right) - \frac{18}{4} + 2$

$y = \frac{9}{4} - \frac{18}{4} + \frac{8}{4}$

$y = \frac{17}{4} - \frac{18}{4} = -\frac{1}{4}$

Range: $(\frac{1}{4}), \infty)$

B) $y = x^2 + 4x - 5$

$y + 5 + 4 = x^2 + 4x + 4$

$y + 9 = (x+2)^2$

$y = (x+2)^2 - 9$

R: $(-9, \infty)$

D: $(-\infty, \infty)$

C) $y = x^2 - 6$

Vertex $= (0, -6)$

Range: $[-6, \infty)$

Domain: $(-\infty, \infty)$

D) $y = -\frac{1}{2}x^2 + 3x - 2$

$-2y = x^2 - 6x + 4$

$-2y - 4 + 9 = x^2 - 6x + 9$

$\left(\frac{-6}{2}\right)^2 = (-3)^2 = 9$

$-2y + 5 = (x-3)^2$

$-2y = (x-3)^2 - 5$

$y = -1/2\,(x-3)^2 - \frac{5}{2}$

R: $(\infty, \frac{5}{2}]$

D: $(-\infty, \infty)$

E) $y = -2x^2$

Vertex: $(0, 0)$

R: $(-\infty, 0]$

D: $(-\infty, \infty)$

F) $y = -3x^2 + 6x - 18$

$\frac{y}{-3} = x^2 - 2x + 6$

$\frac{y}{-3} - 6 + 1 = x^2 - 2x + 1$

$\left(-\frac{2}{2}\right)^2 = (-1)^2 = 1$

$\frac{y}{-3} - 5 = (x-1)^2$

$\frac{y}{-3} = (x-1)^2 + 5$

$y = -3(x-1)^2 - 15$

R: $(-\infty, -15]$

D: $(-\infty, \infty)$

G) $y = \frac{1}{3}x^2 + 2x - 3$

$3y = x^2 + 6x - 9$

$3y + 9 + 9 = x^2 + 6x + 9$

$\left(\frac{b}{2}\right)^2 = (3)^2 = 9$

$3y + 18 = (x+3)^2$

$3y = (x+3)^2 - 38$

$y = \frac{1}{3}(x+3)^2 - 6$

R: $[-6, \infty)$

D: $(-\infty, \infty)$

H) $y = \frac{1}{2}x^2 + \frac{1}{4}x - \frac{1}{3}$

$2y = x^2 + \frac{1}{2}x - \frac{2}{3}$

$2y + \frac{2}{3} + \frac{1}{16} = x^2 + \frac{1}{2}x + \frac{1}{16}$

$\left(\frac{1}{2} \cdot \frac{1}{2}\right)^2 = \left(\frac{1}{4}\right)^2 = \frac{1}{16}$

$2y + \frac{32}{48} + \frac{3}{48} = \left(x + \frac{1}{4}\right)^2$

$2y + \frac{35}{48} = \left(x + \frac{1}{4}\right)^2$

$2y = \left(x - \frac{1}{4}\right)^2 - \frac{35}{48}$

$y = \frac{1}{4}\left(x - \frac{1}{4}\right)^2 - \frac{35}{96}$

R: $[-\frac{35}{96}, \infty)$

D: $(-\infty, \infty)$

I) $y = \frac{1}{3}x^2 + \frac{1}{6}x - \frac{1}{9}$

$3y = x^2 + \frac{1}{2}x - \frac{1}{3}$

$3y + \frac{1}{3} + \frac{1}{16} = x^2 + \frac{1}{2}x + \frac{1}{16}$

$\left(\frac{1}{2} \cdot \frac{1}{2}\right)^2 = \left(\frac{1}{4}\right)^2 = \frac{1}{16}$

$3y + \frac{16}{48} + \frac{3}{48} = \left(x + \frac{1}{4}\right)^2$

$3y + \frac{19}{48} = \left(x + \frac{1}{4}\right)^2$

$3y = \left(x + \frac{1}{4}\right)^2 - \frac{19}{48}$

$y = \frac{1}{3}\left(x + \frac{1}{4}\right)^2 - \frac{19}{144}$

R: $[-\frac{19}{144}, \infty)$

D: $(-\infty, \infty)$

Properties of Parabolas

For each of the following, find the standard form of the function; describe how the graph opens; identify the y-intercept, vertex, x-intercepts (if any), and the range and domain.

A) $y = x^2 + 4x - 2$

$y+2+4 = x^2+4x+4$

$\left(\frac{4}{2}\right)^2 = (2)^2 = 4$

$y+6 = (x+2)^2$

$y = (x+2)^2 - 6$

x-int.

$(x+2)^2 = \sqrt{6}$

$x+2 = \pm\sqrt{6}$

$x = -2 \pm \sqrt{6}$

Standard Form: $y=(x+2)^2-6$

Opens: up

Y-intercept: $(0,-2)$

Vertex: $(-2,-6)$

X-intercepts: $(-2-\sqrt{6},0)(-2+\sqrt{6},0)$

Range: $[-6,\infty)$

Domain: $(-\infty,\infty)$

B) $y = x^2 - 3$

x-int

$x^2-3=0$

$\sqrt{x^2} = \sqrt{3}$

$x = \pm\sqrt{3}$

Standard Form: $y = x^2-3$

Opens: up

Y-intercept: $(0,-3)$

Vertex: $(0,-3)$

X-intercepts: $(\sqrt{3},0)(-\sqrt{3},0)$

Range: $[-3,\infty)$

Domain: $(-\infty,\infty)$

C) $y = -x^2 + 9$

x-int

$-x^2+9=0$

$\sqrt{x^2} = \sqrt{9}$

$x = \pm 3$

Standard Form: $y = -x^2+9$

Opens: down

Y-intercept: $(0,9)$

Vertex: $(0,9)$

X-intercepts: $(-3,0)(3,0)$

Range: $(-\infty,9)$

Domain: $(-\infty,\infty)$

D) $y=-x^2-6x+1$

$-y=x^2+6x-1$

$-y+1+9=x^2+6x+9$

$(\frac{b}{2})^2=(\frac{6}{2})^2=9$

$-y+10=(x+3)^2$

$-y=(x+3)^2-10$

$y=-(x+3)^2+10$

x-int

$\sqrt{10}=\sqrt{(x+3)^2}$

$\pm\sqrt{10}=x+3$

$x=-3\pm\sqrt{10}$

Standard Form: $y=-(x+3)^2+10$

Opens: down

Y-intercept: (0, 1)

Vertex: (-3 , 10)

X-intercepts: $(-3+\sqrt{10},0)$ $(-3-\sqrt{10},0)$

Range: $(-\infty, 10]$

Domain: $(-\infty,\infty)$

E) $y=\frac{1}{2}x^2+4x-1$

$2y=x^2+8x-2$

$2y+2+16=x^2+8x+16$

$(\frac{8}{2})^2=(4)^2=16$

$2y+18=(x+4)^2$

$2y=(x+4)^2-18$

$y=\frac{1}{2}(x+4)^2-9$

x-int

$\sqrt{18}=\sqrt{(x+4)^2}$

$\pm3\sqrt{2}=x+4$

$x=-4\pm3\sqrt{2}$

Standard Form: $y=\frac{1}{2}(x+4)^2-9$

Opens: up

Y-intercept: (0, -1)

Vertex: (-4, -9)

X-intercepts: $(-4-3\sqrt{2},0)(-4,+3\sqrt{2},0$

Range: $[-9,\infty)$

Domain: $(-\infty,\infty)$

F) $y=x^2-x+\frac{25}{4}$

$y-\frac{25}{4}+\frac{1}{4}=x^2-x+\frac{1}{4}$

$(-\frac{1}{2})^2=\frac{1}{4}$

$y-\frac{24}{4}=(x-\frac{1}{2})^2$

$y=(x-\frac{1}{2})^2+\frac{24}{4}$

$y=(x-\frac{1}{2})^2+6$

x-int

$-\frac{24}{4}=(x-\frac{1}{2})^2$

$\sqrt{-6}=\sqrt{(x-1/2)^2}$

$\pm\sqrt{6}i=x-\frac{1}{2}$

$x=\frac{1}{2}\pm\sqrt{6}i$

Standard Form: $y=(x-\frac{1}{2})^2+6$

Opens: up

Y-intercept: $(0,\frac{25}{4})$

Vertex: $(\frac{1}{2},6)$

X-intercepts: none

Range: $[6,\infty)$

Domain: $(-\infty,\infty)$

G) $y = -x^2 - 10x - 25$

$-y = x^2 + 10x + 25$

$-y - 25 + 25 = x^2 + 10x + 25$

$\left(\frac{10}{2}\right)^2 = (5)^2 = 25$

$-y = (x+5)^2$

$y = (x+5)^2$

x-int

$\sqrt{0} = \sqrt{(x+5)^2}$

$0 = x+5$

$-5 = x$

Standard Form: $y = -(x+5)^2$

Opens: down

Y-intercept: $(0,-25)$

Vertex: $(-5,0)$

X-intercepts: $(-5,0)$

Range: $(-\infty, 0]$

Domain: $(-\infty,\infty)$

)

H) $y = -x^2 - 10x - 20$

$-y = x^2 + 10x + 20$

$-y - 20 + 25 = x^2 + 10x + 25$

$\left(\frac{10}{2}\right)^2 = (5)^2 = 25$

$-y + 5 = (x+5)^2$

$-y = (x+5)^2 - 5$

$y = -(x+5)^2 + 5$

x-int

$\sqrt{5} = \sqrt{(x+5)^2}$

$\pm\sqrt{5} = x+5$

$x = -5 \pm \sqrt{5}$

Standard Form: $y = -(x+5)^2 + 5$

Opens: down

Y-intercept: $(0,-20)$

Vertex: $(-5,5)$

X-intercepts: $(-5+\sqrt{5},0)(-5\sqrt{5},0)$

Range: $(-\infty, 5]$

Domain: $(-\infty,\infty)$

I) $y = \frac{1}{3}x^2 + \frac{4}{3}x + \frac{47}{36}$

$3y = x^2 + 4x + \frac{47}{12}$

$-3y - \frac{47}{12} + 4 = 4^2 + 4x + 4$

$\left(\frac{4}{2}\right)^2 = (2)^2 = 4$

$3y = \frac{47}{12} + \frac{48}{12} = (x+2)^2$

$3y = (x+2)^2 - \frac{1}{12}$

$y = \frac{1}{3}(x+2)^2 - \frac{1}{36}$

x-int

$\sqrt{\frac{1}{12}} = \sqrt{(x+2)^2}$

$\pm\frac{1}{2\sqrt{3}} = x+2$

$\pm\frac{\sqrt{3}}{6} = x+2$

$\pm\frac{\sqrt{3}}{6} = x+2$

$x = -2 \pm \frac{\sqrt{3}}{6}$

Standard Form: $y = \frac{1}{3}(x+2)^2 - \frac{1}{36}$

Opens: up

Y-intercept: $(0, \frac{47}{36})$

Vertex: $(-2, -\frac{1}{36})$

X-intercepts: $(-2, -\frac{\sqrt{6}}{6}, 0)(-2+\frac{\sqrt{3}}{6}, 0)$

Range: $[-\frac{1}{36}, \infty)$

Domain: $(-\infty,\infty)$

Graphing Parabolas

Graph each of the following parabolas. Make sure you find all required information, label the vertex, y-intercept, and x-intercepts (if they exist).

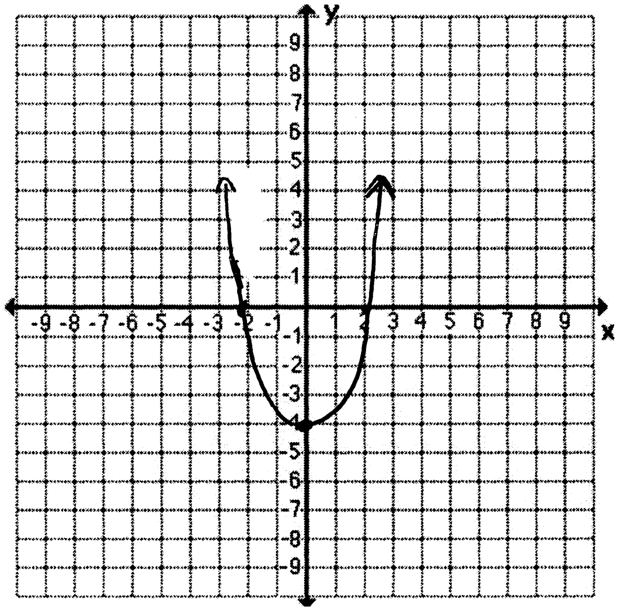
A) $y = x^2 - 4$

x-int

$x^2-4=0$

$\sqrt{x^2}=\sqrt{4}$

$x=\pm 2$



Standard Form: $y=x^2-4$

Opens: up

Y-intercept: $(0,-4)$

Vertex: $(0,-4)$

X-intercepts: $(-2,0)(2,0)$

Range: $[-4,\infty)$

Domain: $(-\infty,\infty)$

B) $y = -x^2 - 6x - 9$

$y=-x^2-6x-9$

$-y=x^2+6x+9$

$-y-9+9=x^2+6x+9$

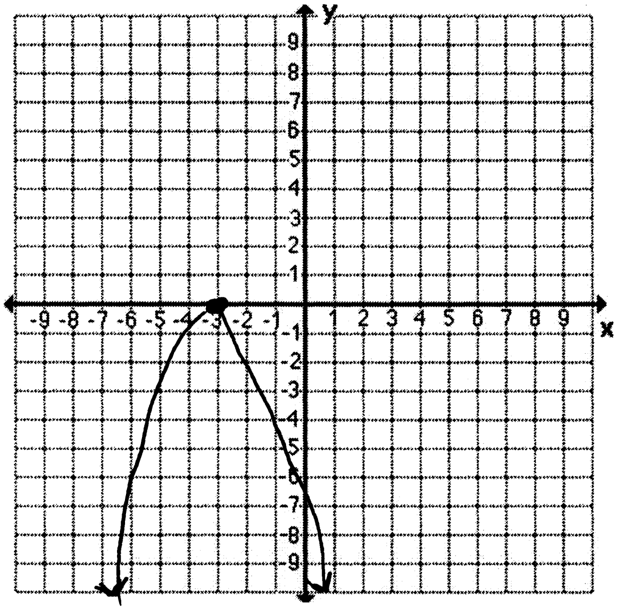
$(\frac{b}{2})^2=(3)^2=9$

$-y=(x+3)^2$

$y=-(x+3)^2$

x-int

$\sqrt{0}=\sqrt{(x+3)^2}$



Standard Form: $y=-(x+3)^2$

Opens: down

Y-intercept: $(0,-9)$

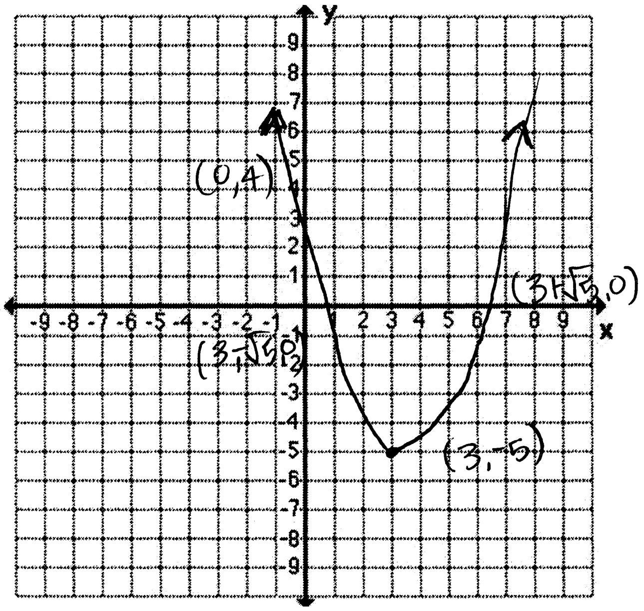
Vertex: $(-3,0)$

X-intercepts: $(-3,0)$

Range: $(-\infty, 0]$

Domain: $(-\infty,\infty)$

C) $y = x^2 - 6x + 4$



Standard Form: $y=(x-3)^2-5$

Opens: up

Y-intercept: $(0,4)$

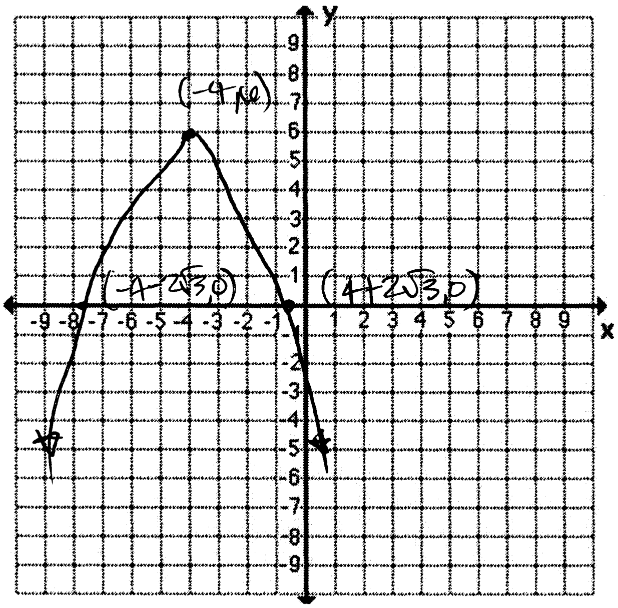
Vertex: $(3,-5)$

X-intercepts: $(3,-\sqrt{5},0)(3+\sqrt{5},0)$

Range: $[-5,\infty)$

Domain: $(-\infty,\infty)$

D) $y = -\frac{1}{2}x^2 - 4x - 2$



Standard Form: $y=-\frac{1}{2}(x+4)^2+6$

Opens: down

Y-intercept: $(0,-2)$

Vertex: $(-4,6)$

X-intercepts: $(-4-2\sqrt{3},0)(-4+2\sqrt{3},0)$

Range: $(-\infty,6]$

Domain: $(-\infty,\infty)$

Intervals

In this section, we will determine the interval in which the value of a function is increasing, decreasing, constant, positive, or negative. In order to do this, the graph of the function is read from the left to the right.

Increasing
To determine the interval in which the value of the function is increasing, read the graph of the function going from left to right. If the graph is rising going from left to right, the value of the function is increasing.

Decreasing
To determine the interval in which the value of the function is decreasing, read the graph of the function going from left to right. If the graph is falling going from left to right, the value of the function is decreasing.

Constant
To determine the interval in which the value of the function is constant, read the graph of the function going from left to right. If the graph is neither rising nor falling going from left to right, the value of the function is constant. In other words, we are looking for a horizontal line.

Positive
To determine the interval in which the value of a function is positive, the x intercepts of the function must be known. If any portion of the graph is above the x axis, this is the interval in which the value of the function is positive.

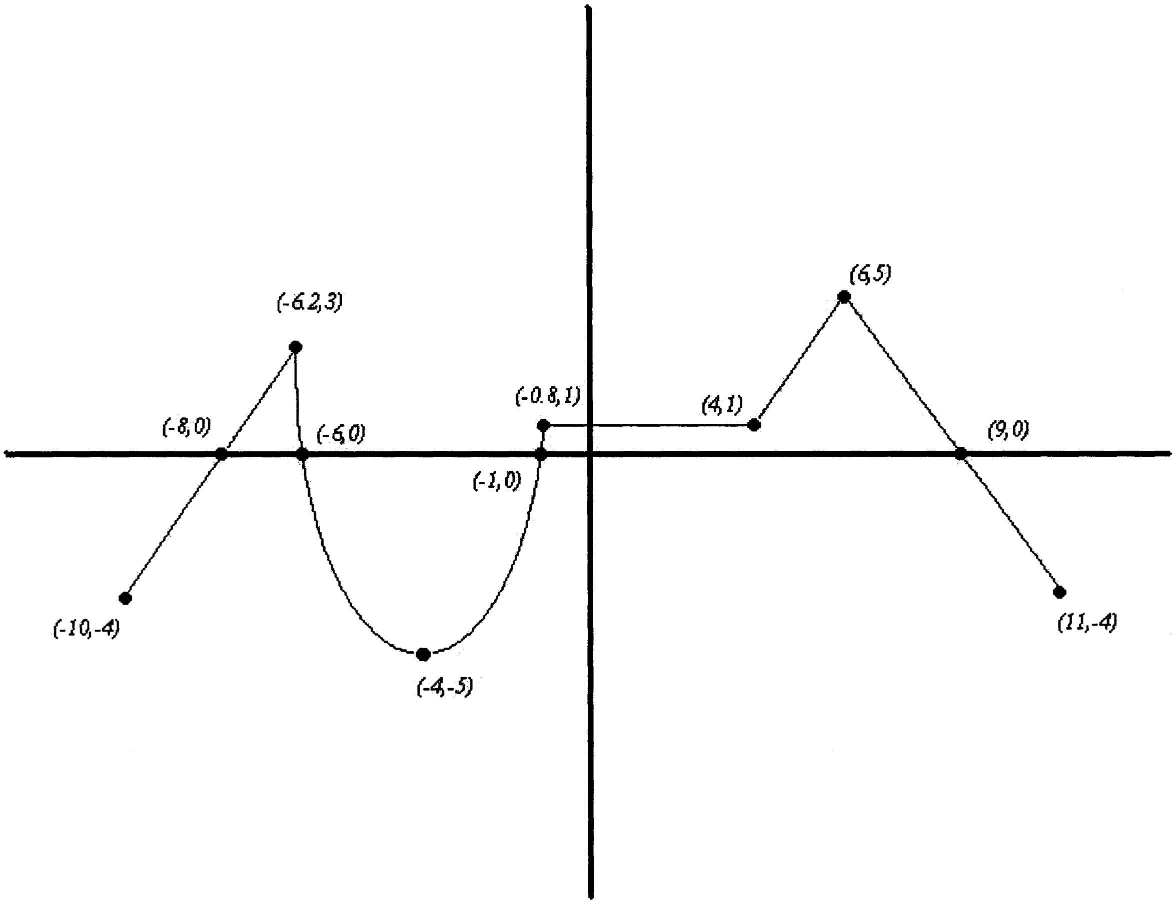
Negative
To determine the interval in which the value of a function is negative, the x intercepts of the function must be known. If any portion of the graph is below the x axis, this is the interval in which the value of the function is negative.

In the case of a parabola, it will not be necessary to see the graph to do this.

Increasing/Decreasing
The interval in which the value of a quadratic function is increasing or decreasing is based on how the parabola opens, and the x value of the vertex.

Positive/Negative
The interval in which the value of a quadratic function is positive or negative is based on the x intercepts of the graph of the function. It is also very important to keep in mind the direction in which the parabola opens. If for example, it is determined that a parabola opens down and has no x intercepts, there is no interval in which the value of the function is positive. This happens because the vertex would be below the x axis.

Here is an example of determine intervals. Notice that only the x values of the coordinates were used to describe the interval, and only parenthesis are used.



Given the above graph of a function, determine the intervals in which the value of the function is increasing, decreasing, constant, positive and negative.

The value of the function is increasing in the intervals:
(-10,-6.2), (-4,-0.8), and (4,6)

The value of the function is decreasing in the intervals:
(-6.2,-4) and (6,11)

The value of the function is constant in the interval:
(-0.8, 4)

The value of the function is positive in the intervals:
(-8,--6) and (-1, 9)

The value of the function is negative in the intervals:
(-10,-8), (-6,-1) and (9,11)

The reason parenthesis must be used when describing these intervals and not brackets is in the meaning of each symbol when using interval notation. A bracket means inclusive. Notice for example the intervals in which the value of the function is increasing and decreasing. The x value of -6.2 is used in both instances. If there had been brackets by the -6.2, it would be stating that the value of the function is both increasing and decreasing at the same time when x is -6.2.

When asked to determine the interval in which the value of the function is increasing, the y values of the function are not used to describe these intervals. The reason these values are not used is because of the question itself. The y coordinates of a function are the values of the function. The question is asking where this is happening, not what the actual values are.

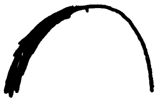
<u>A couple of guidelines to intervals</u>

Make sure to only use the x values of the function to describe the interval.
Never use brackets when answering these types of questions.

Determine the interval in which the value of the function $y=-x^2+2x+6$ is increasing.

$y=-x^2+2x+6$

$-\frac{6}{2a} = -\frac{2}{2(-1)} = \frac{-2}{-2} = 1$

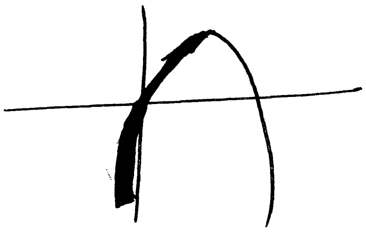


$\boxed{(-\infty, 1)}$

Determine the interval in which the value of the function $y=-2x^2-12x+18$ is increasing.

$y=-2x^2-12x+18$

$\frac{-b}{2a} = \frac{-12}{2(-2)} = -\frac{12}{4} = -3$

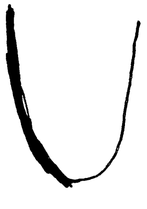


$\boxed{(-\infty, -3)}$

Determine the interval in which the value of the function $y=x^2-3x+4$ is decreasing.

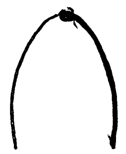
$y=x^2-3x+4$

$-\frac{b}{2a} = -\frac{-3}{2(1)} = \frac{3}{2}$



$\boxed{(-\infty, \frac{3}{2})}$

Determine the interval in which the value of the function $y=-3x^2+18x+21$ is decreasing.

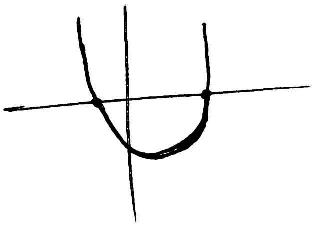


$y=-3x^2+18x+21$

$\frac{-b}{2a}=\frac{-18}{2(-3)}^2=\frac{18}{6}=3$

$\boxed{(3,\infty)}$

Determine the interval in which the value of the function $y=x^2-3x+4$ is negative.



$y=x^2-3x+4$

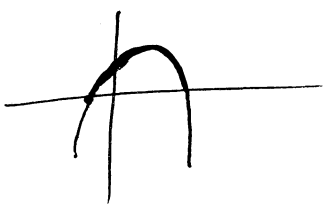
$x^2-3x+4=0$

$x=\frac{3\pm\sqrt{9-4(1)(4)}}{2(1)}$ $9-16$

$x=\frac{3\pm\sqrt{-9}}{2}$

not real, no x-intercepts

Determine the interval in which the value of the function $y=-2x^2-12x+18$ is positive.



$y=-2x^2-12x+18$

$\frac{-2x^2}{-2}\frac{-12x}{-2}\frac{+18}{-2}=\frac{0}{-2}$

$x^2+6x-9=0$

$x^2+6x+9=9+9$

$(\frac{6}{2})^2=(3)^2=9$

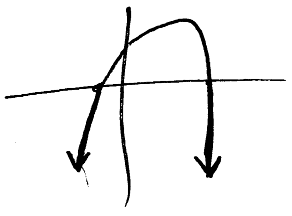
$\sqrt{(x+3)^2}=\sqrt{18}$

$x+3=\pm\sqrt{18}$

$x=-3\pm3\sqrt{2}$

$\boxed{(-3-3\sqrt{2},-3+3\sqrt{2})}$

Determine the interval in which the value of the function $y=-2x^2+12x-12$ is negative.



$y=-2x^2+12x-12$

$\frac{-2x^2}{-2}\frac{+12x}{-2}\frac{-12}{-2}=\frac{0}{-2}$

$x^2-6x+6=0$

$x^2-6x+9=-6+9$

$(-\frac{6}{2})^2=(-3)^2=9$

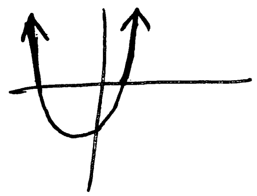
$\sqrt{(x-3)^2}=\sqrt{3}$

$x-3=\pm\sqrt{3}$

$x=3\pm\sqrt{3}$

$\boxed{(-\infty,3-\sqrt{3})\cup(3+\sqrt{3},\infty)}$

Determine the interval in which the value of the function $y=4x^2+16x-84$ is positive.



$y=4x^2+11x-84$

$\frac{4x^2}{4}\frac{+11x}{4}\frac{-84}{4}=\frac{0}{4}$

$x^2+4x-21=0$

$(x+7)(x-3)=0$

$x=-7$ $x=3$

$(-\infty,-7)\cup(3,\infty)$

Finding the Equation of a Parabola

You will be required to find the equation of a parabola with the given information.

Find the equation of a parabola that has x intercepts of $(-3,0)$ ***and*** $(2,0)$.

$(-3,0)$ *and* $(2,0)$.	***Given x intercepts of -3 and 2***
$x=-3 \qquad x=2$	***If the x intercepts are -3 and 2, then the roots of the equation are -3 and 2. You must now set each root equal to zero.***
$(x+3)\ (x-2)$	***For the first root, you will add 3 to both sides of the equal sign.*** ***For the second root, you will subtract 2 to both sides of the equal sign.***
x^2+x-6	***Now multiply the results together, and you have a quadratic expression.***
$y=x^2+x-6$	***Set your expression equal to y, and you have the equation of a parabola.***

In the previous example, all you need to do is set each root equal to x, and work backwards. Imagine solving a quadratic equation in the reverse. You have the solutions, and you work backwards to find out what the original problem is. Now, keep in mind, there are an infinite number of parabolas that can have those two particular x intercepts. If you wanted, for example, to find the equation of a parabola that opens down and has x intercepts of -3 and 2, you would multiply your resultant expression, x^2+x-6, **by (-1).** ***You could also multiply your expression by any other constant, and the parabola would still have the same x-intercepts.***

Find the equation of a parabola that opens up, and has the following x intercepts.

A) (-3,0) and (4,0)

$x=-3 \quad x=4$

$y=(x+3)(x-4)$

$y=4x^2-x-12$

B) (-12,0) and (-3,0)

$x=3/2 \quad x=-3$

$y=-(x+12)(x+3)$

$y=x^2+15x+36$

C) (2,0) and (5,0)

$x=2 \quad x=5$

$y=(x-2)(x-6)$

$y=x^2-7x+10$

Find the equation of a parabola that opens down, and has the following x intercepts.

A) (-2,0) and (6,0)

$x=-2 \quad x=6$

$y=-(x+2)(x-6)$

$y=-(x^2-4x-12)$

$y=-x^2+4x+12$

B) (1,0) and (7,0)

$x=1 \quad x=7$

$y=-(x-1)(x-7)$

$y=-(x^2-8x+7)$

$y=-x^2+8x-7$

C) (5,0)

$y=-(x-5)^2$

$y=-(x^2-10x+25)$

$y=-x^2+10x-25)$

Find the equation of a parabola that has vertex of $(-2,-4)$, and passes through $(1,3)$.

$$y=a(x-h)^2+k$$

Begin with the standard form of a quadratic function.

$$y=a(x-(-2))^2+(-4)$$

Substitute the values of h and k in for the vertex of the parabola.

$$y=a(x+2)^2-4$$

Simplify, now all you need is the value of a to complete the equation.

$$(3)=a((1)+2)^2-4$$

Since you have the values of h and k, you need to find the value of a, so substitute the values of x and y for the point $(1,3)$ into the equation.

$$3=a(3)^2-4$$
$$3=9a-4$$
$$7=9a$$
$$a=\frac{7}{9}$$

Simplify and solve for a.

$$y=\frac{7}{9}(x+2)^2-4$$

Once you have found the value of a, rewrite the completed equation. You have just found the equation of a parabola that has vertex of $(-2,-4)$, and passes through $(1,3)$.

When you encounter these types of problems the most common mistake people make, is they forget to find the value of a.

Find the equation of a parabola that has a vertex of (-3,2) and contains the point (4,7).

$y=a(x-h)^2+k$

$y=a(x+3)^2+2$

$7=a(4+3)^2+2$

$-2-7=a(7)^2$

$49a=5$

$a=\frac{5}{49}$

$y=\frac{5}{49}(x+3)^2+2$

Find the equation of a parabola that has a vertex of (4,5) and contains the point (-2,-2).

$y=a(x-h)^2+k$

$y=a(x-4)^2+5$

$-2=a(-2-4)^2+5$

$-5-7=a(-6)^2$

$36a=-7$

$a=-\frac{7}{36}$

$y=-\frac{7}{36}(x-4)^2+5$

Find the equation of a parabola that has a vertex of (-2,-3) and contains the point (4,1)

$y=a(x-h)^2+k$

$y=a(x+2)^2-3$

$1=a(4+2)^2-3$

$3+1=a(6)^2$

$4=36a$

$a=\frac{1}{9}$

$y=\frac{1}{9}(x+2)^2-3$

Find the equation of a parabola that has a vertex of (0,3) and passes the x axis at (7,0).

$y=a(x-h)^2+k$

$y=a(x)^2+3$

$0=a(7)^2+3$

$-3=49a$

$a=\frac{-3}{49}$

$y=\frac{-3}{49}x^2+3$

Find the equation of a parabola that has a vertex of (3,-1) and has a y intercept of (0,-8).

$y=a(x-h)^2+k$

$y=a(x-3)^2-1$

$-8=a(-3)^2-1$

$1-8=a(-3)^2$

$-7=9a$

$a=\frac{-7}{9}$

$y=\frac{-7}{9}(x-3)^2-1$

Find the equation of a parabola that has a vertex of (5,0) and has a y intercept of (0,-12).

$y=a(x-h)^2+k$

$y=a(x-5)^2$

$-12=a(-5)^2$

$-12=25a$

$a=\frac{-12}{25}$

$y=\frac{-12}{25}(x-5)^2$

On the next page you will learn to find the equation of a parabola given 3 points on the parabola.

Find the equation of a parabola that passes through (1,0), (2,-1) and (3,0).

$$(1,0),\ (2,-1),\ (3,0)$$

Given

$$y = ax^2 + bx + c$$

Use the general form of a parabola

$$\begin{aligned} 0 &= a + b + c \\ -1 &= 4a + 2b + c \\ 0 &= 9a + 3b + c \end{aligned}$$

Substitute the values of x and y into the general form of a parabola for each set of coordinates setting up 3 equations.

$$\begin{array}{ll} Eq_1 & a + b + c = 0 \\ Eq_2 & 4a + 2b + c = -1 \\ Eq_3 & 9a + 3b + c = 0 \end{array}$$

You have now created a system of equations in 3 variables.

$$\begin{array}{ll} Eq_1 & a + b + c = 0 \\ Eq_2 & 4a + 2b + c = -1 \end{array} \qquad \begin{array}{ll} Eq_1 & a + b + c = 0 \\ Eq_3 & 9a + 3b + c = 0 \end{array}$$

One of these equations will be used twice. It doesn't matter which. In this case, equation 1 will be used twice.

$$\begin{array}{ll} -Eq_1 & -a - b - c = 0 \\ +Eq_2 & 4a + 2b + c = -1 \\ \hline Eq_4 & 3a + b = -1 \end{array} \qquad \begin{array}{ll} -Eq_1 & -a - b - c = 0 \\ +Eq_3 & 9a + 3b + c = 0 \\ \hline Eq_5 & 8a + 2b = 0 \end{array}$$

Multiplying equation 1 by -1 and combining the equations yielded two new equations. You <u>must</u> get rid of the same variable each time.

$$\begin{array}{ll} Eq_4 & 3a + b = -1 \\ Eq_5 & 8a + 2b = 0 \end{array}$$

Now we have a system of equations in two variables.

$$\begin{array}{ll} -2Eq_4 & -6a - 2b = 2 \\ +\ Eq_5 & 8a + 2b = 0 \\ \hline & 2a = 2 \\ & a = 1 \end{array}$$

Multiply equation 4 by -2, and add the result to equation 5. This yields a numerical value for the variable a.

$$a = 1$$

Substitute into equation 5

$$\begin{aligned} 8(1) + 2b &= 0 \\ 2b &= -8 \\ b &= -4 \end{aligned}$$

$$a = 1 \quad b = -4$$

Substitute into equation 1

$$\begin{aligned} (1) + (-4) + c &= 0 \\ -3 + c &= 0 \\ c &= 3 \end{aligned}$$

Once the value of the first variable is found, substitute that number, in this case 1, into either equation 4 or 5 and solve for the remaining variable. Now that the value of two of the variables is known, go back to equation 1, substitute and find the value of the third variable.

$$a = 1 \quad b = -4 \quad c = 3$$

The value of all three variables has now been found.

$$y = x^2 - 4x + 3$$

Substitute these values into the general form of a parabola, and the equation of the parabola that passes through (1,0), (2,-1) and (3,0) is found.

A) Find the equation of a parabola that passes through (1,6), (2,5) and (0,5).

$(1,6)$

$(2,5)$

$(0,5)$

$y=ax^2+bx+c$

$a+b+c=6$

$4a+2b+c=5 \rightarrow$

$c=5$

$a+b+5=6$

$4a+2b+5=5 \rightarrow$

$-2(a+b=1)$

$4a+2b=0$

$-2a-2b=-2$

$2a=-2$

$a=-1$

$a+b=1$

$\rightarrow -1+b=1$

$+1 \quad +1$

$b=2$

$a=-1 \quad b=2 \quad c=5$

$y=-x^2+2x+5$

B) Find the equation of a parabola that passes through (0,6), (2,2) and (5,11).

$(2,2)$

$(5,11)$

$(0,6)$

$y=ax^2+bx+c$

$4a+2b+c=2$

$25a+5b+c=11$

$c=6$

$\rightarrow 4a+2b+6=2$

$\rightarrow 25a+5b+6=11$

$\rightarrow -5(4a+2b=-4)$

$2(25a+5b=5)$

$-20a-10b=20$

$50a+10b=10$

$30a=30$

$a=1$

$a=1 \quad b=-4 \quad c=6$

$y=x^2-4x+6$

$4a+2b=-4$

$4(1)+2b=-4$

$4+2b=-4$

$-4 \qquad -4$

$2b=-8$

$b=-4$

C) Find the equation of a parabola that passes through (3,-10), (4,0) and (6,8).

$y=ax^2+bx+c$

(3,-10) $9a+3b+c=-10$

(4,0) $16a+4b+c=0$

(6,8) $36a+6b+c=8$

→

$-(9a+3b+c=-10)$

$16a+4b+c=0$

$-9a-3b-c=10$

$7a+b=10$

$-(9a+3b+c=-10)$

$36a+6b+c=8$

$-9a-3b-c=10$

$27a+3b=18$

$7a+b=10$

$7(-2)+b=10$

$-14+b=10$

$b=24$

$-3(7a+b=10)$

$27a+3b=18$

$-21a-3b=-30$

$\frac{6a}{6}=\frac{-12}{6}$

$a=-2$

$a=-2$ $b=24$ $c=-64$

$y=-2x^2+24x-64$

$9a+3b+c=-10$

$9(-2)+3(24)+c=-10$

$-18+72+c=-10$

$54+c=-10$ $c=-64$

D) Find the equation of a parabola that passes through (0,6), (-6,0) and (2,16).

$y=ax^2+bx+c$

$c=6$

(0,6)

(-6,0)

(2,16)

$36a-6b+c=0$

$4a+2b+c=16$

→

$36a-6b+6=0$

$4a+2b+6=16$

→

$36a-6b=-6$

$3(4a+2b=10)$

→

$4a+2b=10$

$4(\frac{1}{2})+2b=10$

$2+2b=10$

$2b=8$

$b=4$

$36a-6b=-6$

$12a+6b=30$

$\frac{48a}{48}=\frac{24}{48}$

$a=\frac{1}{2}$

$a=\frac{1}{2}$ $b=4$ $c=6$

$y=\frac{1}{2}x^2+4x+6$

Finding a Quadratic Equation

Find a quadratic equation that has the following roots.

A) $-7,\ -8$

$x=-7 \quad x=-8$

$(x+7)(x+8)$

$\boxed{x^2+15x+56=0}$

B) $\frac{4}{5},\ -\frac{2}{3}$

$x=\frac{4}{5} \quad x=-\frac{2}{3}$

$5x=4 \quad 3x=-2$

$(5x-4)(3x+2)$

$15x^2+10x$

$-12x-8$

$\boxed{15x^2-2x-8=0}$

C) $0,\ \frac{1}{8}$

$x=0 \quad x=\frac{1}{8}$

$8x=1$

$x(8x-1)$

$\boxed{8x^2-x=0}$

D) $3+2i,\ 3-2i$

$x=3\pm 2i$

$(x-3)^2=(\pm 2i)^2$

$x^2-6x+9=4i^2$

$x^2-6x+9=-4$

$\boxed{x^2-6x+13=0}$

E) $\frac{1}{2}+\frac{5}{2}i,\ \frac{1}{2}-\frac{5}{2}i$

$x=\frac{1}{2}\pm\frac{5}{2}i$

$2x=1\pm 5i$

$(2x-1)^2=(\pm 5i)^2$

$4x^2-4x+1=25i^2$

$4x^2-4x\ 1=-25$

$\boxed{4x^2-4x+26=0}$

F) $3+\sqrt{2},\ 3-\sqrt{2}$

$x=3\pm\sqrt{2}$

$(x-3)^2=(\pm\sqrt{2})^2$

$x^2-6x+9=2$

$\boxed{x^2-6x+7=0}$

G) $\frac{1+\sqrt{3}}{2},\ \frac{1-\sqrt{3}}{2}$

$x=\frac{1\pm\sqrt{3}}{2}$

$2x=1\pm\sqrt{3}$

$(2x-1)^2=(\pm\sqrt{3})^2$

$4x^2-4x+1=3$

$\boxed{4x^2-4x-2=0}$

H) $\frac{2+2\sqrt{2}}{3},\ \frac{2-2\sqrt{2}}{3}$

$x=\frac{2\pm 2\sqrt{2}}{3}$

$3x=2\pm 2\sqrt{2}$

$(3x-2)^2=(\pm 2\sqrt{2})^2$

$9x^2-12x+4=8$

$\boxed{9x^2-12x-4=0}$

I) $\sqrt{2}i,\ -\sqrt{2}i$

$(x)^2=(\pm\sqrt{2}i)^2$

$x^2=2i^2$

$x^2=-2$

$\boxed{x^2+2=0}$

J) $\frac{\sqrt{3}}{5},\ -\frac{\sqrt{3}}{5}$

$x=\pm\frac{\sqrt{3}}{5}$

$(5x)^2=(\pm\sqrt{3})^2$

$25x^2=3$

$\boxed{25x^2-3=0}$

K) $\frac{1}{2}+\frac{\sqrt{2}}{2}i,\ \frac{1}{2}-\frac{\sqrt{2}}{2}i$

$x=\frac{1}{2}\pm\frac{\sqrt{2}}{2}i$

$2x=1\pm\sqrt{2}i$

$(2x-1)^2=(\pm\sqrt{2}i)^2$

$4x^2-4x+1=2i^2$

$4x^2-4x+1=-2$

$\boxed{4x^2-4x+3=0}$

L) $3+5i,\ 3-5i$

$x=3\pm 5i$

$(x-3)^2=(\pm 5i)^2$

$x^2-6x+9=25i^2$

$x^2-6x+9=-25$

$\boxed{x^2-6x+34=0}$

Word Problems with Quadratics

Solve each of the following.

A) Find two consecutive odd integers whose product is 99.

$x(x+2)=99$

$x^2+2x=99$

$x^2+2x-99=0$

$(x+11)(x-9)=0$

$x=-11 \quad x=9$

-11 and -9
or
9 and 11

B) The sum of two numbers is 40. Find the greatest possible product of the two numbers.

$x+y=40 \quad y=40-x$

$xy=?$

$x(40-x)$

$40x-x^2$

$f(x)=-x^2+40x$

$f(x)=-x^2+40x$

$-f(x)=x^2-40x$

$-f(x)=x^2-40x$

$\left(\frac{b}{2}\right)^2=\left(-\frac{40}{2}\right)^2=(-20)^2=400$

$-f(x)+400=(x^2-40x+400)$

$-f(x)=(x-20)^2-400$

$f(x)=-(x-20)^2+400$

max product
is 400

C) Find the number of units that produce a maximum revenue $R=800x-0.1x^2$, where R is the revenue in dollars, and x is the number of units sold.

$R=800x-0.1x^2$

$R=-0.1x^2+800x$

$-\frac{b}{2a}=\frac{-800}{2(-0.1)}=\frac{800}{0.2}=4000$

4000 units will produce
max revenue

D) The profit a company makes is given by the model $P=-0.4x^2+30x+220$, where P is the profit the company earns and x is the amount spent on advertisement in hundreds of dollars. What amount should the company spend on advertising in order to maximize profits.

$P=-0.4x^2+30x+220$

$-\frac{b}{2a}=\frac{-30}{2(-0.4)}=\frac{30}{0.8}=37.5$

$37.5 \times 100 \leftarrow$ ads in hundreds

\$3,750 should be spent on advertising

E) The more expensive a product, the less you can sell. The relationship between the price, p, and the quantity, q, of sold products is given by the following formula $q=30-2p$. The revenue is given by $R=pq$. For what price will you have the maximum revenue?

$R=p\cdot q \quad q=30-2p$

$R=p(30-2p)$

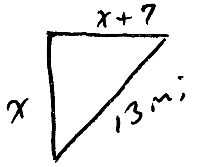
$R=30p-2p^2$

$R=-2p^2+30p$

$-\frac{b}{2a}=\frac{-30}{2(-2)}=\frac{30}{4}=7.5$

Price should
be \$7.50

F) Two cyclists meet at an intersection. The first cyclist travels east, while the second south. The eastbound cyclist is traveling 7 miles per hour faster than the southbound cyclist. After one hour, the two are 13 miles apart. How far have each of the cyclists traveled?



$x^2 + (x+7)^2 = (13)^2$

$x^2 + x^2 + 14x + 49 = 169$

$2x^2 + 14x + 49 = 169$

$2x^2 + 14x - 120 = 0$

$2(x^2 + 7x - 60) = 0$

$2(x+12)(x-5) = 0$

$2 \neq 0$ $\quad x = \cancel{-12}$ $\quad x = 5$

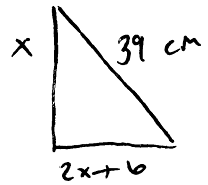
$x = 5$

$x + 7 =$

$(5) + 7 = 12$

Eastbound 12 mi
Southbound 5 mi

G) The base of a right triangle is 6 cm more than twice the length of its leg. Find the length of the two legs of the right triangle if the hypotenuse is 39 cm.



$x^2 + (2x+6)^2 = (39)^2$

$x^2 + 4x^2 + 24x + 36 = 1521$

$5x^2 + 24x - 1485 = 0$

$x = \frac{-24 \pm \sqrt{(24)^2 - 4(5)(-1485)}}{2(5)}$

$x = \frac{-24 \pm \sqrt{30276}}{10}$

$x = \frac{-24 \pm 174}{10}$

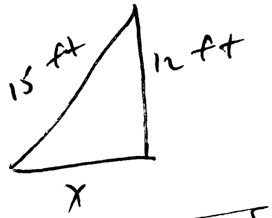
$x = \frac{-24 - 174}{10}$ $\quad x = \frac{-24 + 174}{10}$

$x = \frac{-198}{10}$ $\quad x = \frac{150}{10}$

$x \neq -19.8$ $\quad x = 15$

15 cm and 36 cm

H) A fireman has a 15 foot ladder. He needs to reach a window that is 12 feet high. How far away from the base of the wall should the ladder be placed?



$x^2 + (12)^2 = (15)^2$

$x^2 + 144 = 225$

$x^2 = 81$

$x = \pm 9$

$x \neq -9$ $\quad x = 9$

Ladder must be 9 ft away

I) Find two consecutive even integers whose product is 224.

$x(x+2) = 224$

$x^2 + 2x = 224$

$x^2 + 2x - 224 = 0$

$(x+16)(x-14) = 0$

$x = -16$ $\quad x = 14$

x , $x+2$

-16 and -14
or
14 and 16

Checking Progress

You have now completed the "Quadratic Equations" section of the workbook. The following is a checklist so that you may check your progress. Check off each of the objectives you have accomplished.

The student should be able to…

- ☐ ***Solve quadratic equations by completing the square.***
- ☐ ***Derive the quadratic formula.***
- ☐ ***Solve quadratic equations by using the quadratic formula.***
- ☐ ***Determine the nature of the roots of a quadratic using the discriminant.***
- ☐ ***Put a quadratic function in standard form.***
- ☐ ***Determine how the values of a, h, and k affect the graph of a function in*** $y = a(x-h)^2 + k$ ***form.***
- ☐ ***Find the vertex of a quadratic function.***
- ☐ ***Find the x and y intercepts of a quadratic function.***
- ☐ ***Find the range and domain of a quadratic function without graphing it.***
- ☐ ***Graph a quadratic function.***
- ☐ ***Determine the maximum or minimum value of a quadratic function.***
- ☐ ***Determine the interval in which the value of a function is increasing, decreasing or constant.***
- ☐ ***Determine the interval in which the value of a function is positive or negative.***
- ☐ ***Find the equation of a parabola given the vertex and a point on the parabola.***
- ☐ ***Find the equation of a parabola given the x intercepts of the graph of the function.***
- ☐ ***Find the equation of a parabola given three points on the curve.***

FUNCTIONS

Objectives

The following is a list of objectives for this section of the workbook.

By the time the student is finished with this section of the workbook, he/she should be able to…

- ***Determine the properties of a quadratic function in standard form.***
- ***Find the x and y intercepts of a quadratic function.***
- ***Find the range and domain of a quadratic function.***
- ***Find the vertex of a quadratic function in standard form.***
- ***Graph a quadratic function.***
- ***Determine the properties of an absolute value function in standard form.***
- ***Find the x and y intercepts of an absolute value function.***
- ***Find the range and domain of an absolute value function.***
- ***Find the vertex of an absolute value function.***
- ***Graph an absolute value function.***
- ***Determine the properties of a radical function in standard form.***
- ***Find the x and y intercepts of a radical function.***
- ***Find the range and domain of a radical function.***
- ***Find the point of origin of a radical function.***
- ***Graph a radical function.***
- ***Determine the properties of an exponential function in standard form.***
- ***Find the x and y intercepts of an exponential function.***
- ***Find the range and domain of an exponential function.***
- ***Find the key point of an exponential function.***
- ***Graph an exponential function.***
- ***Determine the properties of a logarithmic function in standard form.***
- ***Find the x and y intercepts of a logarithmic function.***
- ***Find the range and domain of a logarithmic function.***
- ***Find the key point of a logarithmic function.***
- ***Graph a logarithmic function.***
- ***Determine the properties of a cubic function in standard form.***
- ***Find the x and y intercepts of a cubic function.***
- ***Find the range and domain of a cubic function.***
- ***Find the vertex of a cubic function.***
- ***Graph a cubic function.***
- ***Shift the graph of a function without actually knowing the equation, i.e. graphing*** $f_{(x+2)}$.
- ***Graph piece-wise functions.***

Math Standards Addressed

The following state standards are addressed in this section of the workbook.

Algebra II

1.0 Students solve equations and inequalities involving absolute value.

8.0 Students solve and graph quadratic equations by factoring, completing the square, or using the quadratic formula. Students apply these techniques in solving word problems. They also solve quadratic equations in the complex number system.

9.0 Students demonstrate and explain the effect that changing a coefficient has on the graph of quadratic functions; that is, students can determine how the graph of a parabola changes as a, b, and c vary in the equation $y = a(x-b)^2 + c$.

10.0 Students graph quadratic functions and determine the maxima, minima, and zeros of the function.

11.0 Students prove simple laws of logarithms.

> 11.1 Students understand the inverse relationship between exponents and logarithms and use this relationship to solve problems involving logarithms and exponents.
>
> 11.2 Students judge the validity of an argument according to whether the properties of real numbers, exponents, and logarithms have been applied correctly at each step.

12.0 Students know the laws of fractional exponents, understand exponential functions, and use these functions in problems involving exponential growth and decay.

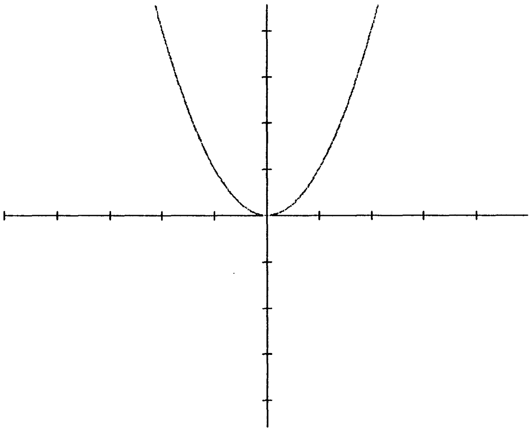
15.0 Students determine whether a specific algebraic statement involving rational expressions, radical expressions, or logarithmic or exponential functions is sometimes true, always true, or never true.

Quadratic Functions

The translation of a function is simply the shifting of a function. In this section, for the most part, we will be graphing various functions by means of shifting the parent function. We will go over the parent function for a variety of algebraic functions in this section. It is much easier to see the effects different constants have on a particular function if we use the parent function. We will begin with quadratics. Observe the following regarding a quadratic function in standard form.

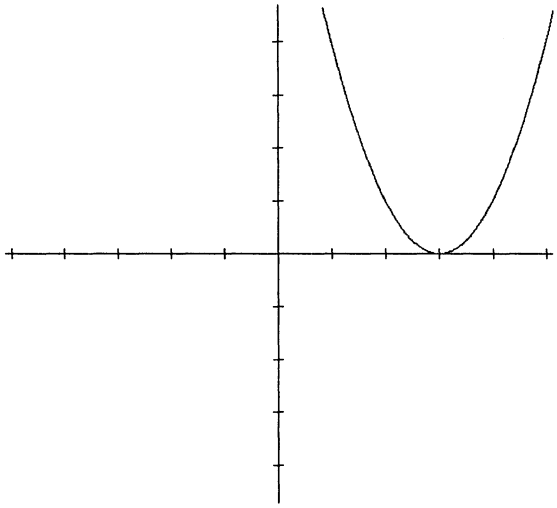
$$y=a(x-h)^2+k$$

$$y=x^2$$



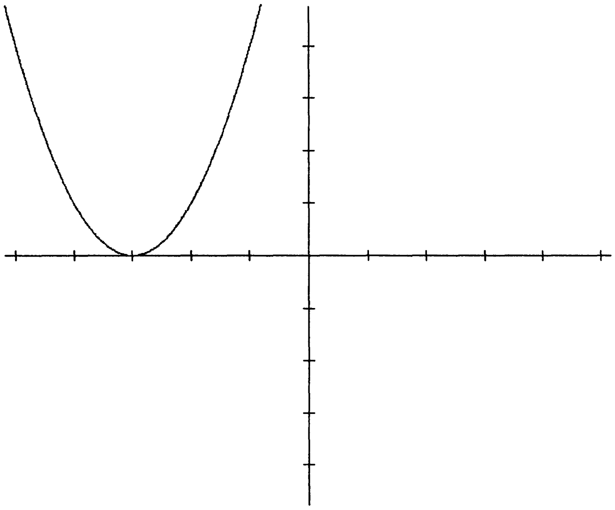
Notice that in the equation above, the h and k values are zero, while the value of a is one. This gives you the parent function for all quadratics. Everything else is merely a manipulation of the parent function.

$$y=(x-3)^2$$



The graph of the function shifts right 3.

$$y=(x+3)^2$$

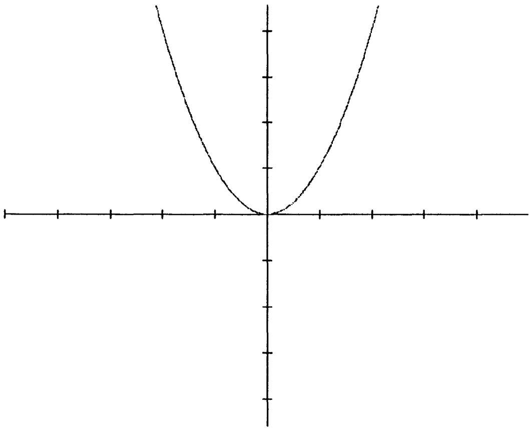


The graph of the function shifts left 3.

The number inside the parenthesis makes the graph shift to the left or right. Remember P.L.N.R., Positive Left Negative Right, tells about the horizontal shift needed to graph the function.

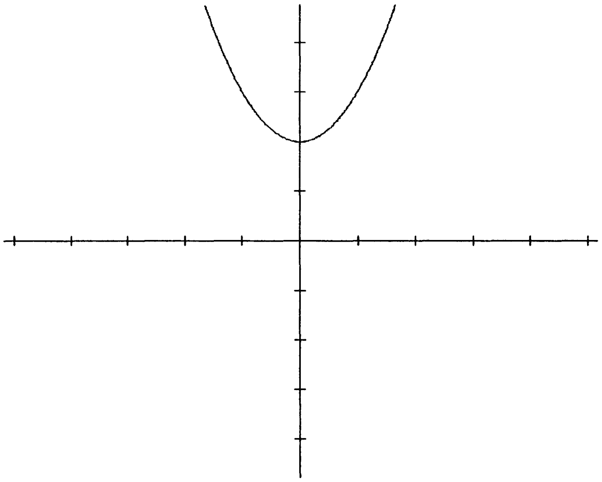
If the function above is $f_{(x)}$, the functions below would be $f_{(x-3)}$ and $f_{(x+3)}$ respectively. This is important to know, because in the future, you will be required to graph functions based solely on the picture provided. No equation will be given. You must rely solely on your knowledge of translating graphs.

$y = x^2$



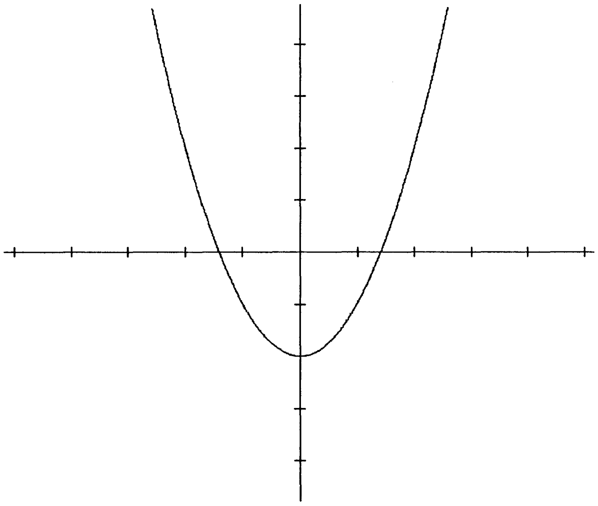
Once again, the parent function is illustrated above, and translations of it below.

$y = x^2 + 2$



The graph of the function shifts up 2.

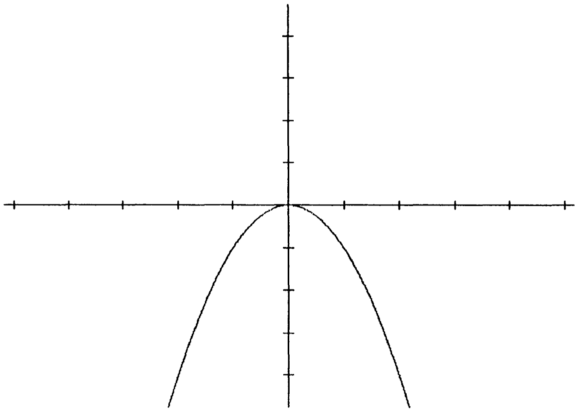
$y = x^2 - 2$



The graph of the function shifts down 2.

In these examples, the k value is what is changing. The value of k dictates a vertical shift of the function. In this case, consider the parent function as being $f_{(x)}$. Given no information regarding the specific equation of the function, the equations for these two translations of $f_{(x)}$ are $f_{(x)} + 2$, and $f_{(x)} - 2$.

$y = -x^2$



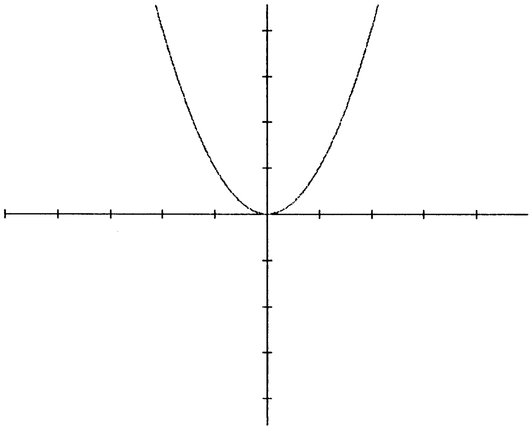
Now, on the left we have the opposite of the parent function. In this particular example, the value of a, in the standard form is -1. A negative reflects the graph of the function about the horizontal axis. Once again, if the parent function given is referred to as $f_{(x)}$, this function is $-f_{(x)}$.

We have seen how to graph a function by shifting the parent function. You may have noticed that we graphed $-f_{(x)}$, but not $f_{(-x)}$. The reason we did not see $f_{(-x)}$, is because this is the graph of an even function. That means that if a –x were plugged in to the function, it would make no difference. The outcome would be the same. However, if we are dealing with a different type of function, one that was not even, $f_{(-x)}$ would cause the graph of the function to reflect about a vertical axis. In other words, if $-f_{(x)}$ makes a graph flip upside down, $f_{(-x)}$ would make the graph flip from right to left, or left to right, whatever may be the case.

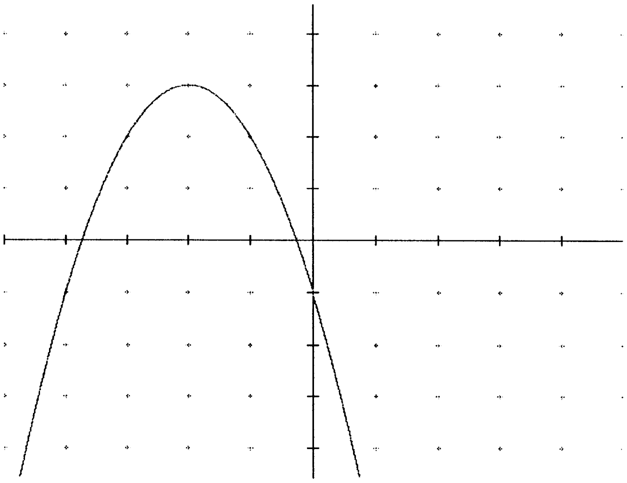
Lets see how differing values of a, h and k will cause various shifts of the function.

$$y = a(x-h)^2 + k$$

Once again, take note of the parent function $y = x^2$



$y = -(x+2)^2 + 3$

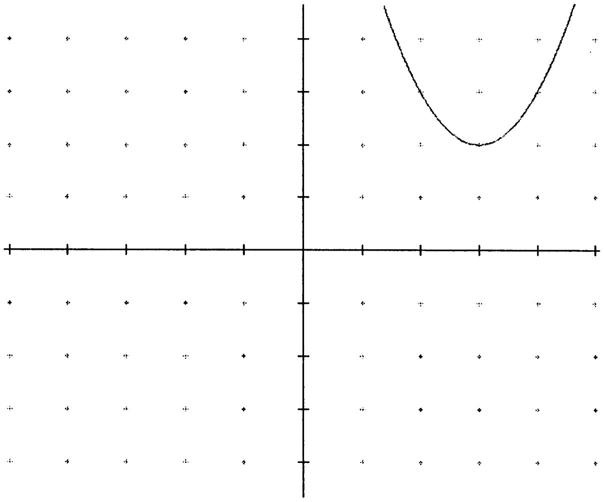


This graph opens down, and shifts left 2, up 3.

If this graph is a translation of the function $f_{(x)}$

It would by written as $-f_{(x+2)} + 3$.

$y = (x-3)^2 + 2$



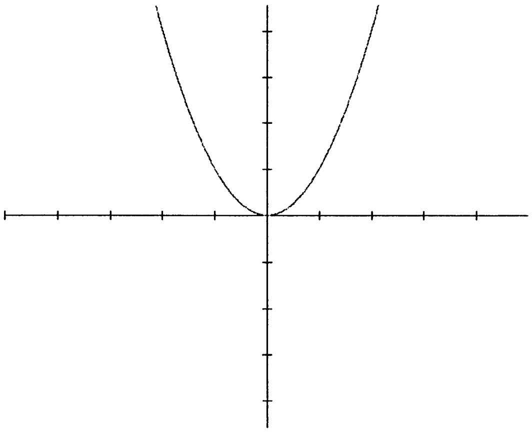
This graph opens up and shifts right 3, and up 2

If this graph is a translation of the function $f_{(x)}$

It would by written as $f_{(x-3)} + 2$.

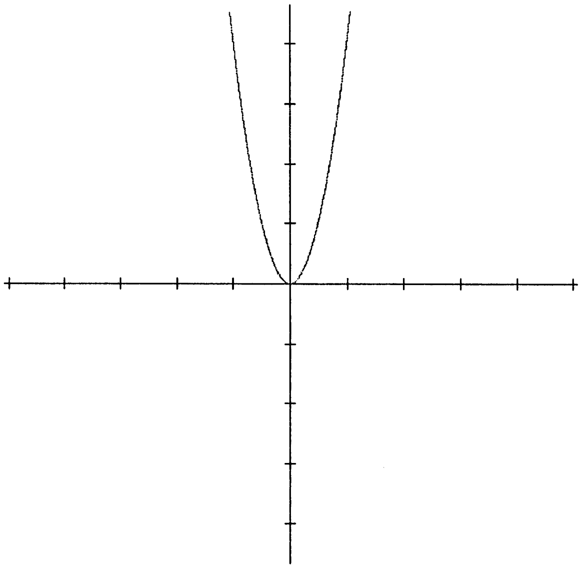
We will be graphing functions using only $f_{(x)}$ in the "translations of functions" section.

$$y=a(x-h)^2+k$$



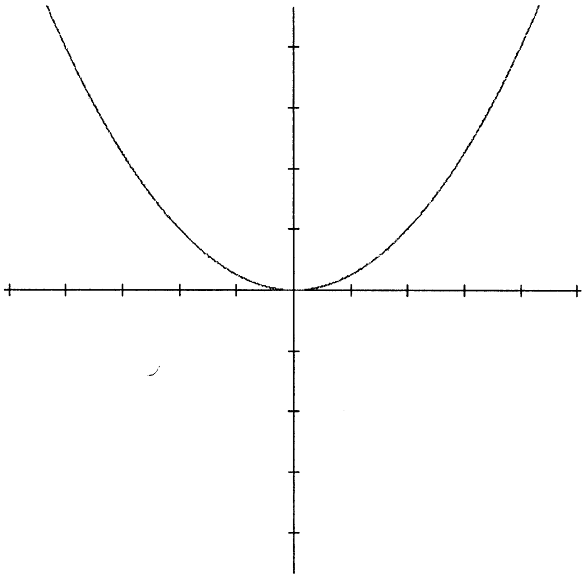
$y=x^2$

Here we will see how the value of a for the quadratic function in standard form affects the graph of the function. To illustrate this, we will look at the graph of a parabola that has its vertex on the origin.



$y=4x^2$

This graph seems very narrow, but what is actually happening, is the value of the function is increasing very rapidly. The y values are increasing at 4 times their normal rate. The rapid increase causes the graph to appear narrow.



$y=\frac{1}{4}x^2$

This graph is wider than the parent function. In this case, the y values of the function are increasing at ¼ their normal rate, causing a more gradual increase.

As you can see, if the value of the leading coefficient is a whole number, the y values of the graph will increase rapidly causing a narrow and steeper curve. In contrast, if the leading coefficient is a fraction, the y values of the function will increase mildly, causing a more gradual curve.

Describe the movement of each of the following quadratic functions. Describe how each opens and if there is any horizontal or vertical movement. Be sure to state how many spaces it moves, for example: ***This graph opens down, and shifts left 2, up 3.***

A) $y=-3(x-4)^2+2$

right 4

up 2

Opens Down

B) $y=2(x+3)^2-8$

Opens up

left 3

down 8

C) $y=\frac{1}{2}(x-3)^2$

opens up

right 3

D) $y=\left(x+\frac{1}{2}\right)^2-\frac{2}{3}$

opens up

left 1/2

down 2/3

E) $y=-(x+5)^2+6$

opens down

left 5

up 6

F) $y=7(x-3)^2+1$

opens up

right 3

up 1

G) $y=-\frac{1}{5}(x-7)^2+4$

opens down

right 7

up 4

H) $y=3(x+6)^2+8$

opens up

left 6

up 8

I) $y=-4(x-3)^2-2$

opens down

right 3

down 2

J) $y=x^2-3$

opens up

down 3

K) $y=-\frac{1}{5}(x+14)^2$

opens down

left 14

L) $y=-2x^2+8$

opens down

up 8

As you describe the graphs of the quadratic functions above, you wrote that it shifts to the left or right, and up or down. What is actually shifting? The vertex

Write the equation for a quadratic function in $y = a(x-h)^2 + k$ form that opens down, shifts left 3 and up 7. $y = -(x+3)^2 + 7$

↑

can be any (-) #

Write the equation for a quadratic function in $y = a(x-h)^2 + k$ form that opens up, shifts right 4 and down 2. $y = 1(x-4)^2 - 2$

↑

can be any (+) #

Write the equation for a quadratic function in $y = a(x-h)^2 + k$ form that opens up, and only shifts down 4. $y = 2x^2 - 4$

↑

can be any (+) #

Write the equation for a quadratic function in $y = a(x-h)^2 + k$ form that opens down and shifts to the left 8 spaces. $y = -(x+8)^2$

↑

can be any (-) #

Write the equation for a quadratic function in $y = a(x-h)^2 + k$ form that opens down and shifts up 7. $y = -x^2 + 7$

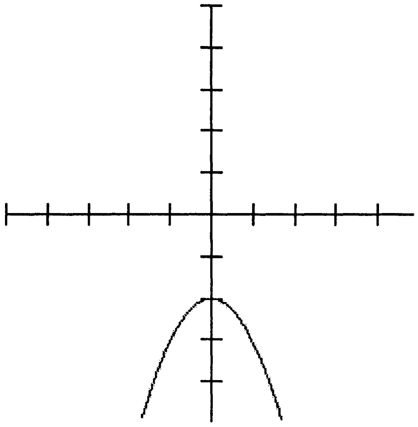
↑

can be any (-) #

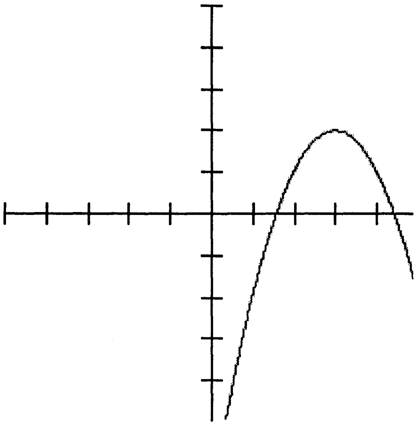
Is a quadratic function a one-to-one function? Why or why not? What does this tell you about the inverse of a quadratic function? No because it fails the horizontal line test. If a function is not one-to-one, the function does not have an inverse

Match the appropriate graph with its equation below. Explain why each of your solutions is true.

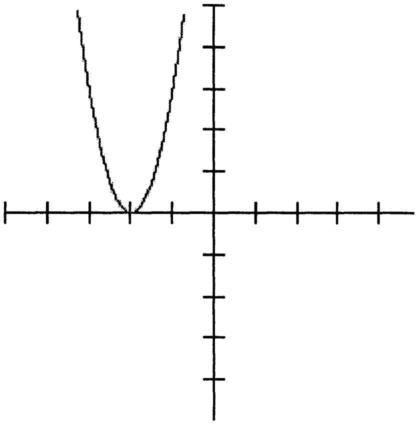
A



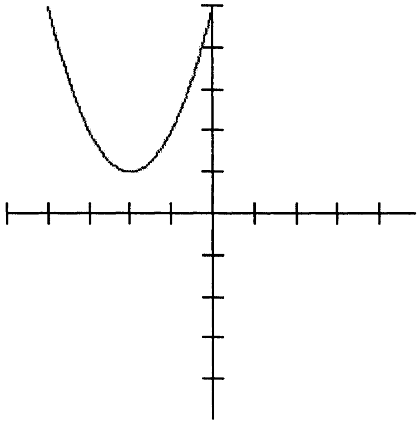
B



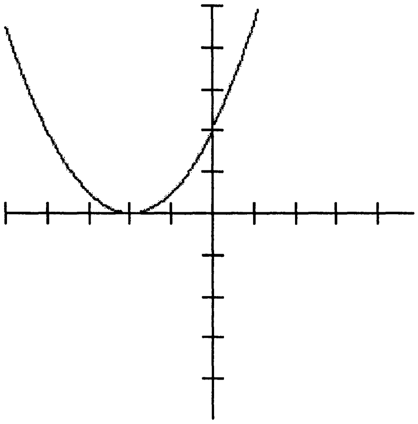
C



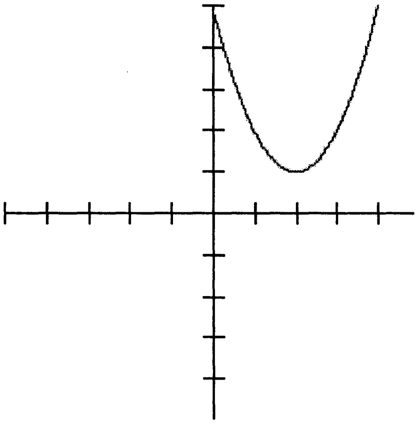
D



E



F



1) $f_{(x)} = \frac{1}{2}(x+2)^2$

E, the vertex of the function is (-2,0) and it opens up. It is also wide

2) $f_{(x)} = -x^2 - 2$

A, The vertex of the function is (0,-2) and it opens down

3) $f_{(x)} = (x+2)^2 + 1$

D, The vertex of the function is (-2, 1) and it opens up

4) $f_{(x)} = (x-2)^2 + 1$

F, The vertex of the function is (2,1) and it opens up.

5) $f_{(x)} = 3(x+2)^2$

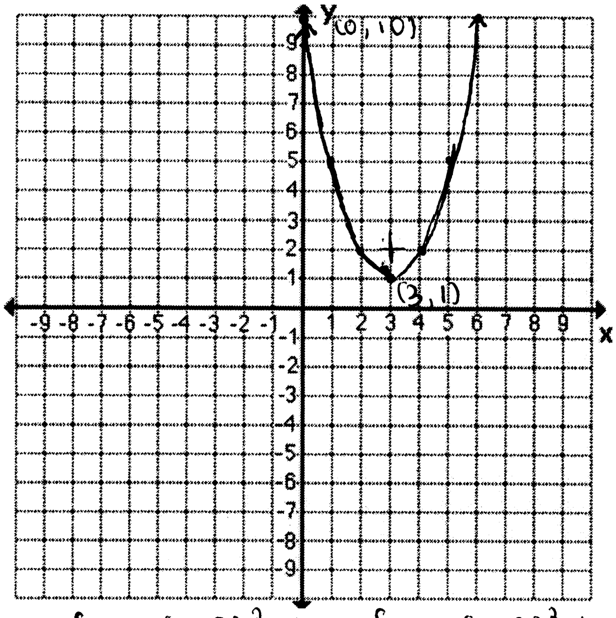
C, The vertex of the function is (-2,0) and it opens up. It is also narrow.

6) $f_{(x)} = -(x-3)^2 + 2$

B, The vertex of the function is (3,2) and the function opens down

Graph each of the following functions. You may need to use an axis of symmetry to graph some of these. **Label the vertex, y-intercept, and all x-intercepts.**

A) $f_{(x)} = (x-3)^2 + 1$



$f(x) = (x-3)^2 + 1$

$0 = (x-3)^2 + 1$

$\sqrt{(x-3)^2} = \sqrt{-1}$

$x - 3 = \pm i$

$x = 3 \pm i$

no x-int

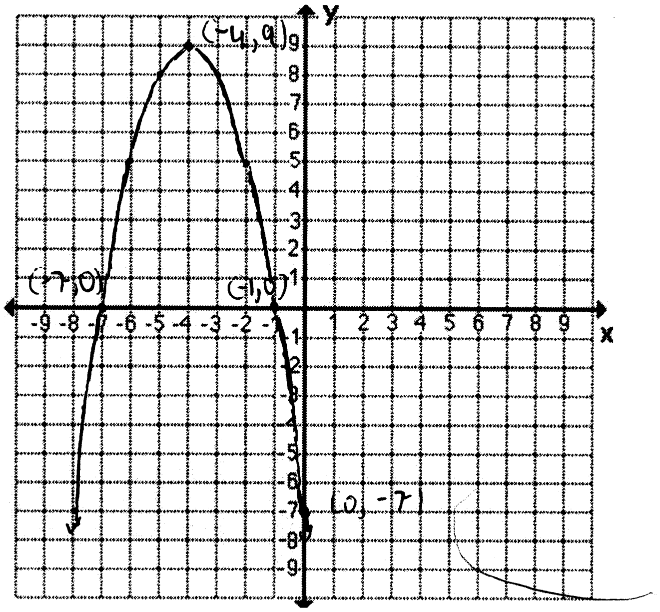
$f(0) = (0-3)^2 + 1$

$f(0) = (9) + 1$

$f(0) = 10$

$(0, 10)$ = y-int

B) $f_{(x)} = -(x+4)^2 + 9$



$f(x) = -(x+4)^2 + 9$

$0 = -(x+4)^2 + 9$

$\sqrt{(x+4)^2} = \sqrt{9}$

$x + 4 = \pm 3$

$x = -7, -1$

x-int = $(-7, 0)$ $(-1, 0)$

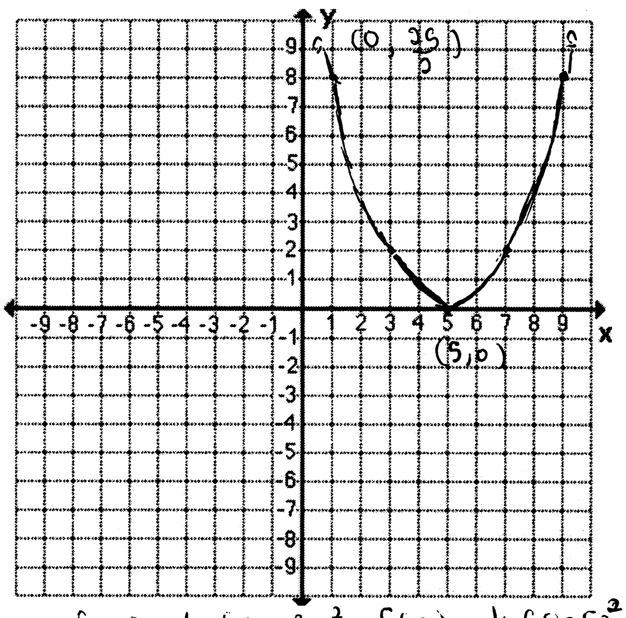
$f(0) = -(0+4)^2 + 9$

$f(0) = -16 + 9$

$f(0) = -7$

$(0, -7)$

C) $f_{(x)} = \frac{1}{2}(x-5)^2$



$f(x) = \frac{1}{2}(x-5)^2$

$0 = \frac{1}{2}(x-5)^2$

$\sqrt{(x-5)^2} = \sqrt{0}$

$x - 5 = 0$

$x = 5$

x-int $(5, 0)$

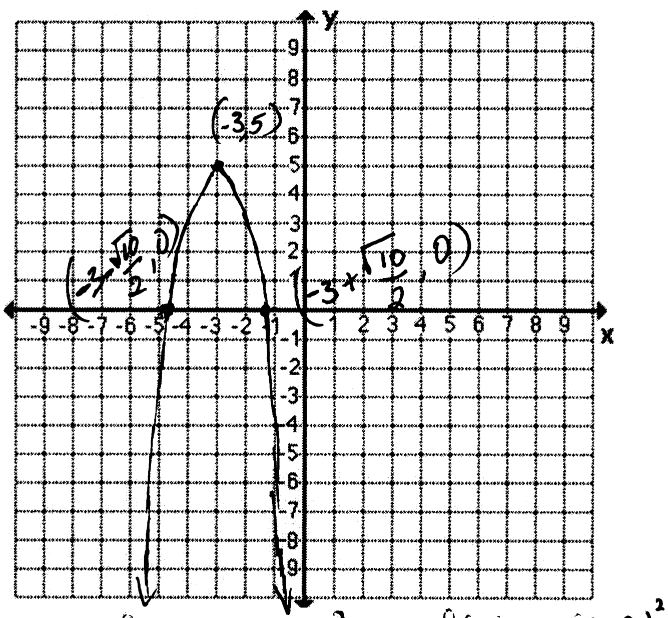
$f(0) = \frac{1}{2}(0-5)^2$

$f(0) = \frac{1}{2}(25)$

$f(0) = \frac{25}{2}$

$(0, \frac{25}{2})$ y-int

D) $f_{(x)} = -2(x+3)^2 + 5$



$f(x) = -2(x+3)^2 + 5$

$0 = -2(x+3)^2 + 5$

$\sqrt{\frac{5}{2}} = \sqrt{(x+3)^2}$

$x + 3 = \pm \frac{\sqrt{10}}{2}$

$x = -3 \pm \frac{\sqrt{10}}{2}$

$f(0) = -2(0+3)^2 + 5$

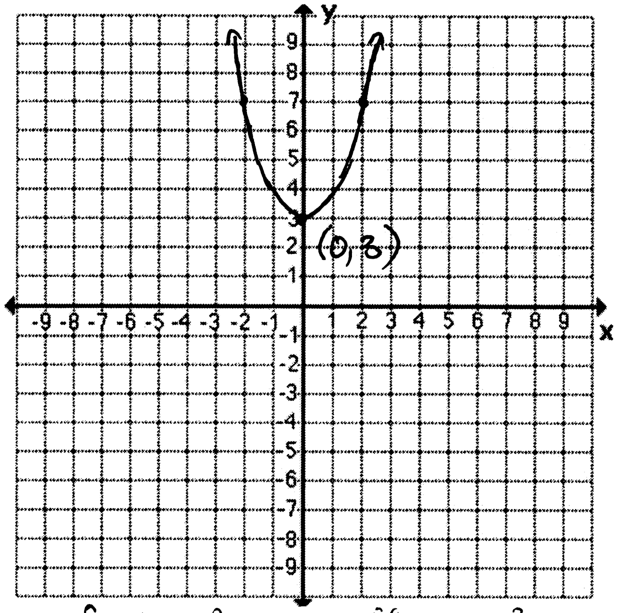
$f(0) = -2(9) + 5$

$f(0) = -13$

y-int $(0, \overset{245}{-13})$

$(-3 - \frac{\sqrt{10}}{2}, 0)$ $(-3 + \frac{\sqrt{10}}{2}, 0)$ x-int

E) $f_{(x)} = x^2 + 3$



$f(x) = x^2 + 3$

$0 = x^2 + 3$

$\sqrt{x^2} = \sqrt{-3}$

$x = \pm\sqrt{3}\, i$

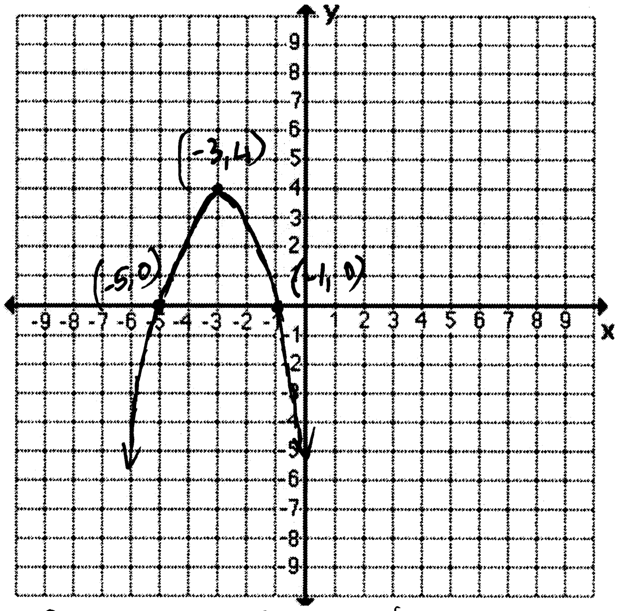
no x-int

$f(0) = (0)^2 + 3$

$f(0) = 3$

y-~~int~~ (0, 3)

F) $f_{(x)} = -(x+3)^2 + 4$



$f(x) = -(x+3)^2 + 4$

$0 = -(x+3)^2 + 4$

$\sqrt{(x+3)^2} = \sqrt{4}$

$x + 3 = \pm 2$

$x = -3 \pm 2$

$x = -5, -1$

x-int (-5, 0) (-1, 0)

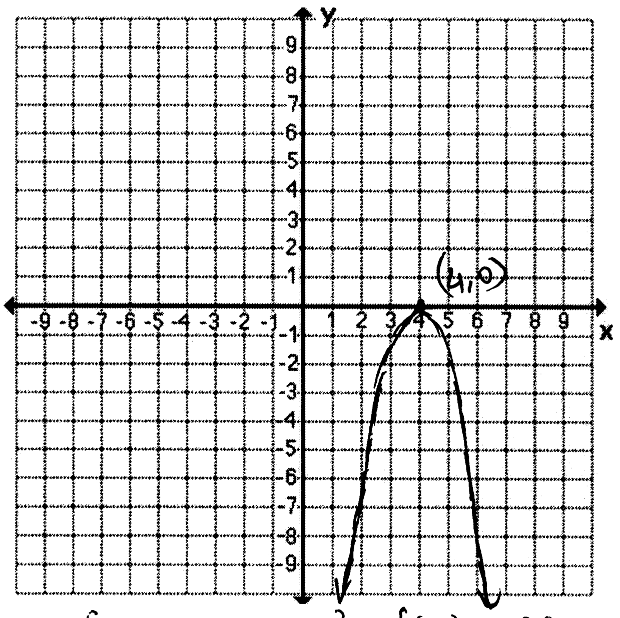
$f(0) = -(0+3)^2 + 4$

$f(0) = -9 + 4$

$f(0) = -5$

y-int (0, -5)

G) $f_{(x)} = -2(x-4)^2$



$f(x) = -2(x-4)^2$

$0 = -2(x-4)^2$

$\sqrt{0} = \sqrt{(x-4)^2}$

$0 = x - 4$

$4 = x$

x-int (4, 0)

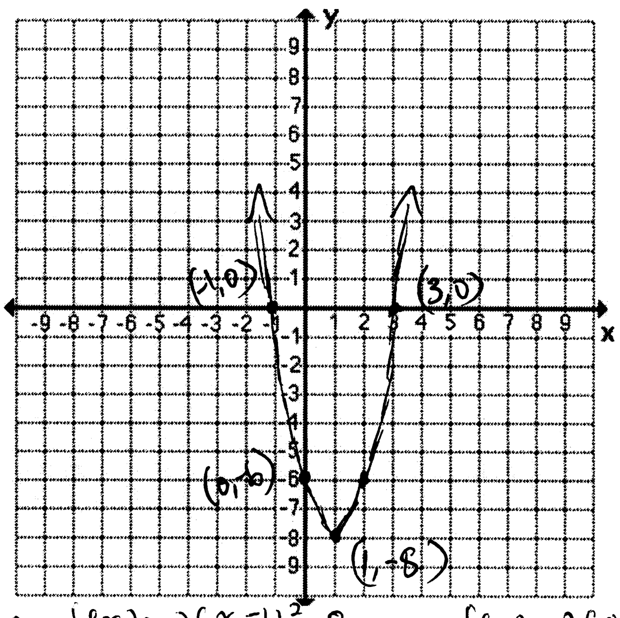
$f(0) = -2(0-4)^2$

$f(0) = -2(16)$

$f(0) = -32$

y-int (0, -32)

H) $f_{(x)} = 2(x-1)^2 - 8$



$f(x) = 2(x-1)^2 - 8$

$0 = 2(x-1)^2 - 8$

$2(x-1)^2 = 8$

$\sqrt{(x-1)^2} = \sqrt{4}$

$x - 1 = \pm 2$

$x = 1 \pm 2$

$x = 3, -1$

x-int

(3, 0)

(1, 0)

$f(0) = 2(0-1)^2 - 8$

$f(0) = 2 - 8 = -6$

y-int = (0, -6)

The quadratic function given by the equation $f_{(x)} = 3(x-2)^2 + 6$ has an axis of symmetry of $x = 2$ **.**

The quadratic function given by the equation $f_{(x)} = -3(x+6)^2 - 4$ has an axis of symmetry of $x = -6$ **.**

The quadratic function given by the equation $f_{(x)} = a(x-h)^2 + k$ has an axis of symmetry of $x = h$ **.**

Considering your answers to the previous questions, we can conclude that the axis of symmetry for any quadratic function is given by the x value of the vertex **.**

Why does the axis of symmetry look as though we are saying x equals a number ($x = \#$)?

because $x = \#$ is the equation for a vertical line.

Why is it sometimes necessary to graph a quadratic function using the axis of symmetry?

Sometimes you may not have any x-intercepts to use for the graph.

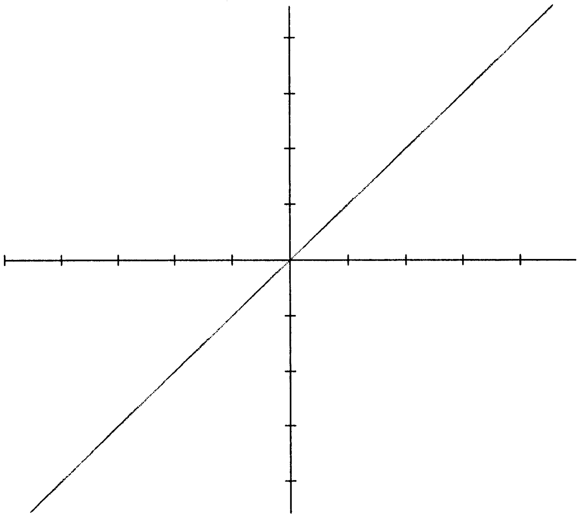
Absolute Value Functions

In order to graph an absolute value function, you will be using many of the same methods you did for quadratics. The standard form of an absolute value function is nearly identical to that of a quadratic function.

$$f_{(x)} = a|x-h| + k$$

The standard graph by which we translate absolute value functions comes from the equation of the diagonal line $y = x$.

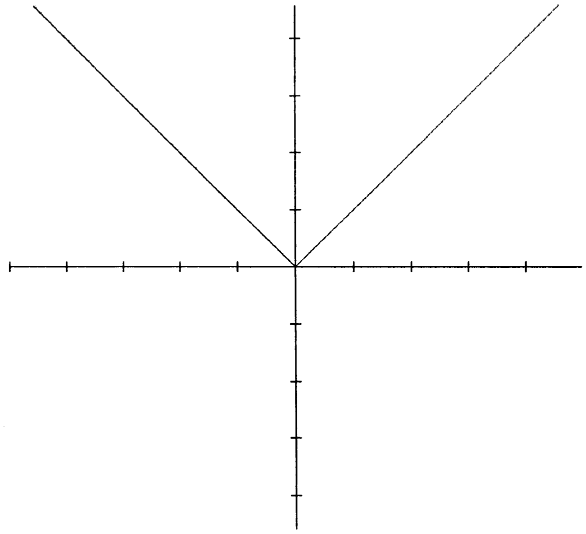
$y = x$



This is the graph of the function $y = x$. ***In this case, the x and y values of coordinates are identical. For example, (-3,-3). You can see the x and y values are the same.***

Now, lets take a look at what happens when I want the absolute value of x.

$y = |x|$

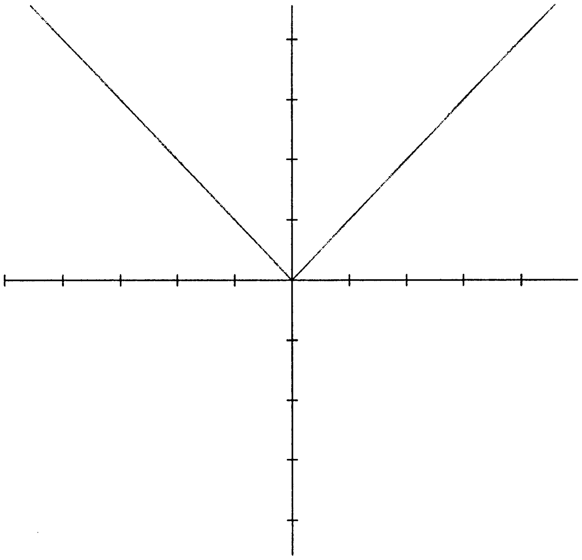


If the graph of $y = x$ ***above is*** $f_{(x)}$, ***the function to the left is*** $|f_{(x)}|$. ***We know that the absolute value of a number cannot be negative. If we take the absolute value of*** $f_{(x)}$, ***it would cause the left portion of the graph above to reflect above the x axis. Now as you can see, all y values of the function are positive. This is where the graph of the absolute value of x comes from.***

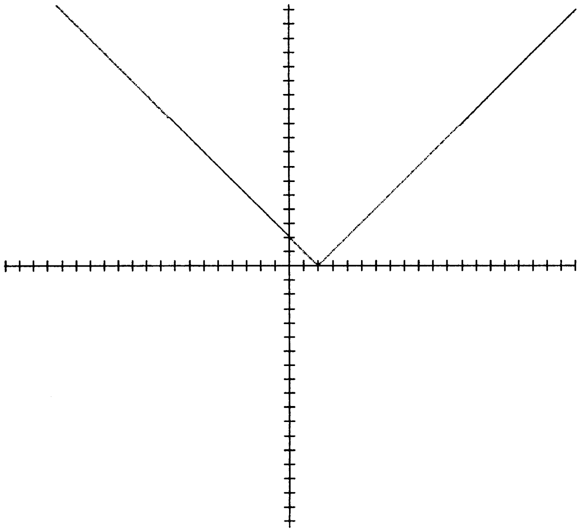
As we look at the following absolute value functions, you will notice how similar they are to quadratic functions. The vertex of an absolute value function is also given by (h,k). Horizontal and vertical shifts are identical, as well as the effect the value of a has on the graph. The rules for finding the range and domain of an absolute value function are also the same as a quadratic. Sometimes, an axis of symmetry must be used to graph your function. Intercepts are found by substituting zero for either x or y, and solving for the remaining variable.

$$f_{(x)} = a|x-h|+k$$

$f_{(x)} = |x|$

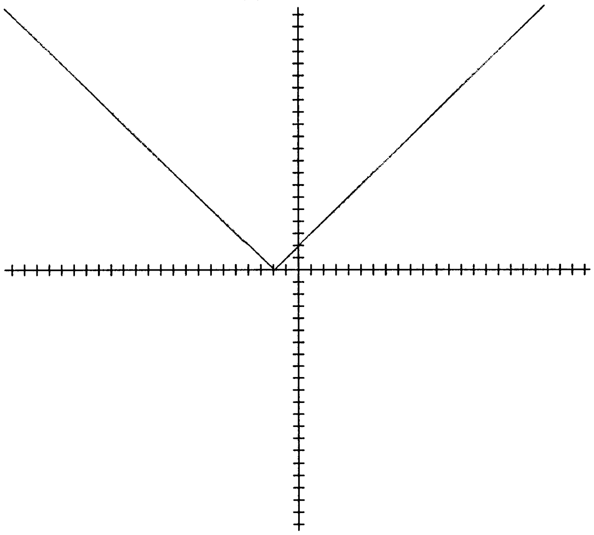


$f_{(x)} = |x-2|$



The graph of this function shifts to the right 2.

$f_{(x)} = |x+2|$

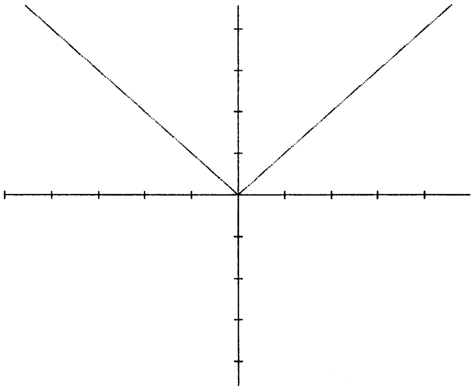


The graph of this function shifts to the left 2.

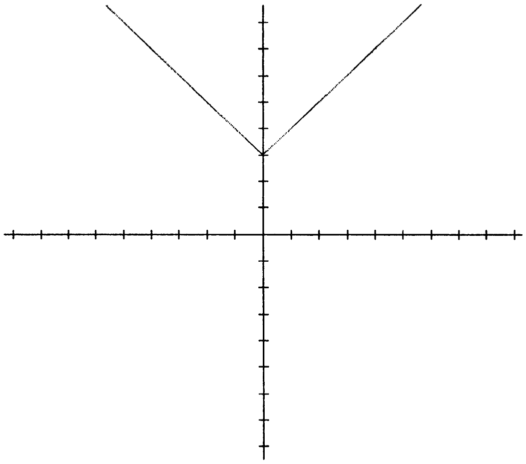
Once again, notice that the value of h determines the horizontal shift of the function. If the function is defined as $f_{(x)}$, the graph on the left is $f_{(x-2)}$, while the graph on the right is $f_{(x+2)}$.

$$f_{(x)} = a|x-h|+k$$

$$f_{(x)} = |x|$$

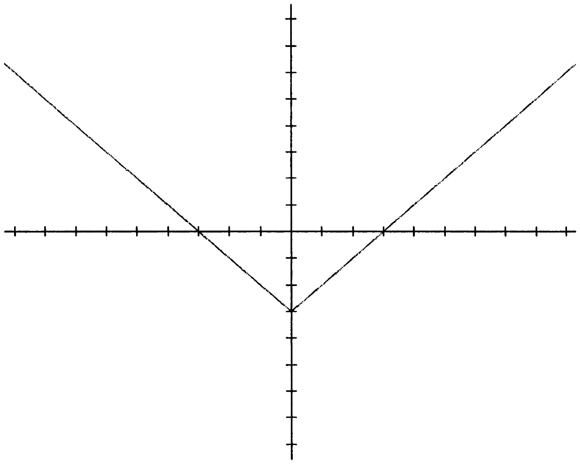


$$f_{(x)} = |x|+3$$



The graph of this function shifts up 3.

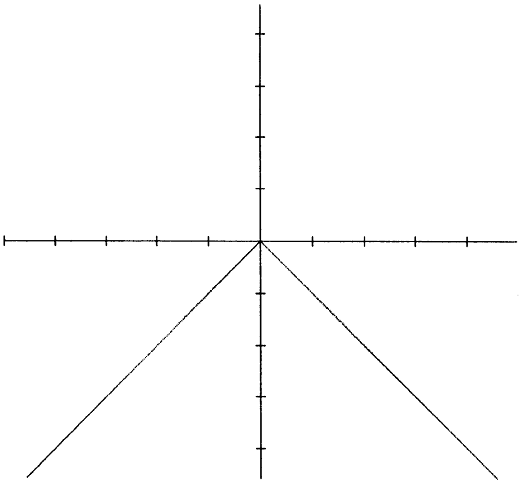
$$f_{(x)} = |x|-3$$



The graph of this function shifts down 3.

The value of k***, for an absolute value function in standard form determines the vertical shift of the function. As before, if the function is simply defined as*** $f_{(x)}$***, we are looking at*** $f_{(x)}+3$ ***and*** $f_{(x)}-3$ ***respectively.***

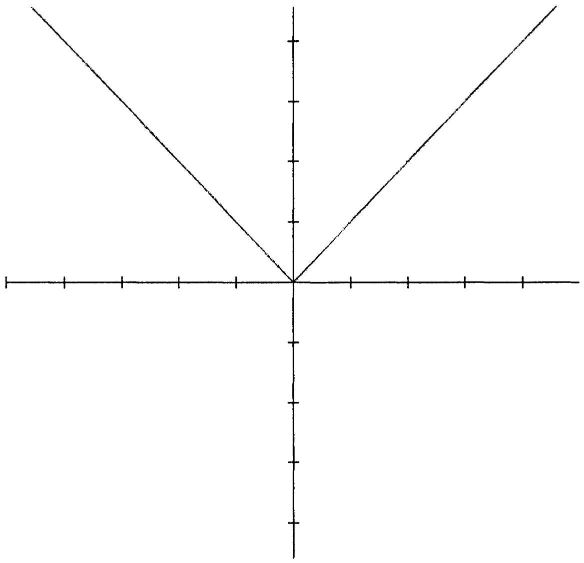
$$f_{(x)} = -|x|$$



On the left we have the opposite of the parent function. In this example, the value of a***, in the standard form is -1. A negative reflects the graph of the function about the horizontal axis. This is read as the opposite of the absolute value of x. If the parent function given is referred to as*** $f_{(x)}$***, this function is*** $-f_{(x)}$***.***

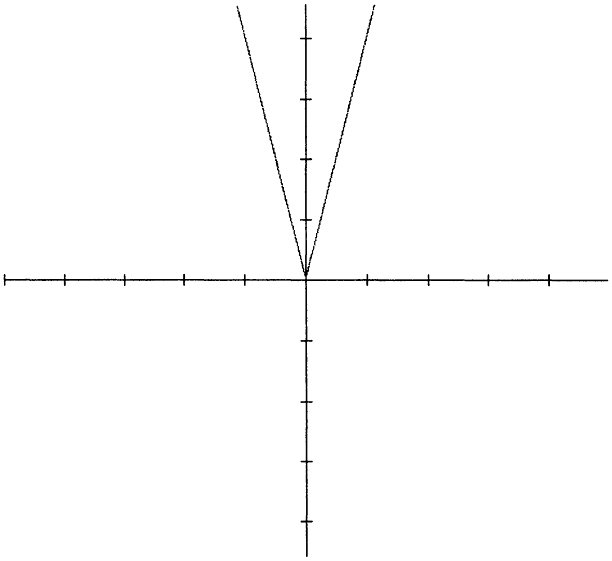
$$f_{(x)} = a|x-h|+k$$

$$f_{(x)} = |x|$$

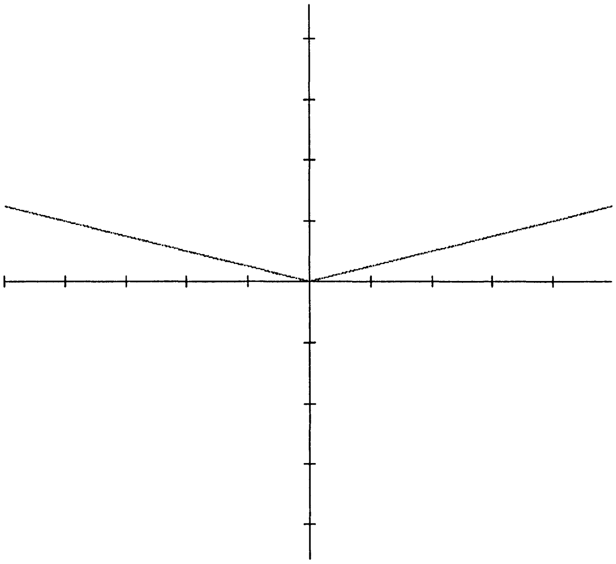


Here we will see how the value of a in an absolute value function in standard form affects the graph of the function. To illustrate this, we will look at the following graphs that have their vertices on the origin.

$$f_{(x)} = 4|x|$$



$$f_{(x)} = \frac{1}{4}|x|$$



This graph seems very narrow, but what is actually happening, is the value of the function is increasing very rapidly. The y values are increasing at 4 times their normal rate. The rapid increase causes the graph to appear narrow.

This graph is wider than the parent function. In this case, the y values of the function are increasing at ¼ their normal rate, causing a more gradual increase.

If the value of the leading coefficient is a whole number, the y values of the graph will increase rapidly causing a narrow graph and more extreme slope. If the leading coefficient is a fraction, the y values of the function will increase mildly, yielding a more gradual slope.

Describe the movement of each of the following absolute value functions. Describe how the graph of the function opens and if there is any horizontal or vertical movement. Be sure to tell identify how many spaces it moves, for example: ***This graph opens up, and shifts left 6, up 3.***

A) $f_{(x)} = 3|x-4|+1$

Opens up
shifts right 4,
up 1

B) $f_{(x)} = -|x+1|+6$

opens down
shifts left 1,
up 6

C) $f_{(x)} = |x|+4$

opens up
shifts up 4

D) $f_{(x)} = |x+5|-2$

opens up
shifts left 5
down 2

E) $f_{(x)} = |x-3|$

opens up
shifts right 3

F) $f_{(x)} = -3|x+1|+3$

opens down
shifts left 1,
up 3

G) $f_{(x)} = \frac{1}{2}|x-3|+2$

opens up
shifts right 3
up 2

H) $f_{(x)} = -|x+6|$

opens down
shifts left 6

I) $f_{(x)} = \frac{2}{3}|x|-4$

opens up
shifts down 4

State the range and domain for each of the following.

A) $f_{(x)} = 3|x-4|+1$

vertex $(4,1)$
R: $[1, \infty)$
D: $(-\infty, \infty)$

B) $f_{(x)} = -|x+1|+6$

vertex $(-1, 6)$
R: $(-\infty, 6]$
D: $(-\infty, \infty)$

C) $f_{(x)} = |x|+4$

vertex $(0, 4)$
R: $[4, \infty)$
D: $(-\infty, \infty)$

D) $f_{(x)} = |x+5|-2$

Vertex: $(-5, -2)$
R: $[-2, \infty)$
D: $(-\infty, \infty)$

E) $f_{(x)} = |x-3|$

vertex: $(3, 0)$
R: $[0, \infty)$
D: $(-\infty, \infty)$

F) $f_{(x)} = -3|x+1|+3$

vertex: $(-1, 3)$
R: $(-\infty, 3]$
D: $(-\infty, \infty)$

G) $f_{(x)} = \frac{1}{2}|x-3|+2$

Vertex $(3, 2)$
R: $[2, \infty)$
D: $(-\infty, \infty)$

H) $f_{(x)} = -|x+6|$

vertex $(-6, 0)$
R: $(-\infty, 0]$
D: $(-\infty, \infty)$

I) $f_{(x)} = \frac{2}{3}|x|-4$

vertex $(0, -4)$
R: $[-4, \infty)$
D: $(-\infty, \infty)$

Find the vertex of each of the following absolute value functions.

A) $f_{(x)} = -|x+3|$

$(-3, 0)$

B) $f_{(x)} = 3|x-2|-4$

$(2, -4)$

C) $f_{(x)} = \frac{1}{2}|x|-2$

$(0, -2)$

D) $f_{(x)} = |x-2|+3$

$(2, 3)$

E) $f_{(x)} = 2|x+6|+9$

$(-6, 9)$

F) $f_{(x)} = -|x-4|+7$

$(4, 7)$

Solve each of the following absolute value equations. ***This is what you will need to do to find the x intercepts of absolute value functions. Remember, first isolate the absolute value, then set up two separate equations to find your solutions.***

$$|x+3|-6=0$$

$$|x+3| \quad =6$$

Set equation equal to 6 and solve by subtracting 3 to both sides.

$$x+3=6$$

$$x=3$$

and

$$x+3=-6$$

$$x=-9$$

Set equation equal to -6 and solve by subtracting 3 to both sides.

So the two solutions are 3 and -9. These would be the x intercepts of the graph of the function $y = |x+3|-6$***. Watch for abnormalities. Iif the absolute value equals a negative number, you cannot create two problems. If this happens, there will be no solutions to the problem, which in terms of the graph of the function, tells you that there are no x intercepts.***

A) $|x-4|-2=0$

$|x-4| = 2$

$x-4 = 2 \qquad x-4 = -2$

$x = 6 \qquad x = 2$

$x = \{2, 6\}$

B) $2|x+4|-12=0$

$2|x+4| = 12$

$|x+4| = 6$

$x+4 = 6 \qquad x+4 = -6$

$x = 2 \qquad x = -10$

$x = \{-10, 2\}$

C) $-\frac{1}{2}|x|+1=0$

$(-2)-\frac{1}{2}|x| = -1(-2)$

$|x| = 2$

$x = 2 \qquad x = -2$

$x = \{\pm 2\}$

D) $-2|x-3|=0$

$\frac{-2|x-3|}{-2} = \frac{0}{-2}$

$|x-3| = 0$

$x-3=0$

$x = 3$

E) $|x-6|-5=0$

$|x-6| = 5$

$x-6 = 5 \qquad x-6 = -5$

$x = 11 \qquad x = 1$

$x = \{1, 11\}$

F) $-3|x-1|+2=0$

$\frac{-3|x-1|}{-3} = \frac{-2}{-3}$

$|x-1| = \frac{2}{3}$

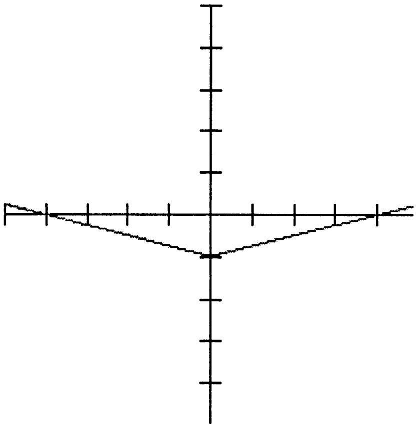
$x-1 = \frac{2}{3} \qquad x-1 = -\frac{2}{3}$

$x = 1\,^2/_3 \qquad x = {}^1/_3$

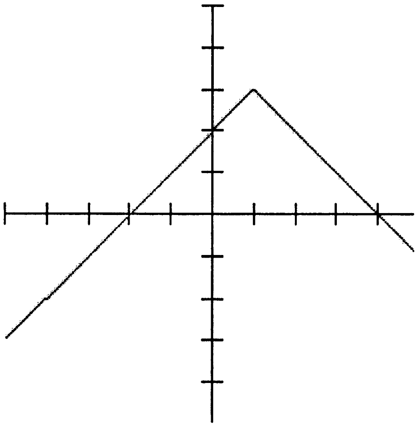
$x = \{{}^1/_3, {}^5/_3\}$

Match the appropriate graph with its equation below. Explain why each of your solutions is true.

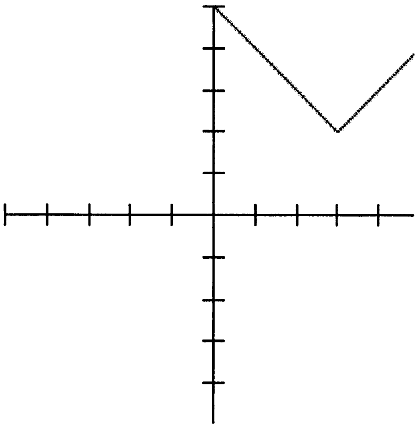
A



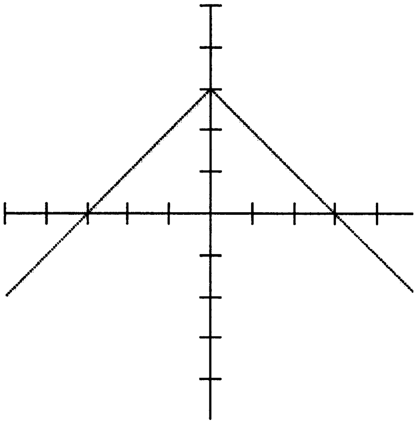
B



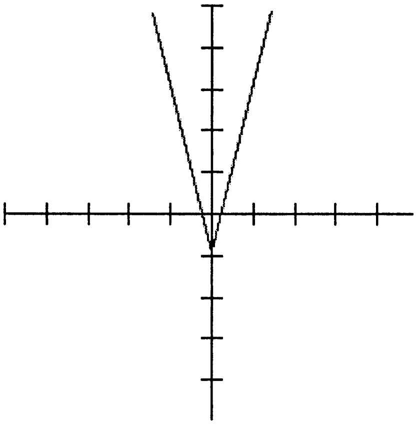
C



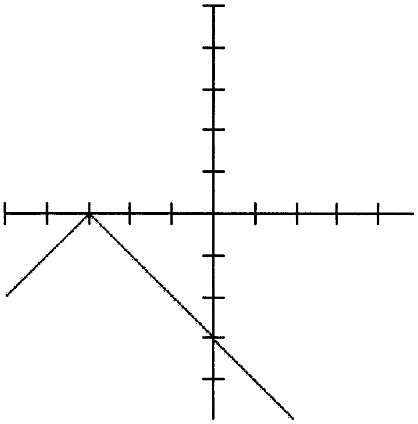
D



E



F



1) $f_{(x)} = 4|x| - 1$

E , vertex of the function is at (0, -1) , opens up and is very narrow

2) $f_{(x)} = \frac{1}{4}|x| - 1$

A , vertex of the function is at (0, -1) , opens up and is very wide.

3) $f_{(x)} = -|x| + 3$

D, vertex of the function is at (0, 3) and it opens down,

4) $f_{(x)} = |x-3| + 2$

C , the vertex of the function is (3, 2) it opens up.

5) $f_{(x)} = -|x-1| + 3$

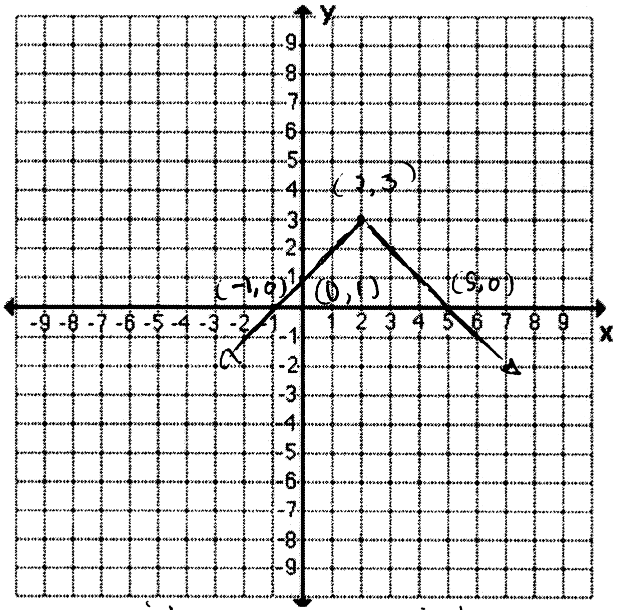
B, the vertex of the function is (1, 3) and it opens down.

6) $f_{(x)} = -|x+3|$

F, the vertex of the function is at (-3, 0) and it opens down.

Graph each of the following functions. You may need to use an axis of symmetry to graph some of these. Label the vertex, y-intercept, and all x-intercepts. Remember, to find the x intercepts of an absolute value function you will need to set the function equal to zero and solve an absolute value equation.

A) $f_{(x)} = -|x-2|+3$



x-int

$0 = -|x-2| + 3$

$-3 = -|x-2|$

$x-2 = 3 \quad x-2 = -3$

$x = 5 \quad x = 1$

(5,0) (1,0)

y-int

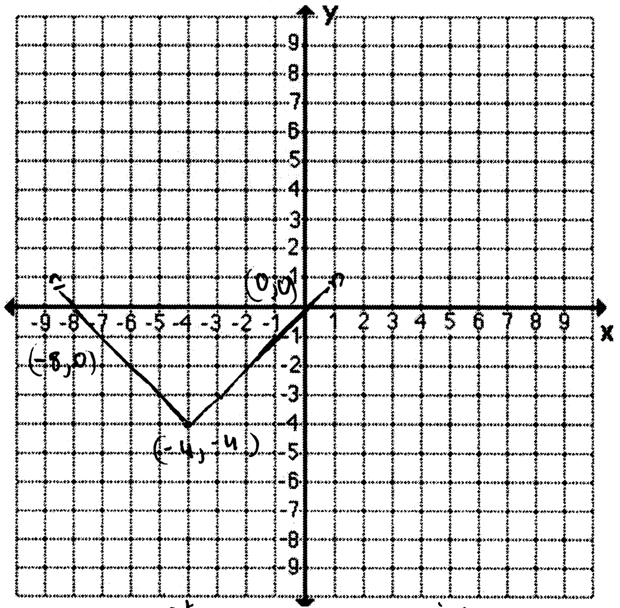
$f(0) = -|0-2| + 3$

$= -|-2| + 3$

$= -2 + 3$

$= 1$

B) $f_{(x)} = |x+4|-4$



x-int

$0 = |x+4| - 4$

$4 = |x+4|$

$x+4 = 4 \quad x+4 = -4$

$x = 0 \quad x = -8$

(0,0)(-8,0)

y-int

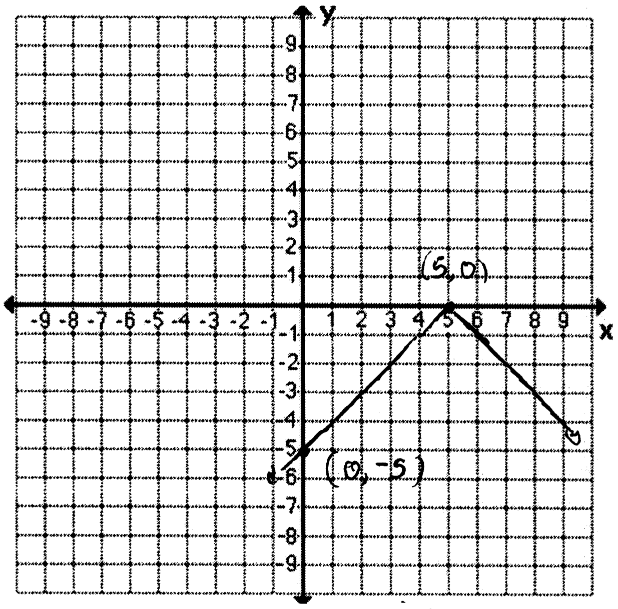
$f(0) = |0+4| - 4$

$= 0+4 - 4$

$= 0$

(0,0)

C) $f_{(x)} = -|x-5|$



x-int

$0 = -|x-5|$

$0 = |x-5|$

$x-5 = 0$

$x = 5$ (5,0)

y-int

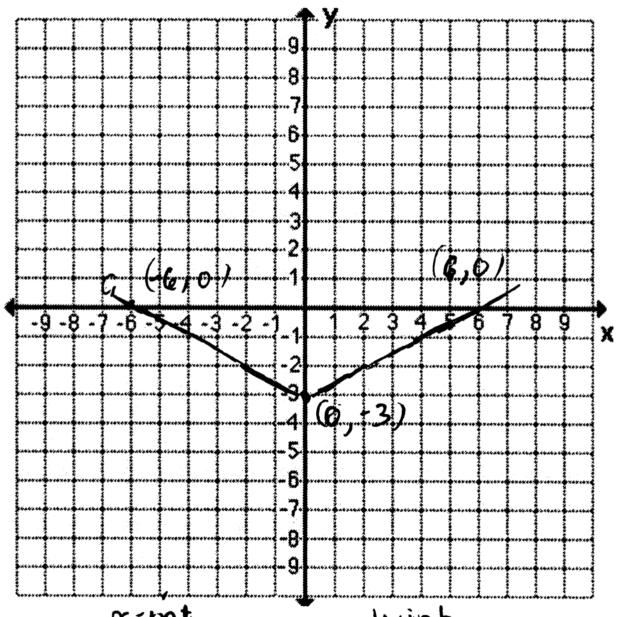
$f(0) = -|0-5|$

$= -|-5|$

$= -5$

(0,-5)

D) $f_{(x)} = \frac{1}{2}|x|-3$



x-int

$0 = \frac{1}{2}|x| - 3$

$3 = \frac{1}{2}|x|$

$6 = |x|$

$x = 6 \quad x = -6$

(6,0) (-6,0)

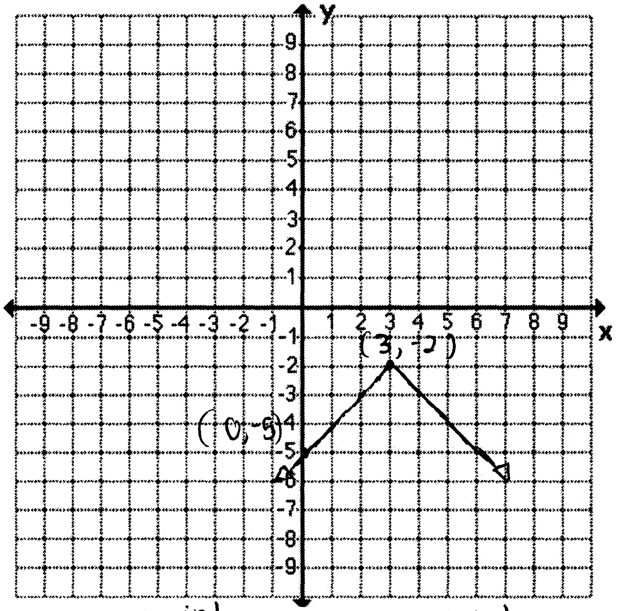
y-int

$f(0) = \frac{1}{2}|0| - 3$

$= -3$

(0,-3)

E) $f_{(x)} = -|x-3| - 2$



x-int

$0 = -|x-3| - 2$

$-2 = |x-3|$

no solution

no x-intercepts

y-int

$f(0) = -|0-3| - 2$

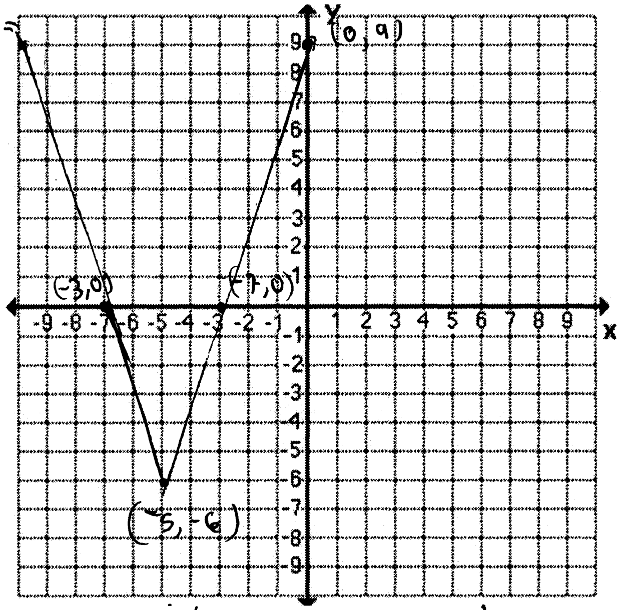
$= -|-3| - 2$

$= -3 - 2$

$= -5$

$(0, -5)$

F) $f_{(x)} = 3|x+5| - 6$



x-int

$0 = 3|x+5| - 6$

$6 = 3|x+5|$

$2 = |x+5|$

$x+5 = 2 \quad x+5 = -2$

$x = -3 \quad x = -7$

$(-3, 0) \; (-7, 0)$

y-int

$f(0) = 3|0+5| - 6$

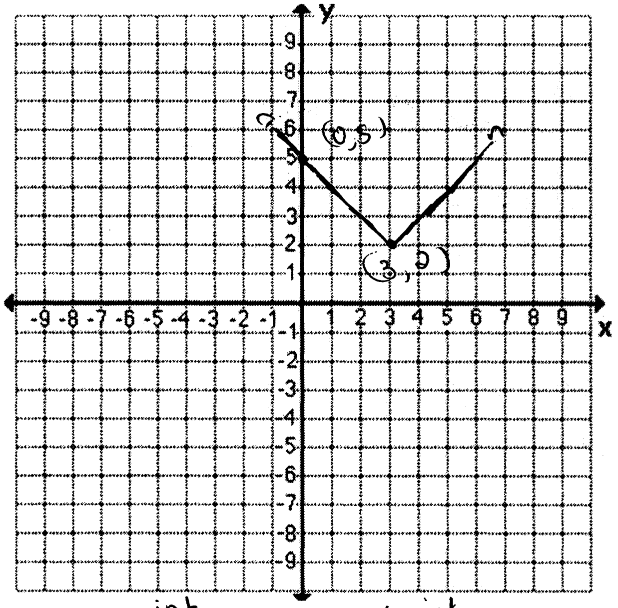
$= 3(5) - 6$

$= 15 - 6$

$= 9$

$(0, 9)$

G) $f_{(x)} = |x-3| + 2$



x-int

$0 = |x-3| + 2$

$-2 = |x-3|$

no solution

no x-int

y-int

$f(0) = |0-3| + 2$

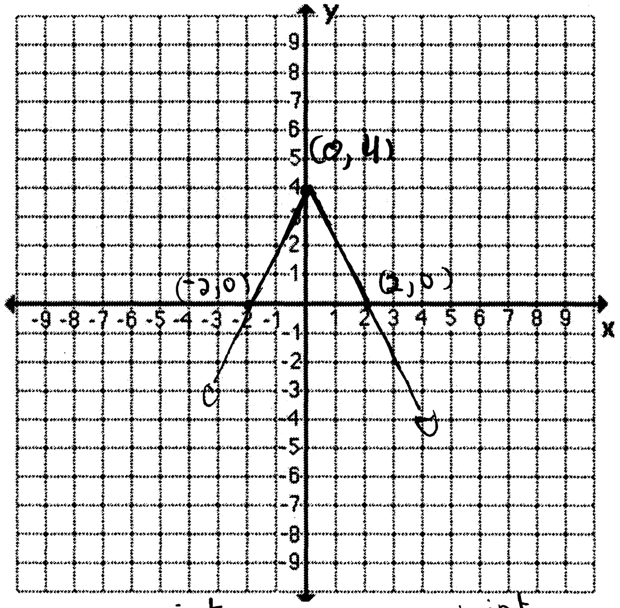
$= |-3| + 2$

$= (3) + 2$

$= 5$

$(0, 5)$

H) $f_{(x)} = -2|x| + 4$



x-int

$0 = -2|x| + 4$

$-4 = -2|x|$

$2 = |x|$

$x = 2, -2$

$(2, 0) \; (-2, 0)$

y-int

$f(0) = -2|0| + 4$

$= 0 + 4$

$= 4$

$(0, 4)$

Translations of Functions

We will now look at graphing a function without actually knowing the equation. Based on the graph of a function, it will be possible to shift, or translate the graph in any manner indicated.

For example, if given the picture of a graph and told "This is the graph of the function $f_{(x)}$." Proceed to first identify the coordinate of any vertex seen. These will serve as a guide for the graph of the function's translation.

To graph the function of $f_{(x+6)}$, the function will need to shift to the left 6 spaces. To accomplish this, subtract 6 from all x values in the original function. The results will be the coordinates for the new graph. Likewise, to graph $f_{(x-4)}$, this function will need to shift to the right 4 spaces, so add 4 to all x values.

In order to graph $f_{(x)}+5$, the function will shift up 5 spaces, requiring that 5 be added to all y values. If asked to graph $f_{(x)}-3$, the will function shift down 3 spaces, meaning subtract 3 from all y values.

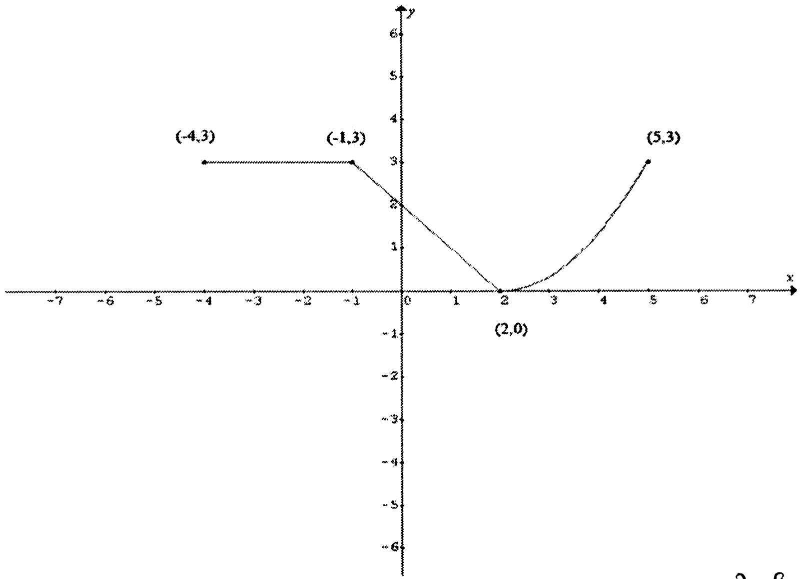
If the number is in the parenthesis, the function is shifting using P.L.N.R.. If the number is after the $f_{(x)}$, simply shift as indicated, + says shift up, - says shift down.

Any number in front of the $f_{(x)}$ will affect the scale of the function. This means it will affect the rate at which the function grows. When graphing, for example, $-f_{(x)}$, change the sign of all y values on the graph of the function. This will cause the graph of the function to flip upside down. A number other than -1 can also be used. Lets say we need to graph $3f_{(x)}$, this means the actual curve will increase 3 times as fast. It will therefore, be necessary to multiply all y values by 3. This will result in the coordinates for the new function. If the 3 were grouped with the x such as $f_{(3x)}$, the horizontal change is the inverse of what it appears to be. So instead of multiplying x values by 3, divide by 3.

When graphing $f_{(-x)}$, take the opposite of the x values of the function. This will cause the graph of the function to flip along a vertical axis.

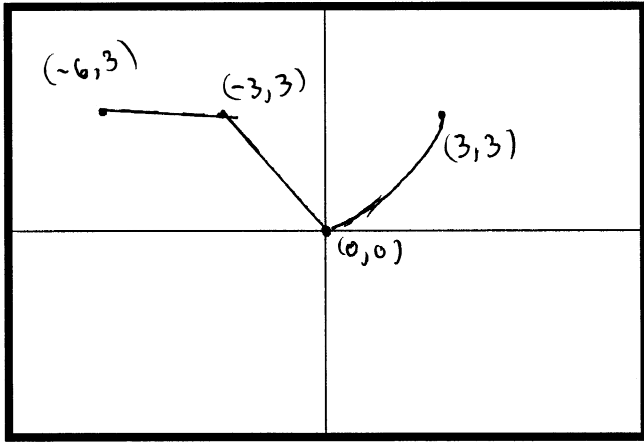
Combinations of these rules will be encountered throughout your study of functions, for example, to shift right 3 and up 6. Just stick with the rules and the graph will be translated to its new location. If faced with a problem such as $2f_{(x)}+3$, follow the order of operations. Multiply all y values by 2 first, then add 3 to each. Referring to the previous two topics, quadratic functions and absolute value functions, you will find references to these rules and examples throughout.

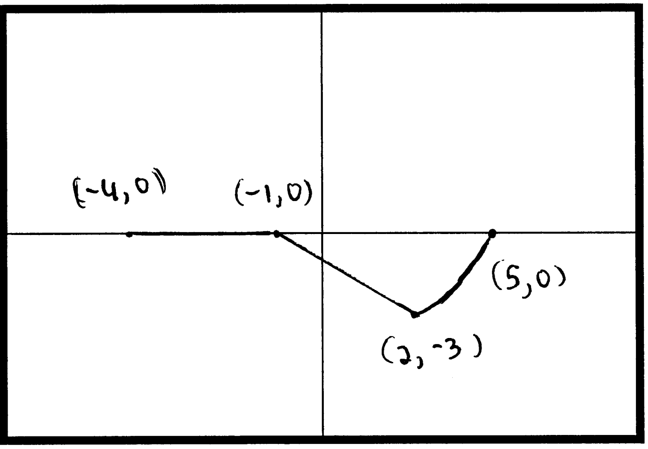
The following is the graph of the function $f_{(x)}$. Use this to graph each function for letters A-D.



A) $f_{(x+2)}$ Subtract 2 from all x-values shift left 2

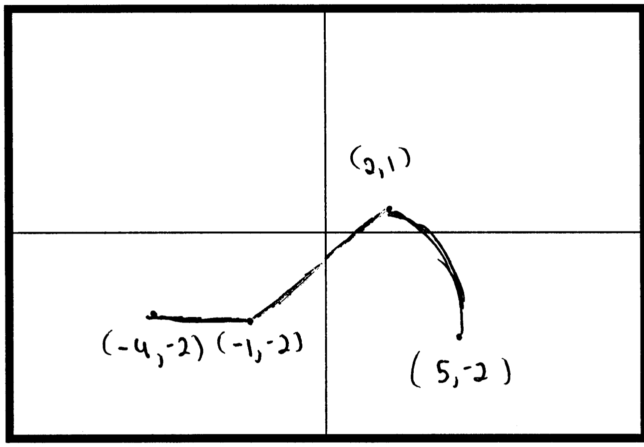
B) $f_{(x)}-3$ Subtract 3 from all y-values shift down 3

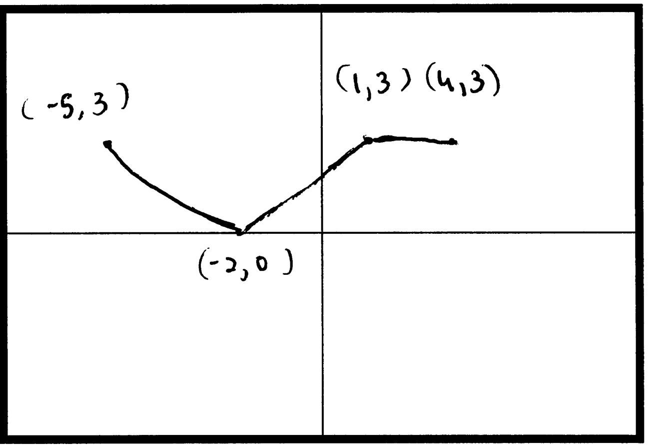




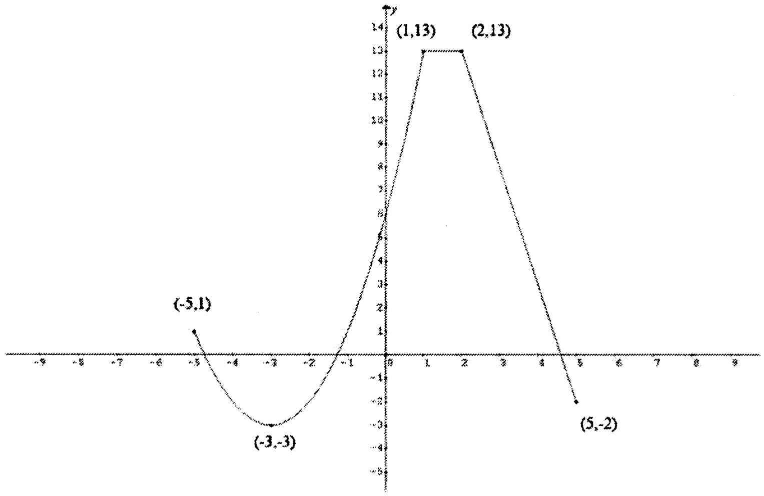
C) $-f_{(x)}+1$ multiply all y-values by -1 first, then add 1

D) $f_{(-x)}$ multiply all x-values by -1



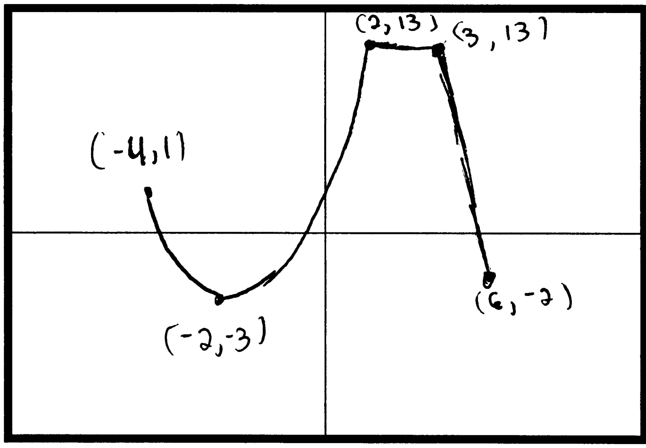


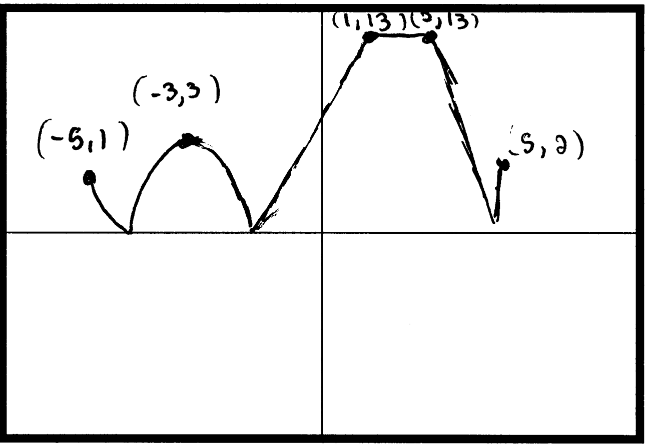
The following is the graph of the function $f_{(x)}$. Use this to graph each function for letters E-H.



E) $f_{(x-1)}$ Shift right 1
Add all x-values by 1

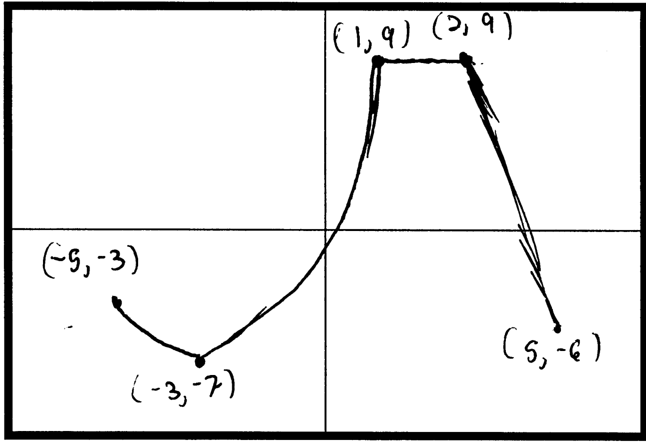
F) $|f_{(x)}|$ reflect function above y-axis

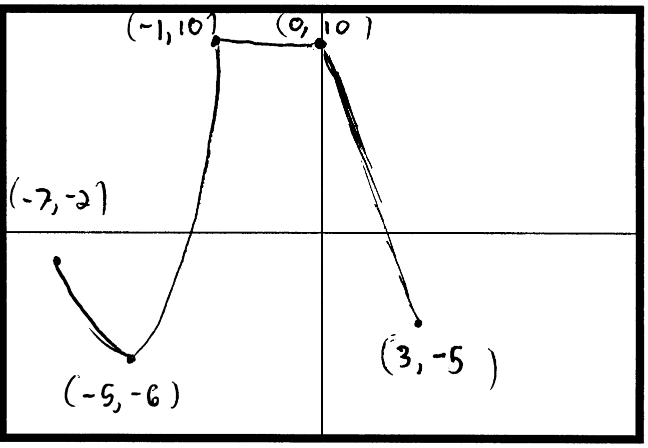




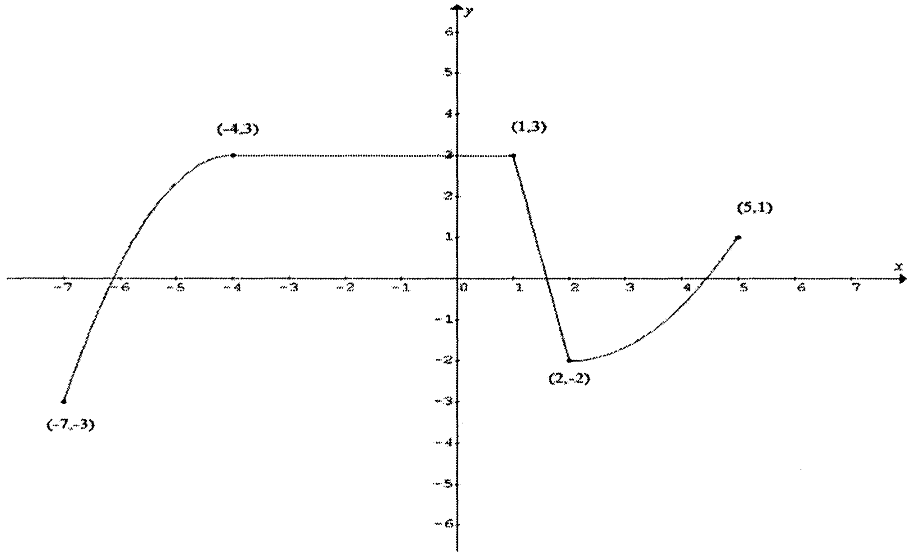
G) $f_{(x)}-4$ Shift down 4
subtract 4 from all y-values

H) $f_{(x+2)}-3$ Shift left 2, down 3
subtract 2 from all x-value, subtract 3 from all y-values



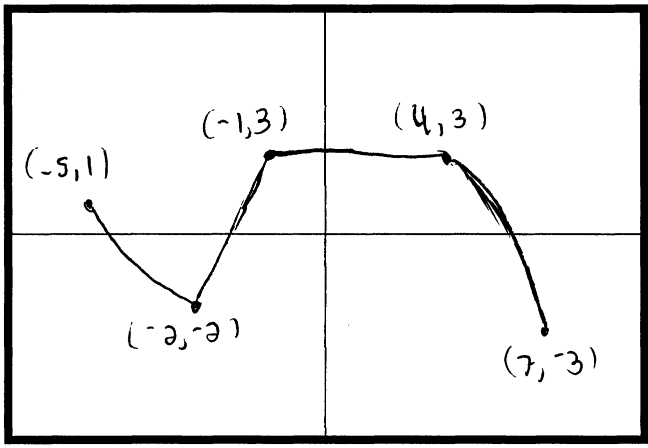


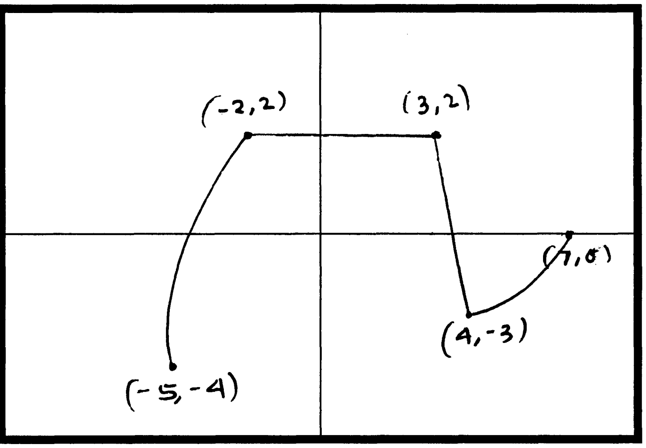
The following is the graph of the function $f_{(x)}$. Use this to graph each function for letters I-L.



I) $f_{(-x)}$ multiply all x-values by -1

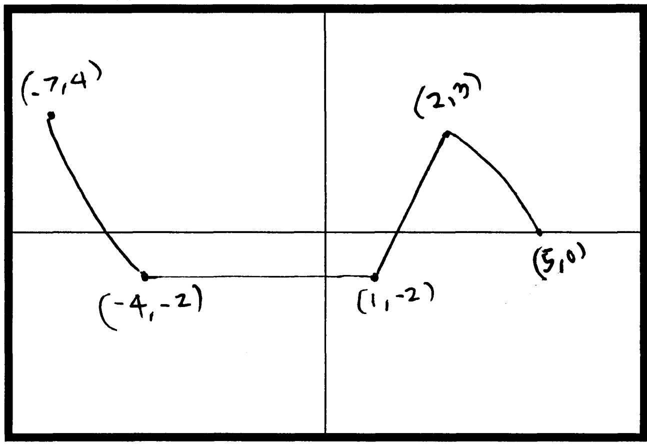
J) $f_{(x-2)}-1$ shift right 2, down 1 add 2 to all x-values, subtract 1 from all y-values

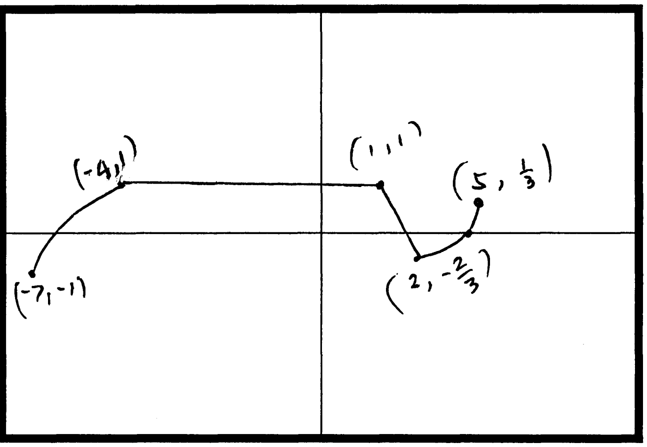




K) $-f_{(x)}+1$ multiply all y-values by -1 Then add 1

L) $\frac{1}{3}f_{(x)}$ multiply all y values by 1/3





Radical Functions

For radical functions we will use the equation $f_{(x)} = a\sqrt{x-h}+k$ to denote the standard form of the equation. Be aware, that the variable x may have a coefficient from time to time. Follow the standard procedure to find the x and y intercepts of any radical function. Set the x or y equal to zero, depending on which one you wish to find, and solve for the remaining variable. Finding the domain of a radical function is a little tricky. To find the domain of any radical function with an even index, set the radicand greater that or equal to zero ($\geq$) and solve. If the radicand is a polynomial, you will need to solve the polynomial inequality by finding critical points, and testing intervals. To find the range of the radical function, find y value of the point of origin, and use the constant a to determine the range of the function.

Given the radical function $f_{(x)} = -\sqrt{x+4}-3$, the following can be determined.

First find the domain of the function. This will give you the x value needed for the point of origin.

$$f_{(x)} = -\sqrt{x+4}-3$$

Finding the domain.

$$x+4 \geq 0$$

$$x \geq -4$$

You can see the domain of the function is $[-4,\infty)$.
The -4 is the x value the point of origin.

Finding the range.

Since the constant a is -1, the function will go downwards. Meaning that the range is $(-\infty,-3]$.

Finding the "point of origin" of a radical function.

To find the point of origin of a radical function use the rules discussed in previous sections. The point of origin for the parent function $y=\sqrt{x}$ is (0, 0). This particular graph will shift left 4 and down 3, so the point of origin is $(-4,-3)$. Be careful when using these rules. Make sure to find the domain of the function before you attempt to find the point of origin. Consider a function such as $y=\sqrt{3-x}$. Since there is a positive 3 inside the radicand, you would normally shift to the left 3. However, If you were to find the domain of this function by setting the radicand $\geq$ 0, You will find the domain is actually $x \leq 3$. This says the graph is shifting to the right 3 spaces.

Finding the x-intercept.

Substitute 0 for y and solve for x.

$$0 = -\sqrt{x+4}-3$$

$$\sqrt{x+4} = -3$$

This is not possible. That means there is no x intercept for this function.

Finding the y-intercept.

Substitute 0 for x and solve for y.

$$y = -\sqrt{(0)+4}-3$$

$$y = -\sqrt{4}-3$$

$$y = -2-3$$

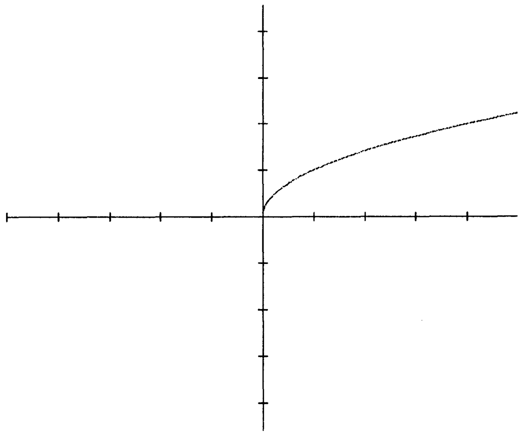
$$y = -5$$

The y intercept of this function is (0, -5).

We will now look at the parent function, and some translations of it.

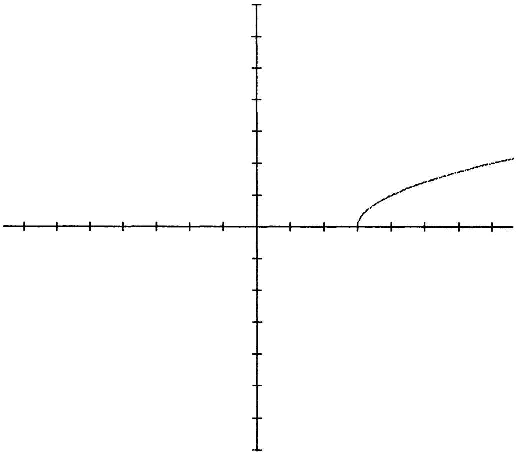
$$f_{(x)} = a\sqrt{x-h} + k$$

$$f_{(x)} = \sqrt{x}$$



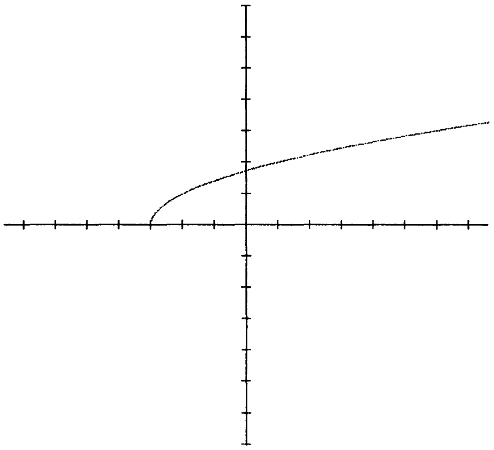
The parent function has the point of origin at (0, 0)

$$f_{(x)} = \sqrt{x-3}$$



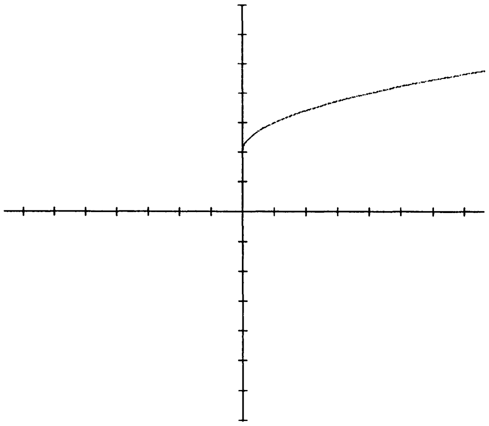
The graph of this function shifts right 3.

$$f_{(x)} = \sqrt{x+3}$$



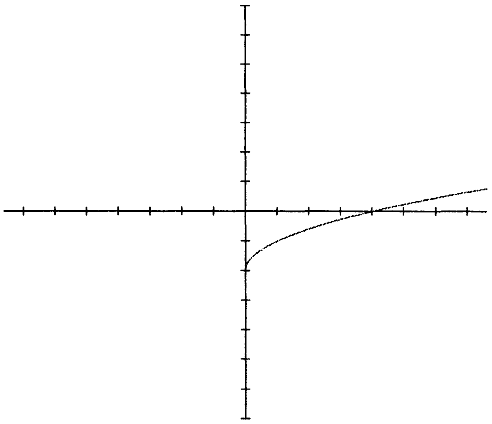
The graph of this function shifts left 3.

$$f_{(x)} = \sqrt{x} + 2$$



Here the graph shifts up 2.

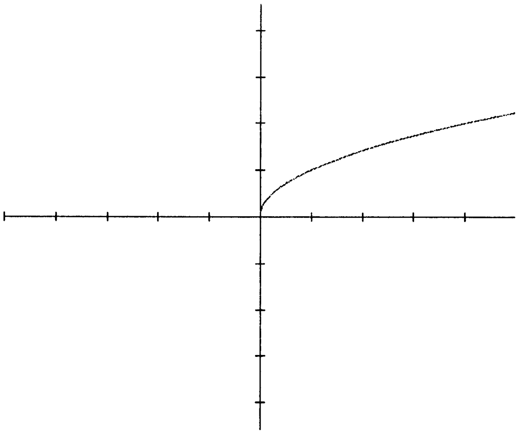
$$f_{(x)} = \sqrt{x} - 2$$



The graph of this function shifts down 2.

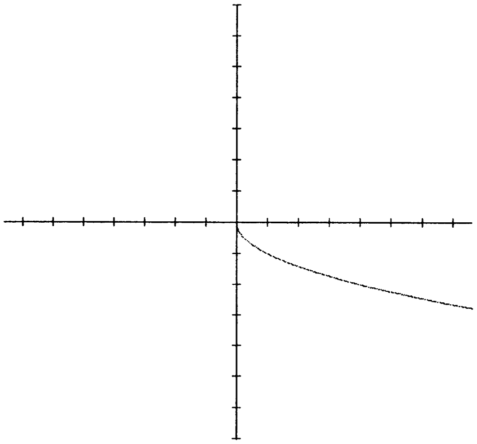
$$f_{(x)} = a\sqrt{x-h}+k$$

$$f_{(x)} = \sqrt{x}$$



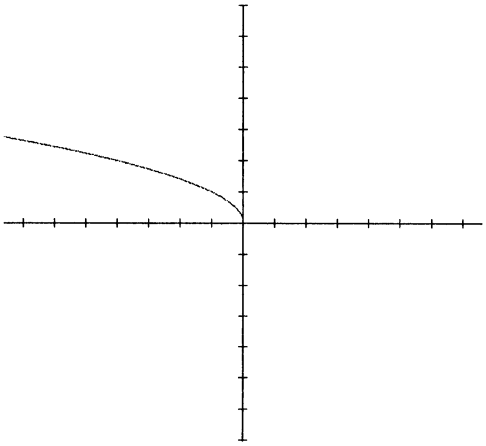
The parent function has the point of origin at (0, 0)

$$f_{(x)} = -\sqrt{x}$$



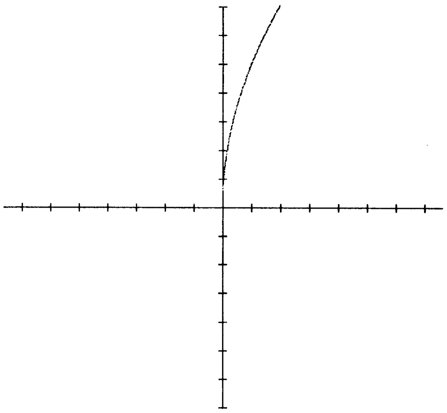
The graph of this function flips upside down.

$$f_{(x)} = \sqrt{-x}$$



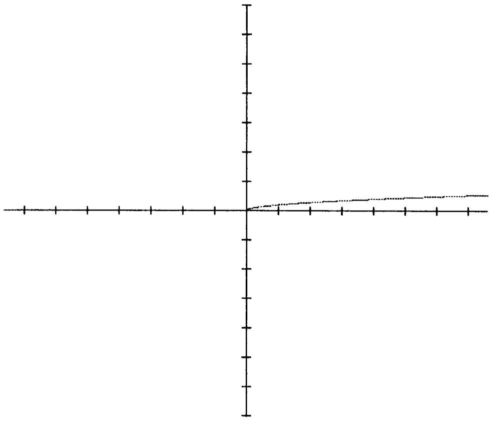
The graph of this function flips from right to left as the –x affects the domain of the function.

$$f_{(x)} = 5\sqrt{x}$$



Scale increased by a factor of 5.

$$f_{(x)} = \frac{1}{5}\sqrt{x}$$



This is 1/5 the normal scale.

Find the domain of each of the following radical functions in interval notation.

A) $f_{(x)} = \sqrt{x+4} - 2$

$x + 4 \geq 0$

$x \geq -4$

$[-4, \infty)$

B) $f_{(x)} = 2\sqrt{4-x} + 1$

$4 - x \geq 0$

$-x \geq -4$

$x \leq 4$

$(-\infty, 4]$

C) $f_{(x)} = \sqrt{2x+3} + 1$

$2x + 3 \geq 0$

$2x \geq -3$

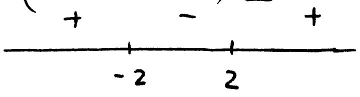
$x \geq -\frac{3}{2}$

$[-\frac{3}{2}, \infty)$

D) $f_{(x)} = \sqrt{x^2 - 4}$

$x^2 + 4x \geq 0$

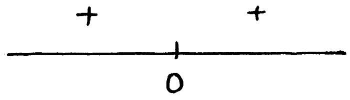
$(x+2)(x-2) \geq 0$



$(-\infty, -2] \cup [2, \infty)$

E) $f_{(x)} = \sqrt{x^2}$

$x^2 \geq 0$



$(-\infty, \infty)$

F) $f_{(x)} = \frac{1}{2}\sqrt{6-x} - 3$

$6 - x \geq 0$

$-x \geq -6$

$x \leq 6$

$(-\infty, 6]$

G) $f_{(x)} = -\sqrt{x+5} - 8$

$x + 5 \geq 0$

$x \geq -5$

$[5, \infty)$

H) $f_{(x)} = \sqrt{2-x} + 1$

$2 - x \geq 0$

$-x \geq -2$

$x \leq 2$

$(-\infty, 2]$

I) $f_{(x)} = 2\sqrt{x+7} - 5$

$x + 7 \geq 0$

$x \geq -7$

$[-7, \infty)$

The range of a radical function in $f_{(x)} = a\sqrt{x-h} + k$ ***form can be found using the value of the "a" term, and the y value of the point of origin.***

If $a > 0$, ***the range of the function is*** $[k, \infty)$.
If $a < 0$, ***the range of the function is*** $(-\infty, k]$.

Find the range for each of the following.

A) $f_{(x)} = \sqrt{x+5} - 3$

$[-3, \infty)$

B) $f_{(x)} = -\sqrt{x-3} + 2$

$(-\infty, 2]$

C) $f_{(x)} = 2\sqrt{x-4} + 3$

$[3, \infty)$

D) $f_{(x)} = -3\sqrt{5-x} + 6$

$(-\infty, 6]$

E) $f_{(x)} = \sqrt{4-x} - 3$

$[-3, \infty)$

F) $f_{(x)} = \sqrt{x-7} + 5$

$[5, \infty)$

Find the point of origin for each of the following radical functions.

A) $f_{(x)} = \sqrt{x+4} - 2$

$(-4, -2)$

B) $f_{(x)} = 2\sqrt{4-x} + 1$

$4 - x = 0$

$x = 4$

$(4, 1)$

C) $f_{(x)} = \sqrt{x} - 4$

$(0, -4)$

D) $f_{(x)} = -\sqrt{x-3}$

$(3, 0)$

E) $f_{(x)} = \sqrt{x^2}$

$(0, 0)$

F) $f_{(x)} = \frac{1}{2}\sqrt{6-x} - 3$

$6 - x = 0$

$x = 6$

$(6, -3)$

G) $f_{(x)} = -\sqrt{x+5} - 8$

$(-5, -8)$

H) $f_{(x)} = \sqrt{2-x} + 1$

$2 - x = 0$

$x = 2$

$(2, 1)$

I) $f_{(x)} = 2\sqrt{x+7} - 5$

$(-7, -5)$

Why is the graph of the function $f_{(x)} = \sqrt{-x}$ moving towards the left rather than the right?

if you let $-x \geq 0$, you must divide both sides by a (-1). This makes the sign change direction to $x \leq 0$; so the domain says I can only substitude values less than or equal to zero.

Explain why the graph of the function $f_{(x)} = \sqrt{x^2}$ is identical to that of $f_{(x)} = |x|$.

This can be explained using a property of radicals $\sqrt{x^2} = |x|$
This only true if the index & power match, and are both even.

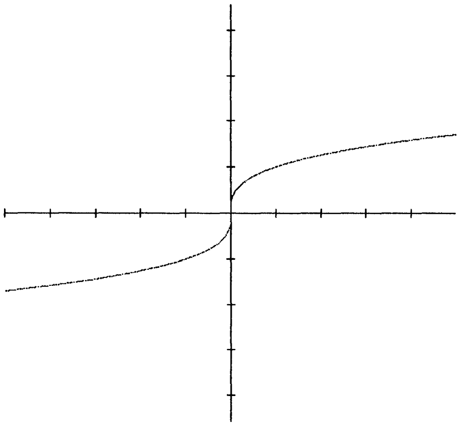
To find the domain of a radical function that has an even index, why do you need to set the radicand ≥ 0?

Because you can't take the square root of a negative # and get a real solution.

We will now look at the cube root function.

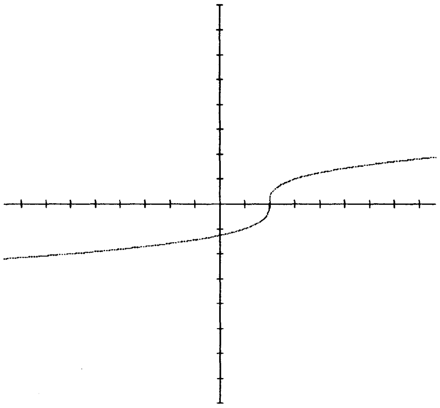
$$f_{(x)} = a\sqrt[3]{x-h}+k$$

$$f_{(x)} = \sqrt[3]{x}$$



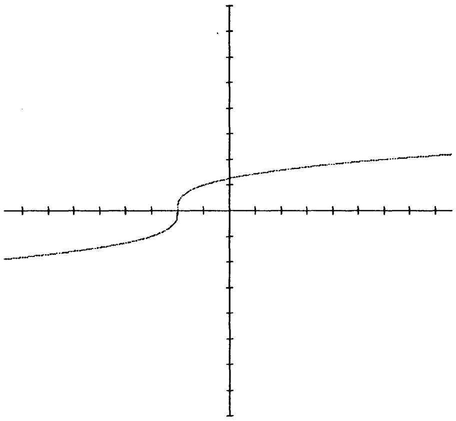
The parent function has the point of origin at (0, 0)

$$f_{(x)} = \sqrt[3]{x-2}$$



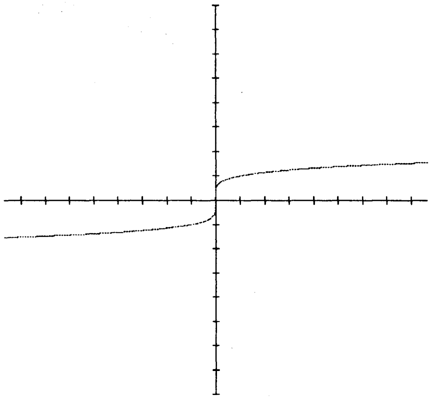
The graph of this function shifts right 2.

$$f_{(x)} = \sqrt[3]{x+2}$$



The graph of this function shifts left 2.

$$f_{(x)} = \sqrt[5]{x}$$

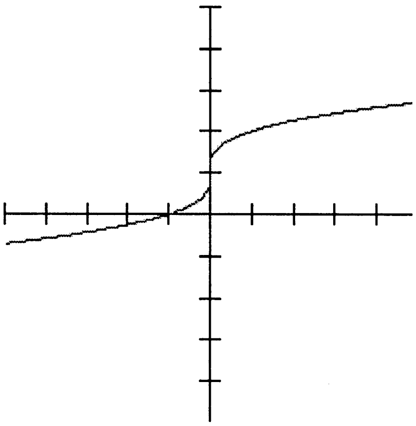


As you can see on the left, the curve is just about the same for a 5th root, verses a cubed root. This will be the same case for any radical function where the index is odd. This also means that any radical function where the index is even will look like a normal square root function. The curves of these functions are a little "flatter" than a regular square root or cubed root.

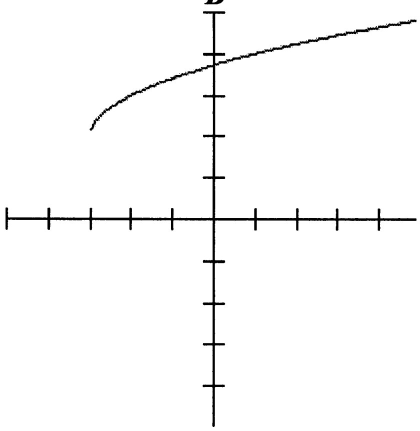
Vertical translations of the function are identical to that of a regular square root function. As you can see, the domain and range of any radical function with an odd index is all real numbers.

Match the appropriate graph with its equation below. Explain why each of your solutions is true.

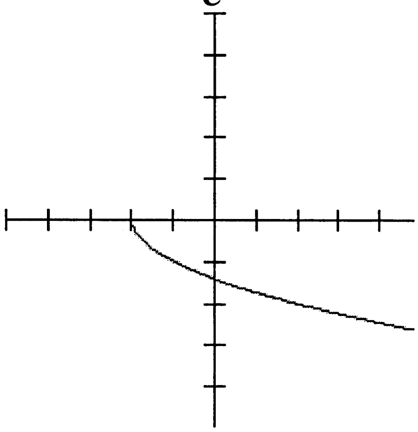
A



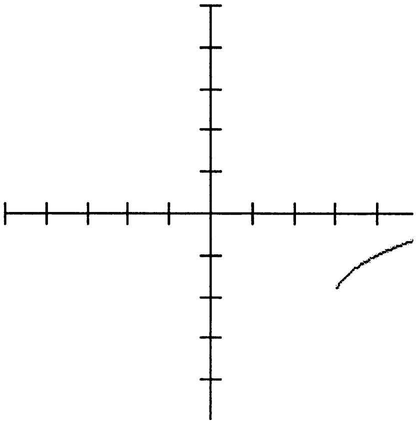
B



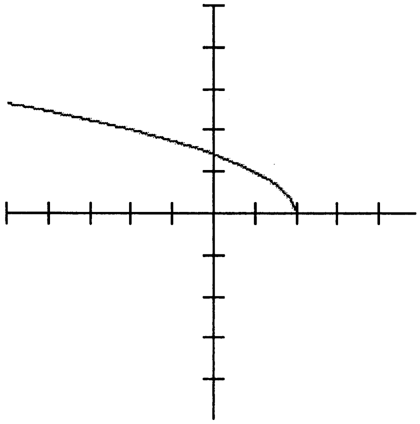
C



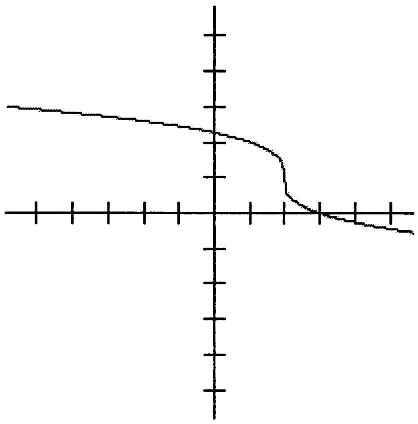
D



E



F



1) $f_{(x)} = \sqrt{x+3}+2$

B, the point of origin is (-3,2) and the function goes to the right

2) $f_{(x)} = \sqrt{x-3}-2$

D, the point of origin is (3,-2). & the function goes to the right.

3) $f_{(x)} = \sqrt[3]{x}+1$

A, the point of origin is (0,1) & it is a standard cubic function.

4) $f_{(x)} = -\sqrt[3]{x-2}+1$

F, the point of origin is (2,1) and the cubic is flipped upside down.

5) $f_{(x)} = -\sqrt{x+2}$

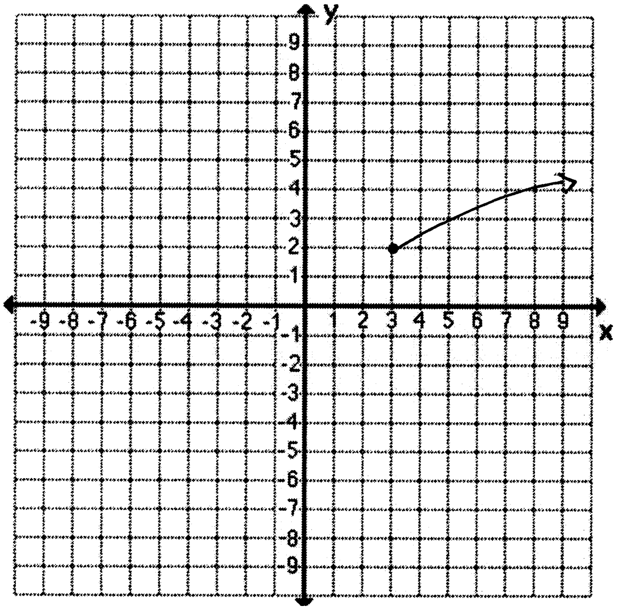
C, the point of origin is (-2,0). & the graph of the function is upside down.

6) $f_{(x)} = \sqrt{2-x}$

E, the point of origin is (2,0) & the function goes to the left.

Graph each of the following radical functions. ***Find all required information.***

A) $f_{(x)} = \sqrt{x-3}+2$



Point of Origin: (3, 2)

Y-intercept: none

X-intercepts: none

Range: $[2, \infty)$

Domain: $[3, \infty)$

y-int

$f(0) = \sqrt{0-3}+2$

$f(0) = \sqrt{-3}+2$

not real

no y-int

x-int

$0 = \sqrt{x-3}+2$

$\sqrt{x-3} = -2$

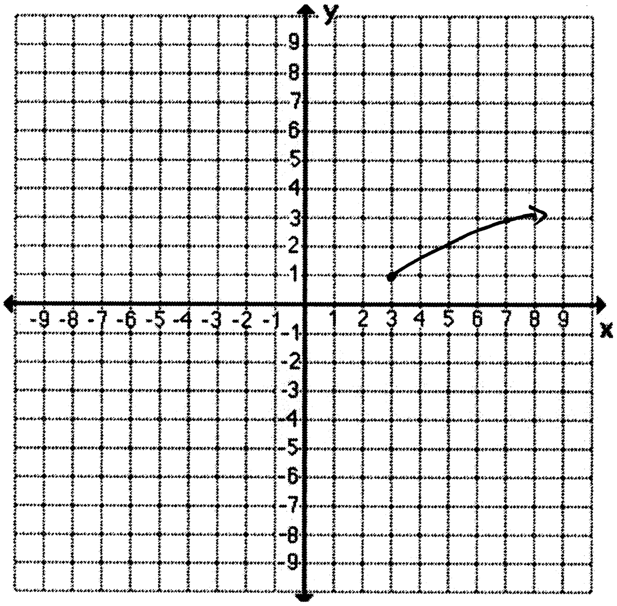
not possible ; no x-int

Domain

$x-3 \geq 0$

$x \geq 3$

B) $f_{(x)} = -\sqrt{x-3}+1$



Point of Origin: (3, 1)

Y-intercept: none

X-intercepts: (4, 0)

Range: $(-\infty, 1]$

Domain: $[3, \infty)$

y-int

$f(0) = -\sqrt{-3}+1$

not real

no y-int

x-int

$0 = -\sqrt{x-3}+1$

$(\sqrt{x-3})^2 = (1)^2$

$x-3 = 1$

$x = 4$

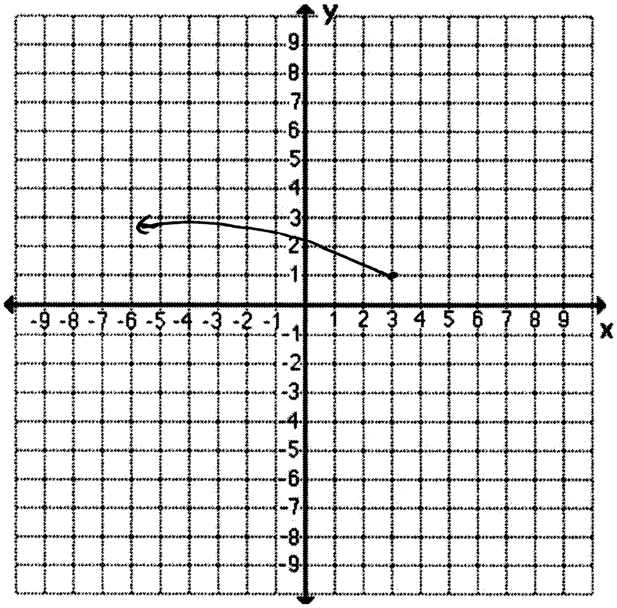
(4, 0)

Domain

$x-3 \geq 0$

$x \geq 3$

C) $f_{(x)} = \sqrt{3-x} + 1$



Point of Origin: $(3, 1)$

Y-intercept: $(0, 1+\sqrt{3})$

X-intercepts: none

Range: $[1, \infty)$

Domain: $(-\infty, 3]$

y-int

$f(0) = \sqrt{3-0} + 1$

$F(0) = 1+\sqrt{3}$

$(0, 1+\sqrt{3})$

X-int

$0 = \sqrt{3-x} + 1$

$\sqrt{3-x} = -1$

no solution

no x-int.

Domain

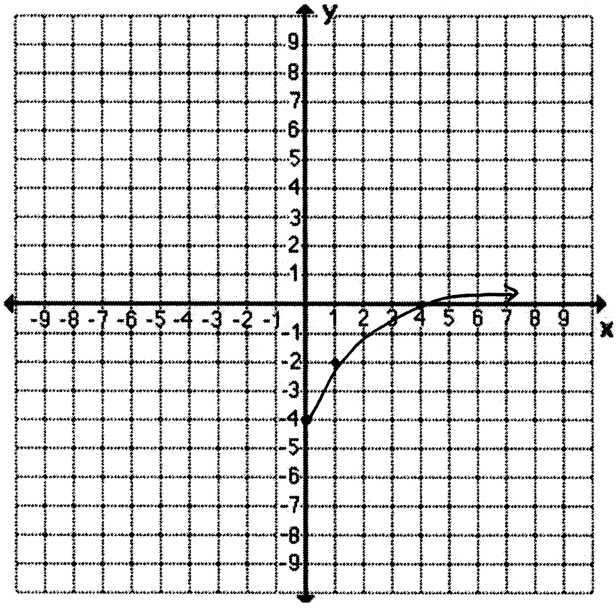
$3 - x \geq 0$

$-x \geq -3$

$x \leq 3$

$(-\infty, 3]$

D) $f_{(x)} = 2\sqrt{x} - 4$



Point of Origin: $(0, -4)$

Y-intercept: $(0, -4)$

X-intercepts: $(4, 0)$

Range: $[-4, \infty)$

Domain: $[0, \infty)$

y-int

$f(0) = 2\sqrt{0} - 4$

$F(0) = -4$

$(0, -4)$

x-int

$0 = 2\sqrt{x} - 4$

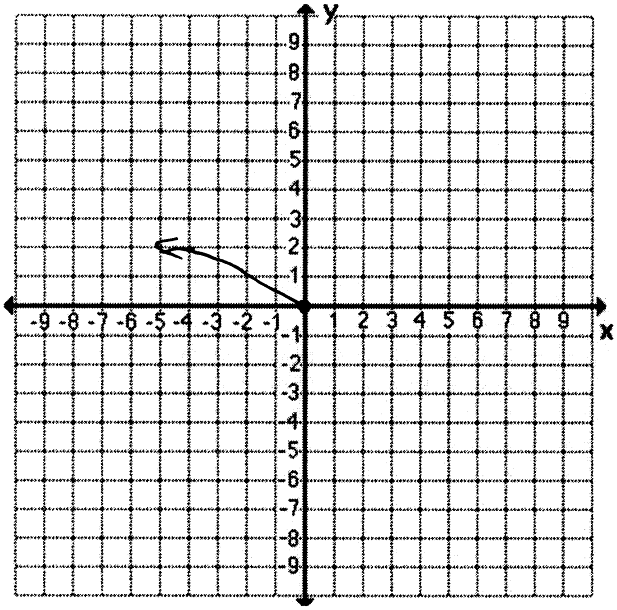
$4 = 2\sqrt{x}$

$(2)^2 = (\sqrt{x})^2$

$4 = x$

$(4, 0)$

E) $f_{(x)} = -\sqrt{-x}$



Point of Origin: $(0,0)$

Y-intercept: $(0,0)$

X-intercepts: $(0,0)$

Range: $(-\infty, 0]$

Domain: $(-\infty, 0]$

Y-int

$f(0) = -\sqrt{0}$

$f(0) = 0$

$(0,0)$

Domain

$-x \geq 0$

$x \leq 0$

x-int

$0 = -\sqrt{-x}$

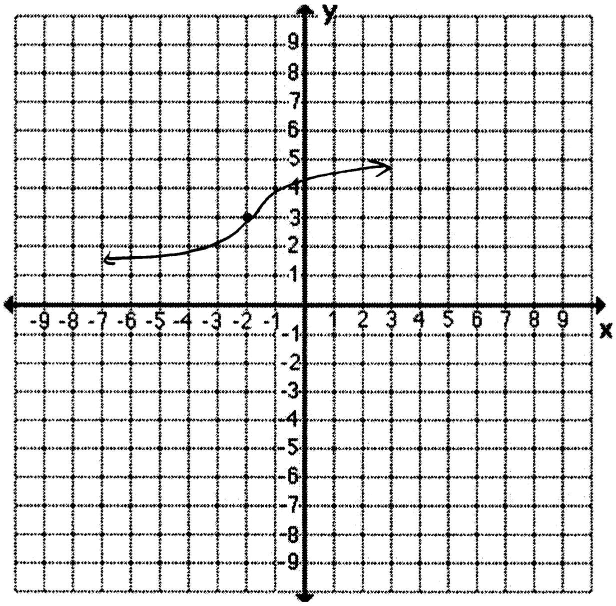
$(\sqrt{-x})^2 = (0)^2$

$-x = 0$

$x = 0$

$(0,0)$

F) $f_{(x)} = \sqrt[3]{x+2} + 3$



Point of Origin: $(-2,3)$

Y-intercept: $(0, 3 + \sqrt[3]{2})$

X-intercepts: $(-29,0)$

Range: $(-\infty, \infty)$

Domain: $(-\infty, \infty)$

x-int

$f(0) = \sqrt[3]{2} + 3$

$(0, \sqrt[3]{2} \underset{3.3}{+} 3)$

x-int

$0 = \sqrt[3]{x+2} + 3$

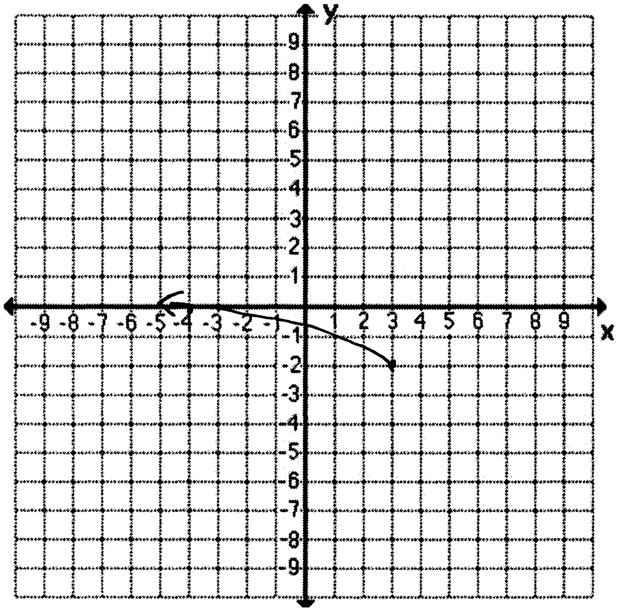
$(\sqrt[3]{x+2})^3 = (-3)^3$

$x + 2 = -27$

$x = -29$

$(-29,0)$

G) $f_{(x)} = -\sqrt[3]{x-3}-2$



Point of Origin: $(3, -2)$

Y-intercept: $(0, -2 - \sqrt[3]{-3})$

X-intercepts: $(-5, 0)$

Range: $(-\infty, \infty)$

Domain: $(-\infty, \infty)$

y-int

$f(0) = -\sqrt[3]{0-3} + 2$

$f(0) = -\sqrt[3]{-3} - 2$

$(0, -2 - \sqrt[3]{-3})$

x-int

$0 = -\sqrt[3]{x-3} - 2$

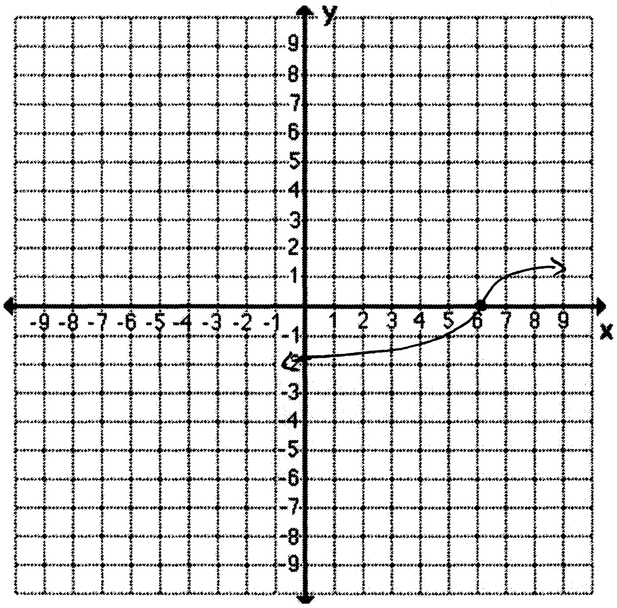
$(\sqrt[3]{x-3})^3 = (-2)^3$

$x - 3 = -8$

$x = -5$

$(-5, 0)$

H) $f_{(x)} = \sqrt[3]{x-6}$



Point of Origin: $(6, 0)$

Y-intercept: $(0, \sqrt[3]{-6})$

X-intercepts: $(6, 0)$

Range: $(-\infty, \infty)$

Domain: $(-\infty, \infty)$

y-int

$f(0) = \sqrt[3]{-6}$

$(0, \sqrt[3]{-6})$

x-int

$(0)^3 = (\sqrt[3]{x-6})^3$

$x - 6 = 0$

$x = 6$

$(6, 0)$

Why are the graphs of $y = \sqrt[3]{x}$ and $y = -\sqrt[3]{-x}$ identical?

If I evaluate & simplify $\sqrt[3]{-x}$, I can look at this as $\sqrt[3]{(-1)x}$ which simplifies to $-\sqrt[3]{x}$ which is $-(-\sqrt[3]{x})$, which is $\sqrt[3]{x}$.

Exponential Functions

This information was covered in a previous section of the workbook, but it won't hurt to go over it again.

Standard exponential function

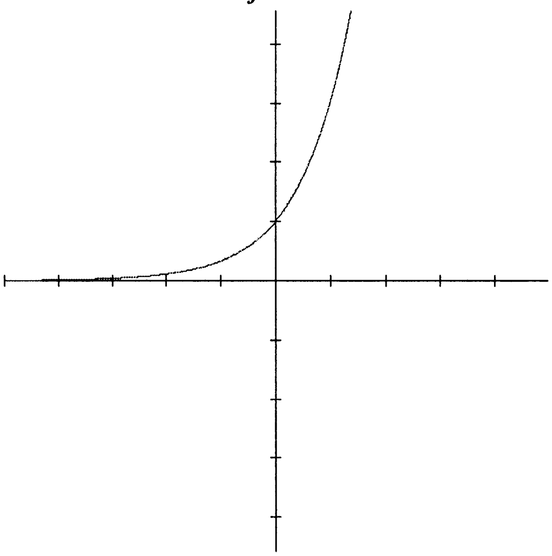
$$f_{(x)} = ca^{x-h} + k$$

The c term is a constant that can make to graph reflect about a horizontal axis or change to scale of the graph of the function proportionately. If c is a positive value, then you will have a standard looking growth or decay curve. If c is negative, the growth or decay curve will flip upside down. We will get into the effects different values of h and k have on this function shortly. What we will concentrate on here is identifying an exponential function as being growth or decay, and finding the range, domain and key point of the function.

Exponential Growth

$$f_{(x)} = ca^{x-h} + k$$

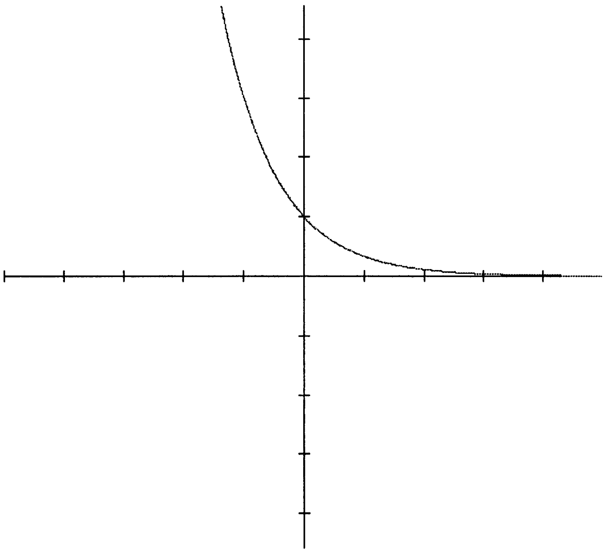
If a>1



Exponential Decay

$$f_{(x)} = ca^{x-h} + k$$

If 0<a<1



Obviously if $a = 1$***, we are raising 1 to various powers, and we wind up getting a horizontal line because no matter what you do, raising one to any power still yields a result of one. Pay special attention to the exponential decay function. The statement*** $0 < a < 1$ ***is saying that the value of*** a ***is a fraction whose value is between zero and one. Do not make the mistake of just looking for a fraction to determine whether or not the function is decay. Make sure the value of the fraction is between zero and one.***

The values for variables $h,\ k,\ and\ \ c$ ***act to make the graph shift left/right, up/down, change the scale or will reflect the function about a horizontal axis.***

Notice the key point for each of these functions is the point $(0,1)$. This information is vital. This key point will shift depending on the values of h, k and c. To find the x value of the key point, evaluate $x-h=0$. In other words, find the value of x that would create a problem such as 3 to the zero power. This number is the x value of the key point. To find the y value, substitute the x value back in. Refer to the following example.

$f_{(x)}=2^{x-3}+5$

to find the key point evaluate $x-3=0$

$x-3=0$

$x=3$ *this is the x value of the key point*

now substitute 3 *back into the problem for* x

$f_{(3)}=2^{3-3}+5$

$f_{(3)}=2^{0}+5$

$f_{(3)}=1+5$

$f_{(3)}=6$

so the key point is $(3,6)$

$$f_{(x)}=ca^{x-h}+k$$

As we work to translate these functions, use $(0,1)$, as the default key point to any exponential growth or decay curve that is above the horizontal asymptote where the value of "c" is 1. In other words, if the value of "c" is positive one, use the point $(0,1)$ to assist you in shifting the function. If the graph of the function is below the horizontal asymptote, and the "c" value is -1, you will use $(0,-1)$ as the key point. If the value of "c" is any other number, you must find the key point algebraically. Consider the example above.

$f_{(x)}=2^{x-3}+5$

The first thing I did was notice that the graph of this function shifts right 3 and up 5. Pay special attention to the value of k. That tells you where your new horizontal asymptote is going to be, in this case, at $y=5$.

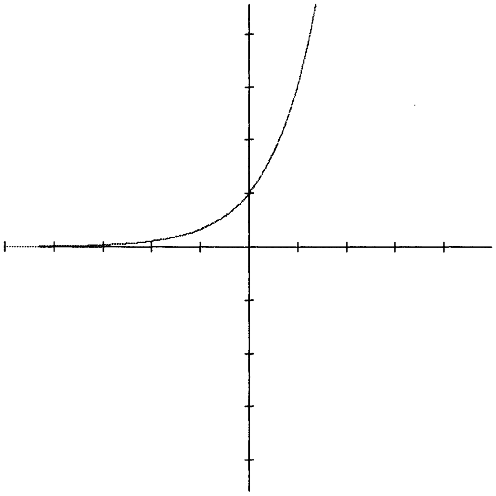
Since the value of c is positive one, begin at the key point $(0,1)$. Since the graph will shift right 3 and up 5, simply add 3 to the x value, and 5 to the y value of the key point. This produces a new key point of $(3,6)$. Observe how this information matches the work above. The key point in these functions acts as the vertex in a parabola. It gives you a point of reference with which to shift the function. Make sure the correct key point is used from the beginning, either $(0,1)$ or $(0,-1)$. Remember, if the "c" term is a number other that 1 or -1, the key point is actually multiplied by that number. For example, the function $f_{(x)}=3(2)^{x}$ has a key point of $(0,3)$, not $(0,1)$.

Here we will look at how to graph these functions by means of translation.

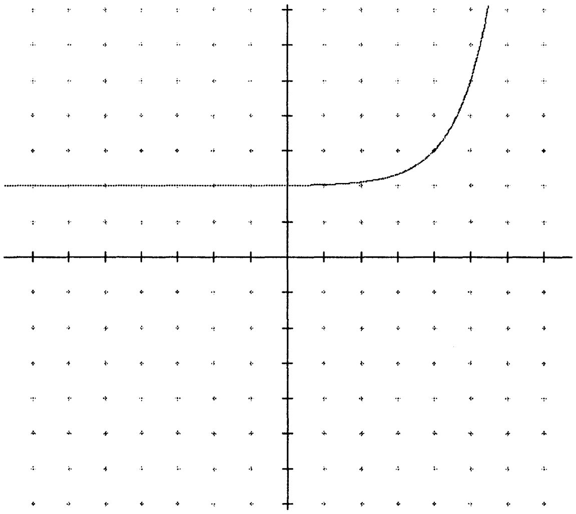
Exponential Growth

$$f_{(x)} = ca^{x-h} + k$$

$$f_{(x)} = 3^x$$

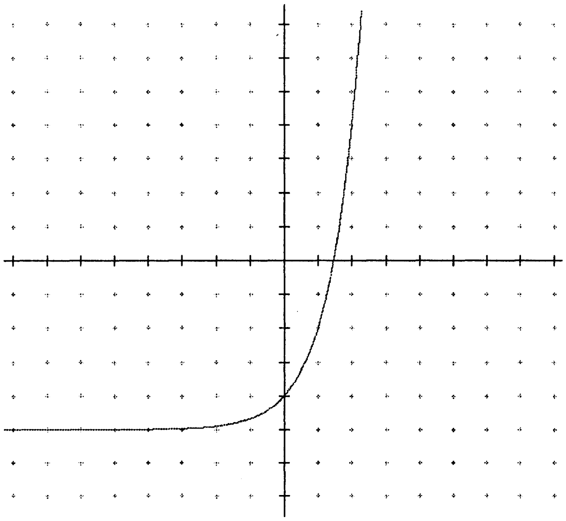


$$f_{(x)} = 3^{x-4} + 2$$



Notice that the horizontal asymptote is at y=2. Since the graph of this function is going to be above the x axis, begin with the key point (0,1). This function shifts right 4 and up 2. The dots have been left on the graph so it would be easier to see. Adding 4 to the x value of the key point, and 2 to the y value, the new key point is at (4,3). Just remember where to begin, and do not cross the horizontal asymptote.

$$f_{(x)} = 3^x - 5$$



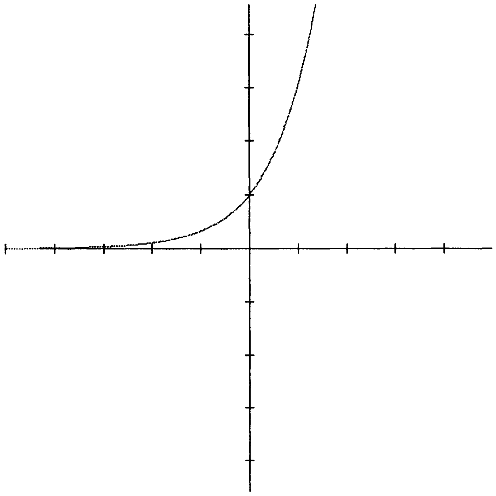
In this function, the value of k is -5. This tells you the new horizontal asymptote will be at y = -5. Since the value of the constant "c" is a positive one, begin with the key point (0,1). This function will only shift down 5 spaces. Therefore, subtract 5 from the y value of the key point which is 1. This results in: (1 – 5 = -4) therefore, the new key point is at (0,-4).

Graphing exponential functions by translation is relatively simple. The most difficult part will be finding the x and y intercepts as the x-intercept will involve the use of logarithms.

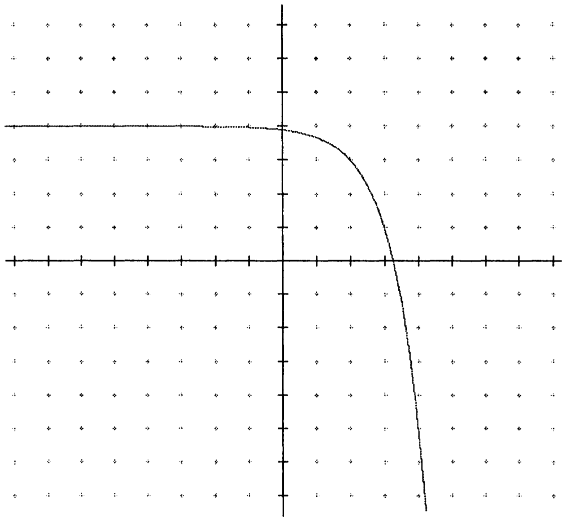
Exponential Growth

$$f_{(x)} = ca^{x-h} + k$$

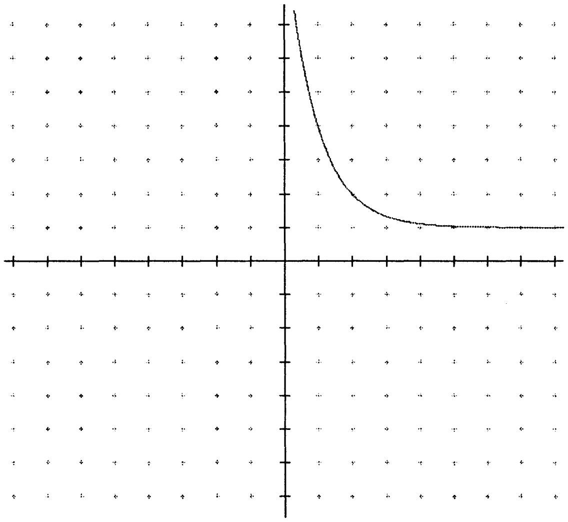
$$f_{(x)} = 3^x$$



$$f_{(x)} = -3^{x-2} + 4$$



$$f_{(x)} = 3^{2-x} + 1$$



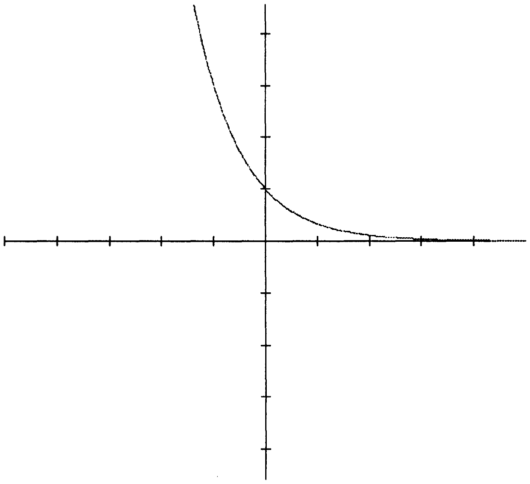
Since the value of "c" in the equation of this function is -1, we must begin with the key point of (0,-1). This is the key point, because that value of "c" caused the graph to reflect about the horizontal asymptote. The entire function will shift up 4, so the new horizontal asymptote is y = 4. The curve is going to shift right 2 and up 4. By adding 2 to the x value of the key point, and 4 to the y value, the new point can be found at (2,3). Notice the graph runs right though that point.

Here we have something that looks like decay. The value of "a" in this function is greater that one, so it should be growth. What really happened here, is the laws of exponents went to work. 3^{2-x} ***Is the same thing as*** $3^{-(x-2)}$. ***The power of a power rule says this can be seen as*** $\left(3^{-1}\right)^{x-2}$. ***This simplifies to*** $\left(\frac{1}{3}\right)^{x-2}$ ***, a decay curve. OK, so we begin with a decay curve that has a key point of (0,1). Add 2 to the x value, and 1 to the y value of the key point, and the new key point is (2,2), with a horizontal asymptote of y = 1.***

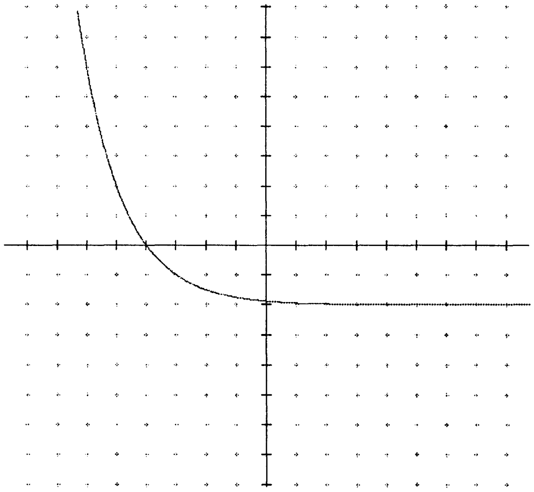
Exponential Decay

$$f_{(x)} = ca^{x-h} + k$$

$$f_{(x)} = \left(\frac{1}{2}\right)^{x}$$

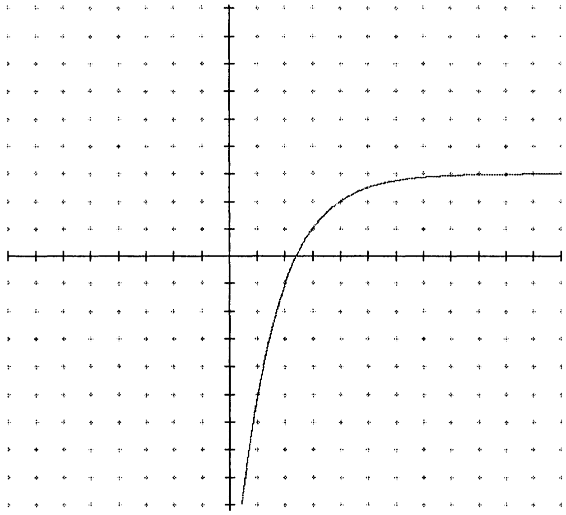


$$f_{(x)} = \left(\frac{1}{2}\right)^{x+3} - 2$$



This is an exponential decay curve. The original graph will lie above the x axis. Therefore, begin with the key point (0,1). The key point will shift to the left 3, and down 2, so subtract 3 from the x value and 2 from the y value of the key point, and the coordinates of the new point will be at (-3,-1). The graph of this function shifts down 2, so the horizontal asymptote of this function is y = -2.

$$f_{(x)} = -\left(\frac{1}{2}\right)^{x-4} + 3$$



This is an exponential decay function that is reflected and lies below the horizontal asymptote. The initial key point here is (0,-1). Since this graph will shift right 4 and up 3, add 4 to the x value of the key point and 3 to the y value. This yields a result of (4,2). Since the entire graph shifted upwards 3 spaces, the horizontal asymptote is y = 3. Once again, when graphing, do not cross the horizontal asymptote.

The translations of these functions are very similar to that of other functions we have seen. A point of reference with which to shift is all that is needed. Most important is to make sure to always use the appropriate key point to start with. <u>Draw the horizontal asymptote first</u>, that way the graph of the function does not accidentally cross it.

$$f_{(x)} = ca^{x-h} + k$$

The domain of any exponential function is $(-\infty, \infty)$. The values of c and k terms will determine the range of the function. Since the horizontal asymptote of an exponential function is given by y=k, the value of k will determine where the horizontal asymptote of the function lies, whereas the value of c will determine if the function is above or below that asymptote. Be careful not to use brackets when describing the range of an exponential function. The horizontal asymptote must not be touched, so only parenthesis may be used to describe the range in interval notation.

Find the range and domain of each of the following exponential functions.

A) $f_{(x)} = 2^{x+6} - 4$

H· A $y=-4$

R: $(-4, \infty)$

D: $(-\infty, \infty)$

B) $f_{(x)} = -\left(\frac{1}{2}\right)^{x-1} + 3$

H· A : $y=3$

R: $(-\infty, 3)$

D: $(-\infty, \infty)$

C) $f_{(x)} = 2(3)^{x+1} - 5$

H · A $y = -5$

R: $(-5, \infty)$

D: $(-\infty, \infty)$

D) $f_{(x)} = 5^{-x} - 3$

H · A $y=-3$

R: $(-3, \infty)$

D: $(-\infty, \infty)$

E) $f_{(x)} = -2(5)^{x+2} - 3$

H · A $y = -3$

R: $(-\infty, -3)$

D: $(-\infty, \infty)$

F) $f_{(x)} = e^{x+2} - 3$

H A : $y = -3$

R: $(-3, \infty)$

D: $(-\infty, \infty)$

G) $f_{(x)} = \left(\frac{5}{4}\right)^{x-8} + 2$

H A $y = 2$

R: $(2, \infty)$

D: $(-\infty, \infty)$

H) $f_{(x)} = -2^{x-3} - 7$

H A $y=-7$

R: $(-\infty, -7)$

D: $(-\infty, \infty)$

I) $f_{(x)} = -4^{3-x} + 2$

$f(x) = -(4^{-1})^{x-3} + 2$

$f(x) = -(\frac{1}{4})^{x-3} + 2$

R: $(-\infty, 2)$

D: $(-\infty, -)$

J) $f_{(x)} = 2\left(\frac{1}{3}\right)^{x-5} + 1$

H · A $y = 1$

R: $(1, \infty)$

D: $(-\infty, \infty)$

K) $f_{(x)} = -6^{x-7} - 1$

H A $y = -1$

R: $(-\infty, -1)$

D: $(-\infty, \infty)$

L) $f_{(x)} = -e^{x-2} + 3$

H A : $y=3$

R: $(-\infty, 3)$

D: $(-\infty, \infty)$

Here is an example of finding the x and y intercept of an exponential function.

$$f_{(x)} = 3^{x+2} - 4$$

Finding the x intercept.

Begin by substituting 0 for $f_{(x)}$

$$0 = 3^{x+2} - 4$$

$$4 = 3^{x+2}$$

$$\log 4 = \log 3^{x+2}$$

$$\log 4 = (x+2)\log 3$$

$$\log 4 = x\log 3 + 2\log 3$$

$$\log 4 - 2\log 3 = x\log 3$$

Now divide both sides by log 3.

$$\frac{\log 4 - 2\log 3}{\log 3} = \frac{x\log 3}{\log 3}$$

$$x = \frac{\log 4 - 2\log 3}{\log 3}$$

$$x \approx -0.7381$$

Finding the y intercept.

Begin by substituting 0 for x.

$$f_{(x)} = 3^{0+2} - 4$$

$$f_{(x)} = 3^2 - 4$$

$$f_{(x)} = 9 - 4$$

$$f_{(x)} = 5$$

As you can see, this function has an x intercept of approximately (-0.74,0), and a y intercept of (0,5).

Find the key point to each of the following functions.

A) $f_{(x)} = 3^{x+4} - 2$

$x + 4 = 0$

$x = -4$

$f(-4) = 3^{-1+4} - 2$

$f(-4) = 3^0 - 2$

$f(-4) = 1 - 2$

$f(-4) = -1$

key pt: $(-4, -1)$

B) $f_{(x)} = -4^{x-2} + 1$

$x - 2 = 0$

$x = 2$

$f(2) = -4^{2-2} + 1$

$f(2) = -4^0 + 1$

$f(2) = -1 + 1$

$f(2) = 0$

key pt: $(2, 0)$

C) $f_{(x)} = 2^{4-x} + 5$

$4 - x = 0$

$x = 4$

$f(4) = 2^{4-4} + 5$

$f(4) = 2^0 + 5$

$f(4) = 1 + 5$

$f(4) = 6$

key. pt: $(4, 6)$

D) $f_{(x)} = 3(2)^{x+1} - 5$

$f(-4) = 3^{-1+4} - 2$

$f(-4) = 3^0 - 2$

$f(-4) = 1 - 2$

$f(-4) = -1$

E) $f_{(x)} = 2\left(\frac{1}{2}\right)^{x+4} - 3$

$x + 4 = 0$

$x = -4$

$f(-4) = 2\left(\frac{1}{2}\right)^{-4+4} - 3$

$f(-4) = 2\left(\frac{1}{2}\right)^0 - 3$

$f(-4) = 2(1) - 3$

$f(-4) = 2 - 3$

$f(-4) = -1$

Key pt: $(-4, -1)$

F) $f_{(x)} = -3^{x+2} - 4$

$x + 2 = 0$

$x = -2$

$f(-2) = -3^{-2+2} - 4$

$f(-2) = -(3)^0 - 4$

$f(-2) = -1 - 4$

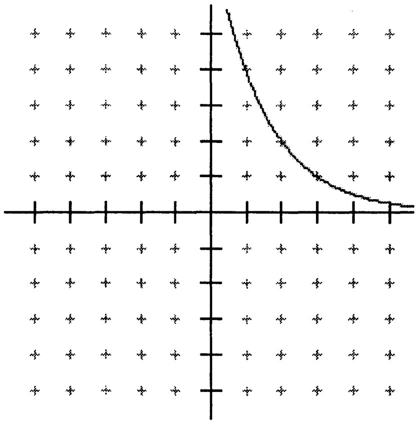
$f(-2) = -1 - 4$

$f(-2) = -5$

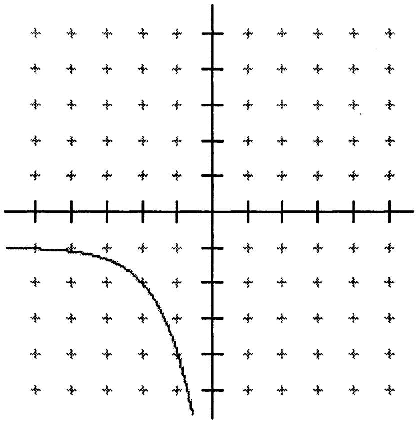
Key. pt: $(-2, -5)$

Match the appropriate graph with its equation below. Explain why each of your solutions is true.

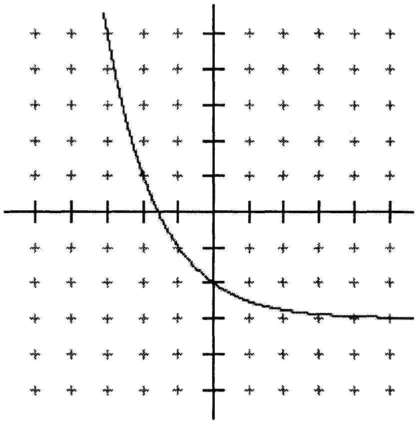
A



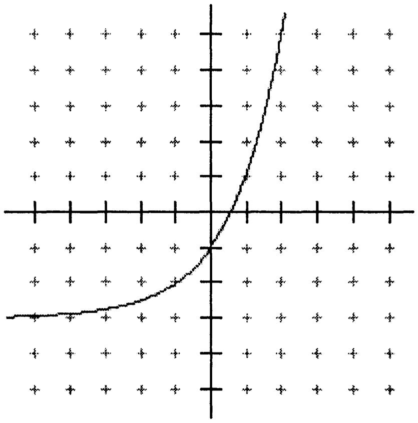
B



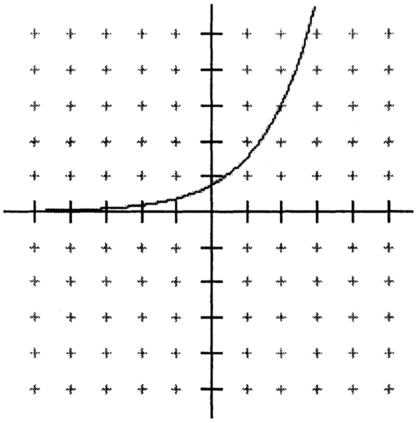
C



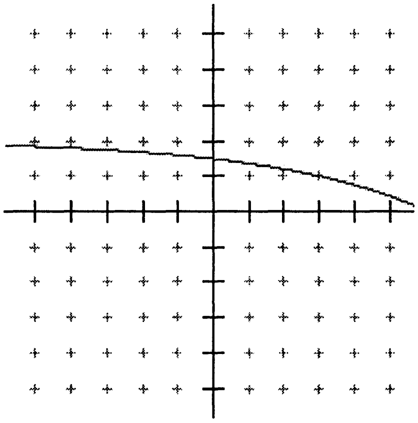
D



E



F



1) $f_{(x)} = 2^{x+1} - 3$

D, exponential growth
shifts left 1, down 3,
new key pt is (-1,-2)

2) $f_{(x)} = \left(\frac{1}{2}\right)^{x} - 3$

C, exponential decay,
shifts down 3,
new key pt. (0,-2)

3) $f_{(x)} = 3(2)^{x-2}$

E, exponential growth,
shifts right 2,
$x-2=0$
$x=2$
$f(2)=3(1)$
$f(2)=3$
key pt. (2,3)

4) $f_{(x)} = -3^{x+2} - 1$

B, exponential growth
upside down shifts
left 2, down 1
key pt : (-2,-2)

5) $f_{(x)} = -\left(\frac{5}{4}\right)^{x-3} + 2$

F, exponential growth
upside down,
shifts right 3, up 2
new key pt. (3,1)

6) $f_{(x)} = 2^{3-x}$

A, exponential decay
shifts right 3,
new key pt. (3,1)
$(2^{-1})^{x-3}$
$(\frac{1}{2})^{x-3}$

Graph each of the following exponential functions. Be sure to label the key point of the function. Find the x intercept (if it exists) and y intercept of each function.

A) $f_{(x)} = 3^{x-2} - 1$

Y-int

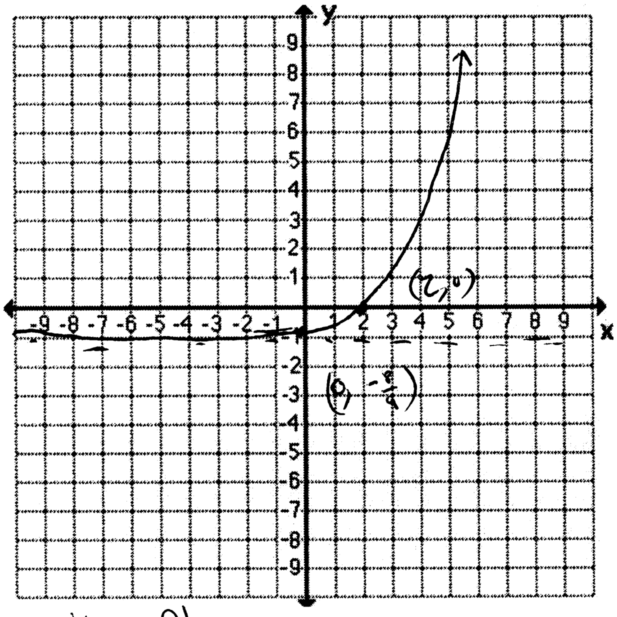
$f(0) = 3^{0-2} - 1$

$f(0) = 3^{-2} - 1$

$f(0) = \frac{1}{9} - 1$

$f(0) = \frac{-8}{9}$

$(0, -\frac{8}{9})$



Y-intercept: $(0, -\frac{8}{9})$

X-intercepts: (2,0)

Range: $(-1, \infty)$

Domain: $(-\infty, \infty)$

X-int

$0 = 3^{x-2} - 1$

$\log 3^{x-2} = \log 1$

$(x-2)\log 3 = \log 1$

$x\log 3 - 2\log 3 = 0$

$\frac{x\log 3}{\log 3} = \frac{2\log 3}{\log 3}$

$x = 2$

key pt.

(0,1)

+2 -1

(2,0)

B) $f_{(x)} = -2^{x+3} - 4$

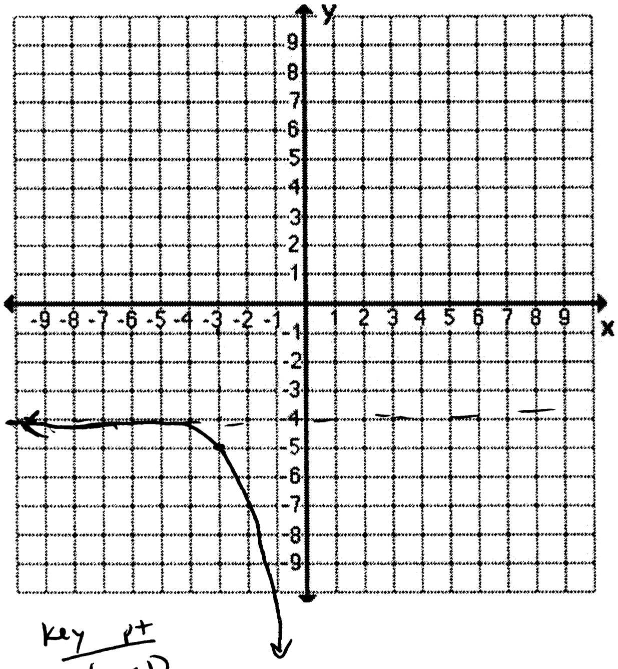
y int

$f(0) = -2^{3} - 4$

$-8 - 4$

-12

$(0, -12)$



Y-intercept: (0, -12)

X-intercepts: none

Range: $(-\infty, -4)$

Domain: $(-\infty, \infty)$

x int

$0 = -2^{x+3} - 4$

$\log 2^{x+3} = \log -4$

$(x+3)\log 2 = \log(-4)$

not possible

key pt

(0, -1)

-3 -4

(-3, -5)

C) $f_{(x)} = \left(\frac{1}{2}\right)^{x+2} - 3$

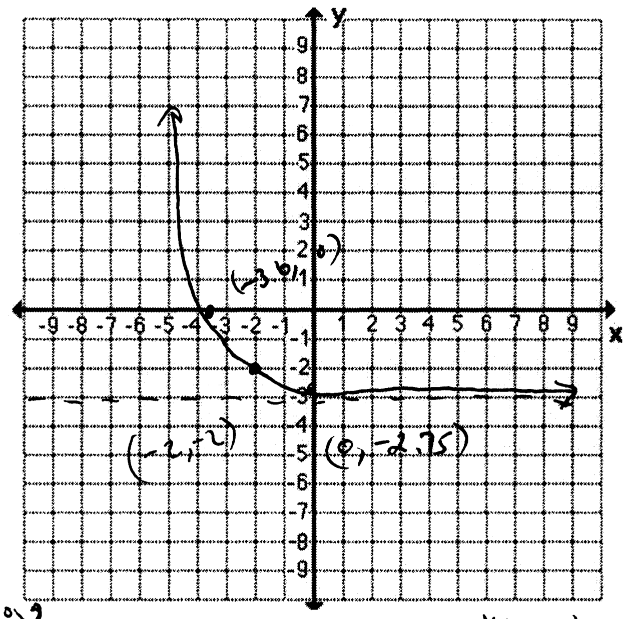
y int

$f(0) = \left(\frac{1}{2}\right)^{2} - 3$

$\frac{1}{4} - 3$

$f(0) = -2\frac{3}{4}$

$(0, -2.75)$



x int

$0 = \left(\frac{1}{2}\right)^{x+2} - 3$

$\log\left(\frac{1}{2}\right)^{x+2} = \log 3$

$(x+2)\log\frac{1}{2} = \log 3$

$x\log\frac{1}{2} + 2\log\frac{1}{2} = \log 3$

$x = \dfrac{\log 3 - 2\log\frac{1}{2}}{\log\frac{1}{2}}$ $(-3.6, 0)$

key pt

$(0, 1)$

$-2 \quad -3$

$(-2, -2)$

Y-intercept: $(0, -2.75)$

X-intercepts: $(-3.6, 0)$

Range: $(-3, \infty)$

Domain: $(-\infty, \infty)$

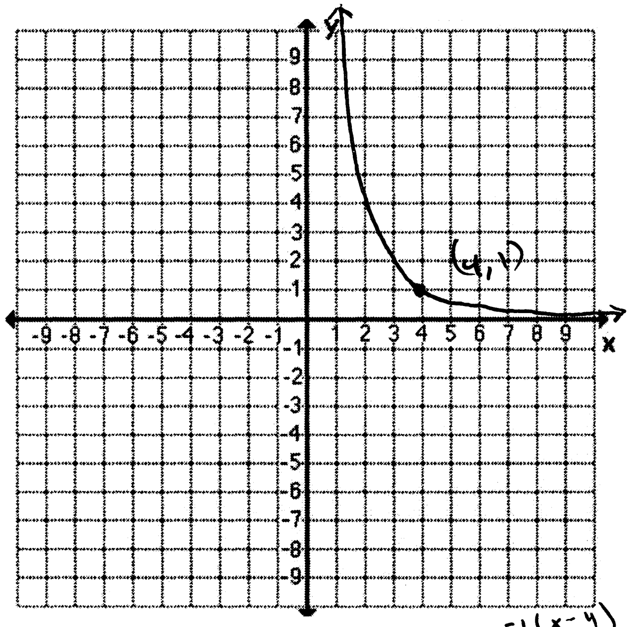
D) $f_{(x)} = 2^{4-x}$

y int

$f(0) = 2^{4}$

$= 16$

$(0, 16)$



x int

$0 = 2^{4-x}$

$\log 0 = \log 2^{(4-x)}$

↑ not possible

no x int

key pt

$(0, 1)$

$+4$

$(4, 1)$

$f(x) = 2^{-1(x-4)}$

$\left(\frac{1}{2}\right)^{x-4}$

Y-intercept: $(0, 16)$

X-intercepts: none

Range: $(0, \infty)$

Domain: $(-\infty, \infty)$

E) $f_{(x)} = -\left(\frac{1}{3}\right)^{x-2} - 3$

y int

$f(0) = -\left(\frac{1}{3}\right)^{-2} - 3$

$-(3)^2 - 3$

$-9 - 3$

-12

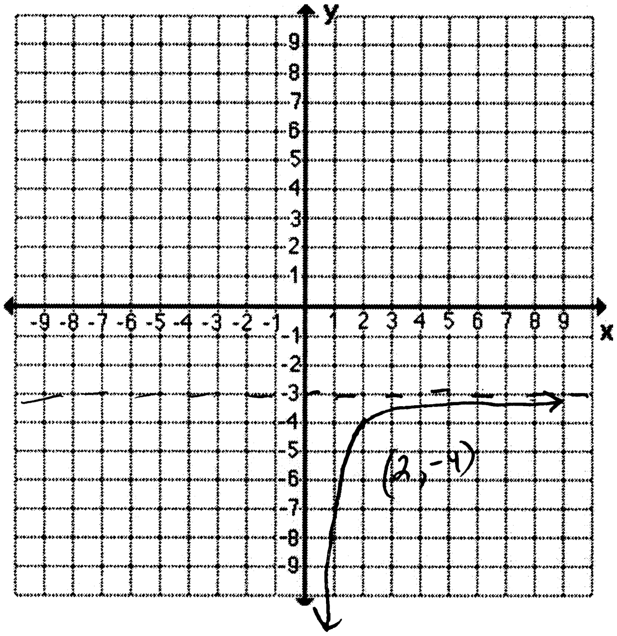
$(0, -12)$

x int

$0 = -\left(\frac{1}{3}\right)^{x-2} - 3$

$\log \left(\frac{1}{3}\right)^{x-2} = \log -3$

↑ not possible



Y-intercept: $(0, -12)$

X-intercepts: none

Range: $(-\infty, -3)$

Domain: $(-\infty, \infty)$

Key pt

$(0, -1)$

$+2 \quad -3$

$(2, -4)$

F) $f_{(x)} = 2(3)^{x-2} + 1$

y int

$f(0) = 2(3)^{-2} + 1$

$2\left(\frac{1}{9}\right) + 1$

$\frac{2}{9} + 1$

$1\,2/9$

$(0, 1\frac{2}{9})$

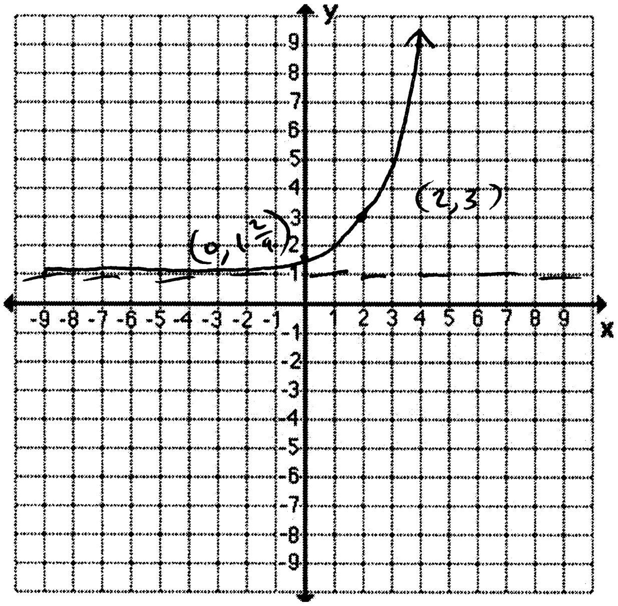
x int

$0 = 2(3)^{x-2} + 1$

$2(3)^{x-2} = -1$

$\log 3^{x-2} = \log -\frac{1}{2}$

↑ not possible



Y-intercept: $(0, 1\frac{2}{9})$

X-intercepts: none

Range: $(1, \infty)$

Domain: $(-\infty, \infty)$

Key pt

$x - 2 = 0$

$x = 2$

$f(2) = 2(3)^0 + 1$

$2 + 1$

3

$(2, 3)$

Logarithmic Functions

$$f_{(x)} = a\log_n(bx+c)+d \qquad f_{(x)} = a\ln(bx+c)+d$$

Logarithmic functions will be graphed in the same manner as radical functions. It is first necessary to find the domain of the logarithmic function. The range of a logarithmic function is all real numbers, so only the domain needs to be found. To find the domain of a logarithmic function evaluate $bx+c>0$***. Remember, this is not ≥, because you cannot take the log of zero. Once the domain is found, it will tell in which direction the function is moving. This inequality will also help find the vertical asymptote for the function.***

If the x inside the log does not have a negative coefficient, the curve will be on the right side of the vertical asymptote. If the coefficient in front of x is 1, begin with the key point of (1,0). From that point on, treat the function just like an exponential function. Adding or subtracting to either the x or y values to find the new key point making the graph shift.

If the x inside the log has a negative coefficient, the curve will be on the left side of the vertical asymptote. If the coefficient in front of x is -1, begin with the key point of (-1,0) and shift from there.

***Once again, just like exponential growth and decay functions, watch the value of "a", as it affects the scale of the function. If the value of "a" is some number other that 1 or -1, find the key point algebraically before you translate the function.**

As the function shifts, it will be helpful to draw a broken line for both the horizontal and vertical asymptotes. It is OK to cross the horizontal asymptote, as you will find the key point always rests on it. The vertical asymptote, however, may never be crossed.

$$f_{(x)} = a\log_n(bx+c)+d \qquad f_{(x)} = a\ln(bx+c)+d$$

Solving for $bx+c=0$***, will yield the equation for the vertical asymptote. The equation for the horizontal asymptote is*** $y=d$***.***

$$f_{(x)} = \log_3(x-4)+2$$

Finding the domain.

$$x-4>0$$
$$x>4$$

Notice the similarity in the procedures.

Finding the vertical asymptote.

$$x-4=0$$
$$x=4$$

Finding the horizontal asymptote.

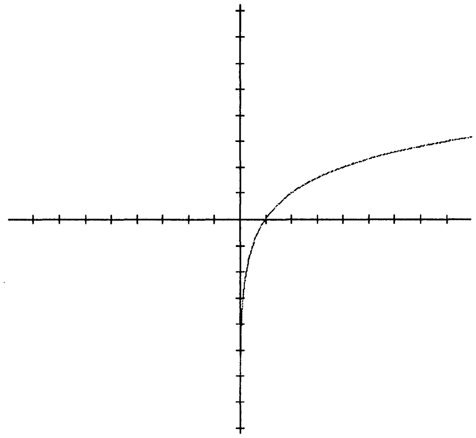
$$y=2$$

There is no real work involved with finding the horizontal asymptote. Identify the vertical shift. This is the equation of the horizontal asymptote.

***If the variable x inside the log has a coefficient other than 1 or -1, the key point will be different. The key point must then be found algebraically. To find the x value of the key point solve for $bx+c=1$. Substitute that solution back into the problem to find the y value.**

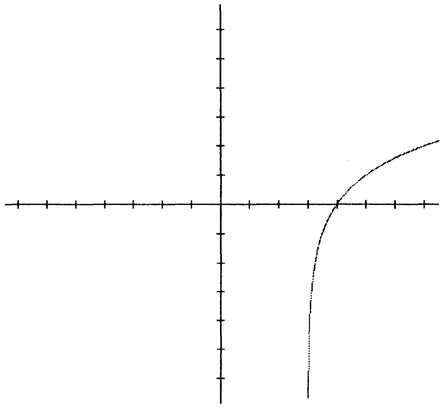
$$f_{(x)} = a\log_n\left(bx+c\right)+d \qquad f_{(x)} = a\ln\left(bx+c\right)+d$$

$$f_{(x)} = \log_2 x$$



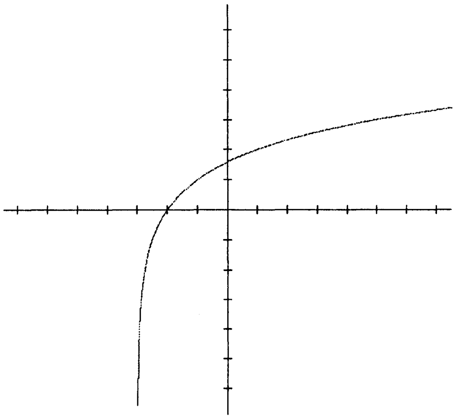
The parent function has the key point at (1, 0)

$$f_{(x)} = \log_2\left(x-3\right)$$



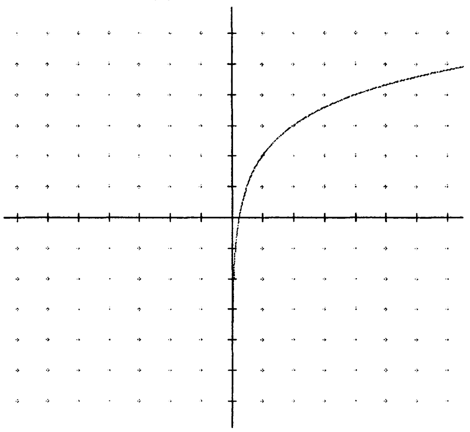
The graph of this function shifts right 3. Notice the key point moved to the right 3 places to (4,0).

$$f_{(x)} = \log_2\left(x+3\right)$$



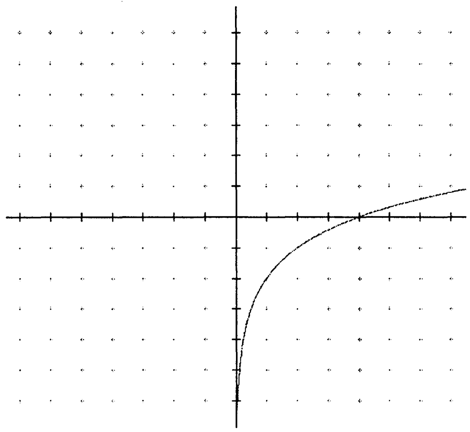
The graph of this function shifts to the left 3. The new key point is (-2,0).

$$f_{(x)} = \log_2 x + 2$$



This function shifts up 2. Add 2 to the y value of the key point, and it is now at (1,2).

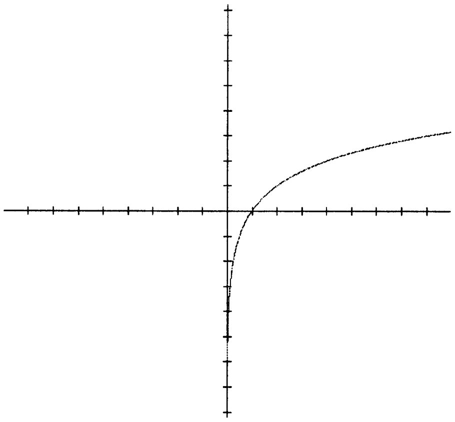
$$f_{(x)} = \log_2 x - 2$$



This function shifts down 2. Subtracting 2 from the y value of the key point results in (1,-2).

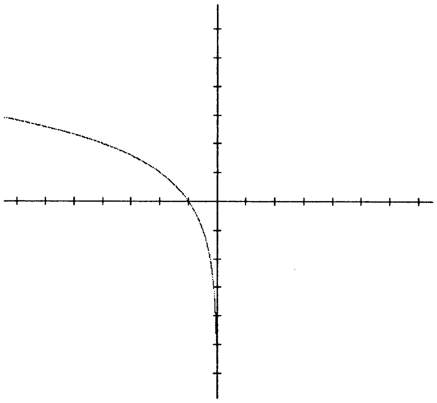
$f_{(x)} = a\log_n(bx+c)+d$ $\qquad$ $f_{(x)} = a\ln(bx+c)+d$

$f_{(x)} = \log_2 x$



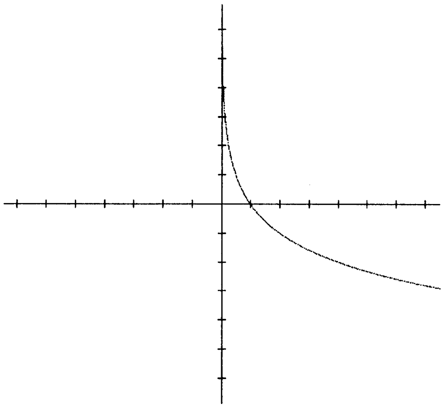
The parent function has the key point at (1, 0)

$f_{(x)} = \log_2(-x)$



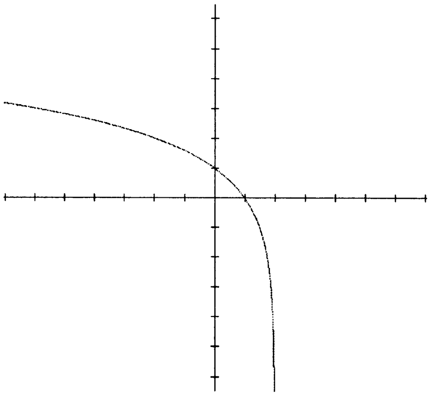
The graph of this function reflects about the vertical asymptote. Key point is now (-1,0).

$f_{(x)} = -\log_2 x$



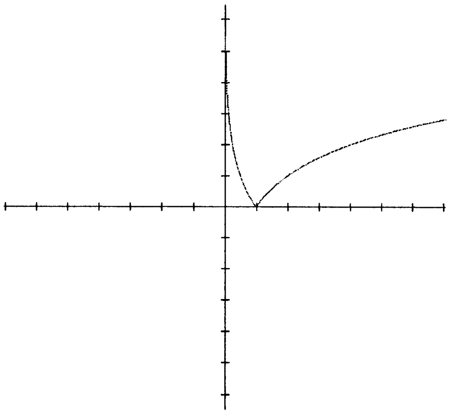
The graph of this function is reflected about the horizontal asymptote. Key point is still at (1,0).

$f_{(x)} = \log_2(2-x)$



Since the coefficient of x is -1, this graph will be on the left side of the vertical asymptote. Begin with the key point (-1,0), and shift right 2 because it is positive. Add 2 to the x value of the key point. The new key point is (1,0).

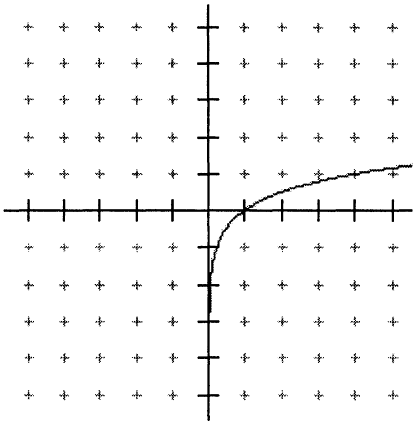
$f_{(x)} = |\log_2 x|$



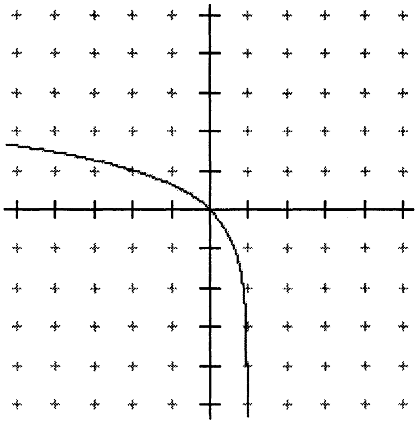
Notice the negative portion of the graph reflected above the x axis.

Match the appropriate graph with its equation below. Explain why each of your solutions is true.

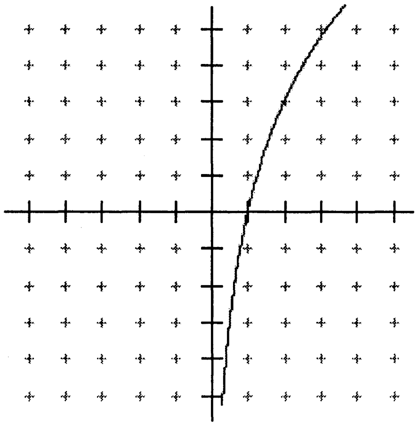
A



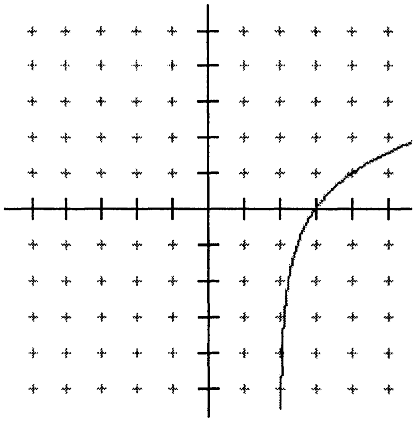
B



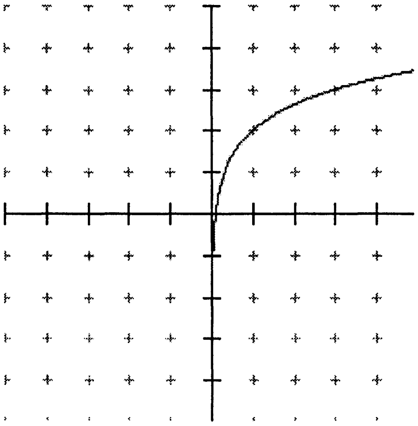
C



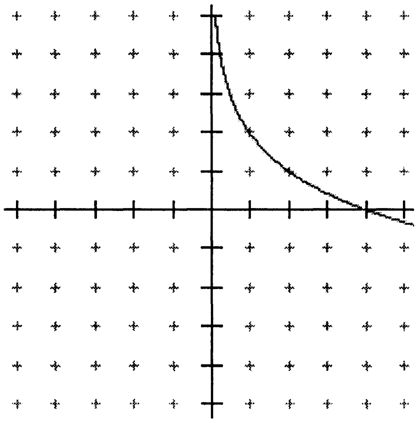
D



E



F



1) $f_{(x)} = \log_2(x-2)$

D, std. log function that shifts rt. 2, The new key point is (3,0)

2) $f_{(x)} = \log_3(1-x)$

domain $1-x>0$ $x<1$

Key pt $1-x=1$ $x=0$

B, function curves to the left. Domain is $(-\infty,1)$ and key point is (0,0)

3) $f_{(x)} = -\log_2 x + 2$

F, function is upside down and shifts up 2. The new key point is (1,2)

4) $f_{(x)} = \log_3 x + 2$

E, std. log function that shifts up 2, the new key point is (1,2)

5) $f_{(x)} = \frac{1}{2}\log_2 x$

A, function whose scale has changes by a factor of ½. The new key point is (1,0)

6) $f_{(x)} = 3\log_2 x$

C, the scale of this function increases by a factor of 3. Key point is (1,0)

The translation of a logarithmic function is almost identical to that of an exponential function. Just make sure to identify on which side of the vertical asymptote the graph of the function will reside. This will determine which key point to begin with. Remember to draw both asymptotes to graph the function and watch for the value of "a" which will affect key point.

Graph each of the following logarithmic functions by finding the asymptotes and labeling the key point. Be sure to find the x intercept and y intercept (if they exist).

A) $f_{(x)} = \log_3 x + 2$

y int

$f(x) = \log_3 0 + 2$

not possible

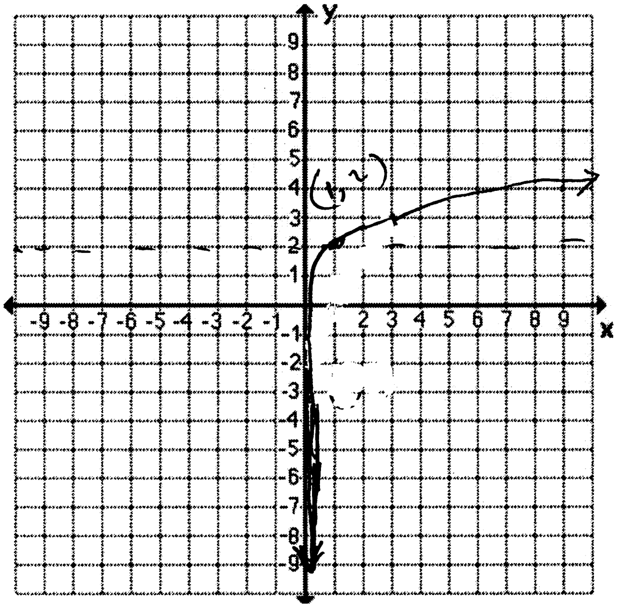
x int

$0 = \log_3 x + 2$

$\log_3 x = -2$

$3^{-2} = x$

$x = 1/9$



Y-intercept: None

X-intercepts: (1/9, 0)

Range: (-∞, ∞)

Domain: (0, ∞)

Key pt

(1, 0)

+2

(1, 2)

B) $f_{(x)} = \log_2 (x+3)$

y int

$f(0) = \log_2 3$

$f(0) \approx 1.6$

x int

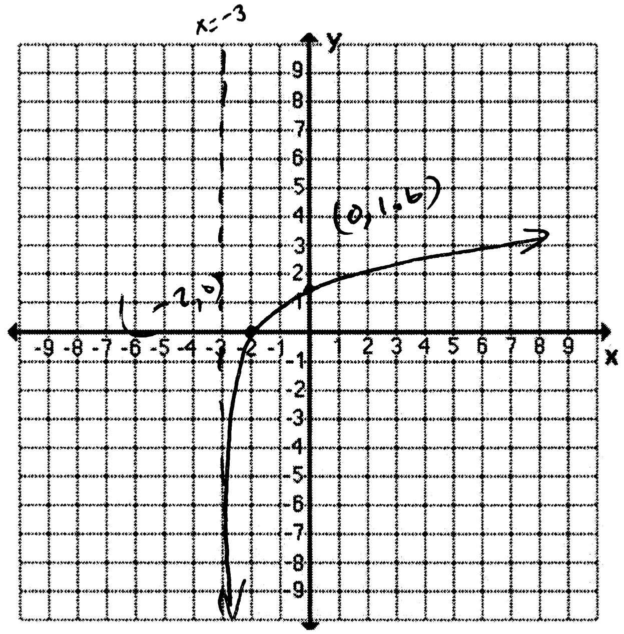
$0 = \log_2 (x+3)$

$2^0 = x+3$

$1 = x+3$

$-3 \quad -3$

$x = -2$



Y-intercept: (0, 1.6)

X-intercepts: (-2, 0)

Range: (-∞, ∞)

Domain: (-3, ∞)

Key pt

(1, 0)

-3

(-2, 0)

C) $f_{(x)} = -\log_2 x + 4$

y int

$f(0) = -\log_2 0 + 4$
$\uparrow$
not possible

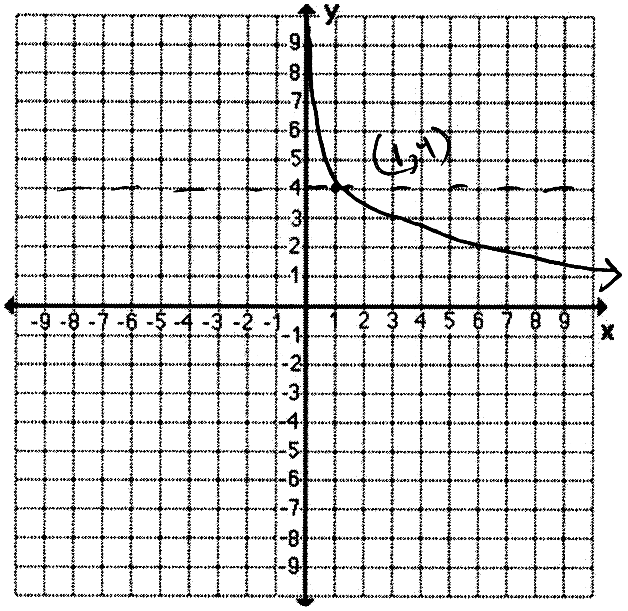
x int

$0 = -\log_2 x + 4$

$\log_2 x = 4$

$2^4 = x$

$x = 16$



Y-intercept: none

X-intercepts: (16, 0)

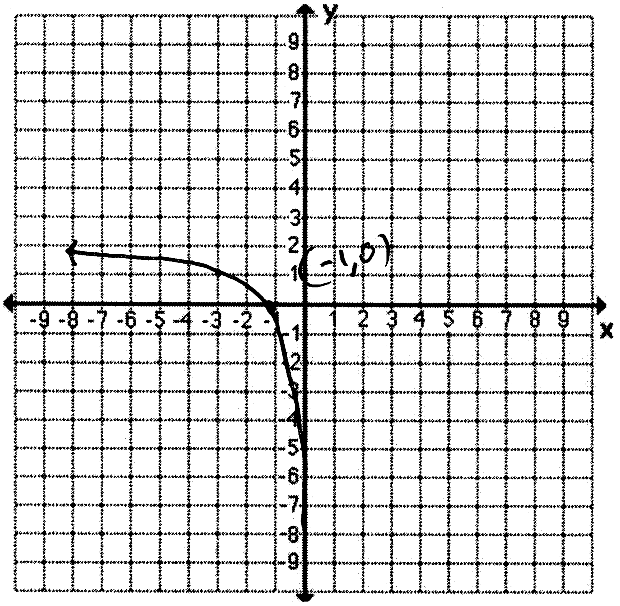
Range: $(-\infty, \infty)$

Domain: $(0, \infty)$

Key pt

(1, 0)
+4
(1, 4)

D) $f_{(x)} = \ln(-x)$



Y-intercept: none

X-intercepts: (-1, 0)

Range: $(-\infty, \infty)$

Domain: $(-\infty, 0)$

Domain

$\frac{-x}{-1} > \frac{0}{-1}$

$x < 0$

Key pt

(-1, 0)

E) $f_{(x)} = -\ln(x-2)$

y int

$f(0) = -\ln(-2)$

↗

not possible

x int

$\frac{0}{-1} = \frac{-\ln(x-2)}{-1}$

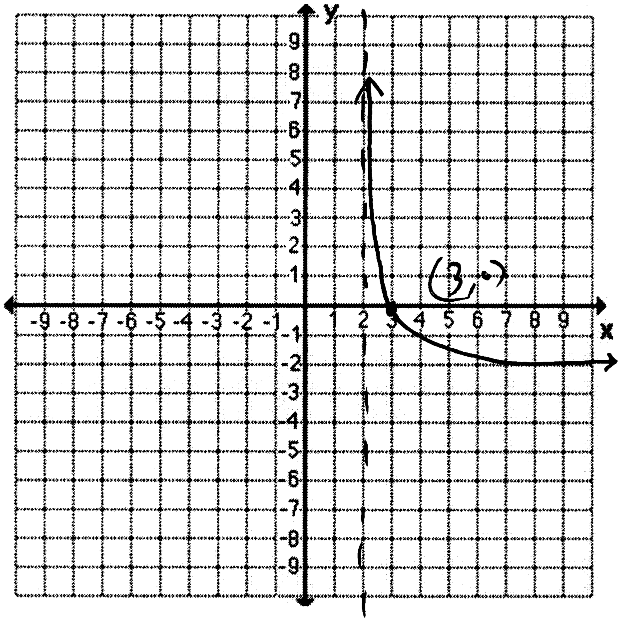
$\ln(x-2) = 0$

$e^0 = x-2$

$1 = x-2$

$+2 \quad +2$

$x = 3$



Y-intercept: none

X-intercepts: (3, 0)

Range: $(-\infty, \infty)$

Domain: $(2, \infty)$

Key pt

(1, 0)

+2

(3, 0)

Domain

$x-2>0$

$x>2$

F) $f_{(x)} = \log_2(3-x)+2$

y int

$f(0) = \log_2 3 + 2$

$f(0) \approx 3.6$

x int

$0 = \log_2(3-x)+2$

$\log_2(3-x) = -2$

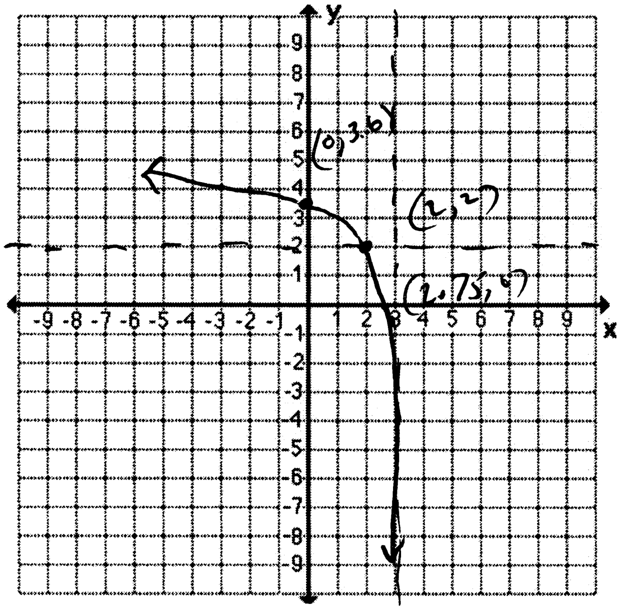
$2^{-2} = 3-x$

$\frac{1}{4} = 3-x$

$-3 \quad -3$

$\frac{-2\frac{3}{4}}{-1} = \frac{-x}{-1}$

$x = 2.75$



Y-intercept: (0, 3.6)

X-intercepts: (2.75, 0)

Range: $(-\infty, \infty)$

Domain: $(-\infty, 3)$

Domain

$3-x>0$

$3>x$

$x<3$

Key pt

$3-x = 1$

$-3 \quad -3$

$-x = -2$

$x = 2$

$\log_2(3-2)+2$

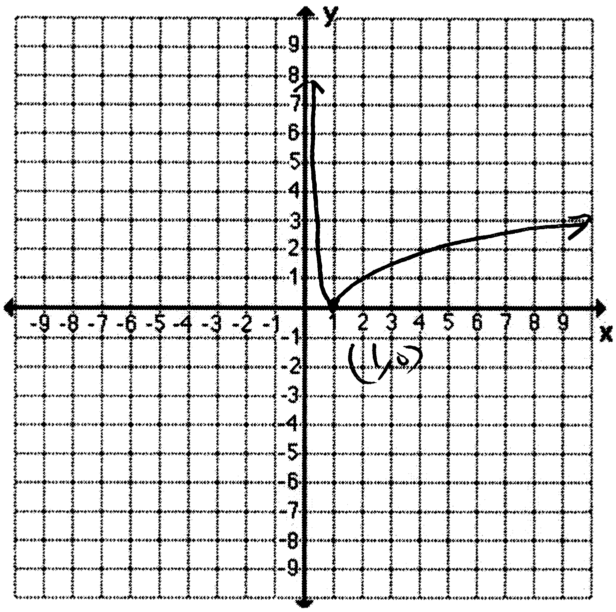
$\log_2 1 + 2$ (2, 2)

$0+2$

2

G) $f_{(x)} = |\ln x|$

Std ln function (-) values reflected upwards because of abs. value around entire function



Y-intercept: none

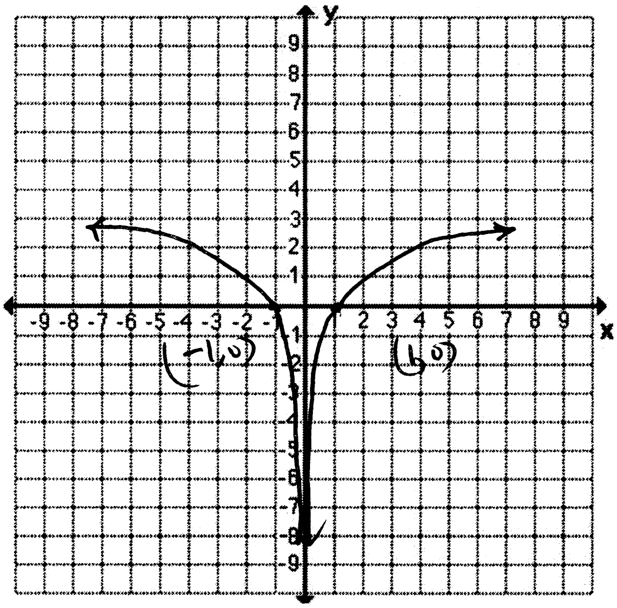
X-intercepts: (1, 0)

Range: [0, ∞)

Domain: (0, ∞)

H) $f_{(x)} = \ln|x|$

because abs. value is around x, (-) values of x are ok.



Y-intercept: none

X-intercepts: (-1,0) (1,0)

Range: (-∞, ∞)

Domain: (-∞,0) ∪ (0,∞)

All standard logarithmic functions (meaning a function without absolute value symbols), must have an x intercept. All standard exponential growth and decay functions must have a y intercept. Are these two statements true? Why or why not?

This is true because the range of a logarithmic function is $(-\infty,\infty)$ and the domain of an exponential function is $(-\infty,\infty)$. The two functions are inverses of each other.

In order to find the domain of the logarithmic function $f_{(x)} = \log_4(x+5)-3$, we need to evaluate $x+5>0$. Why must we use this inequality?

This inequality must be used because we cannot evaluate the log of zero or a negative number.

What is the problem with relying on a graphing calculator to graph a logarithmic function?

Most calculators will not show the tail end of the function as it approaches the asymptote.

Cubic Functions

The cubic function is similar to the cubed root. You will notice similarities in the shape of the curve. Translations are the same as any standard function. The range and domain of any cubic function is all real numbers.

$$f_{(x)} = a(x-h)^3 + k$$

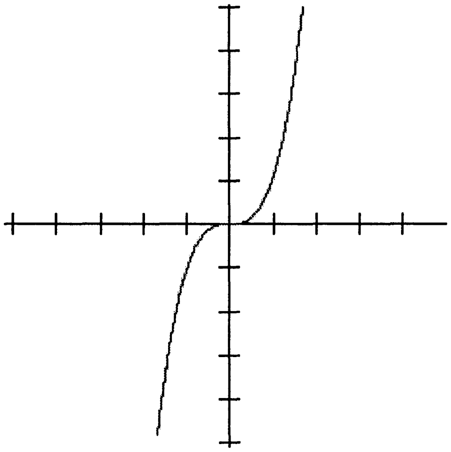
Let us look at this as the standard form of a cubic function. The center of the cubic function is given by (h,k). To find the x and y intercepts of the function, follow the standard procedures of substituting zero for one of the values, and solving for the remaining variable.

All cubic functions in this section will be given to you in this standard form. In the next section of the workbook, we will address polynomial functions that are greater than 2nd degree. These functions will have no standard from with which to work. We will be graphing them by alternative means.

For now we will concentrate on the function $f_{(x)} = x^3$, and translations of this.

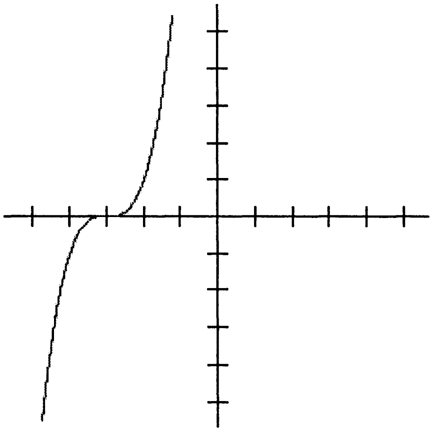
$$f_{(x)} = a(x-h)^3 + k$$

$$f_{(x)} = x^3$$



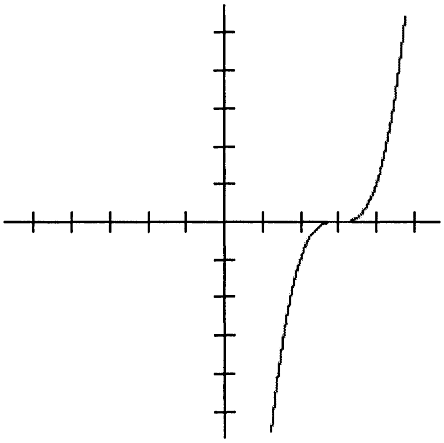
The parent function has the point of origin at (0, 0)

$$f_{(x)} = (x+3)^3$$



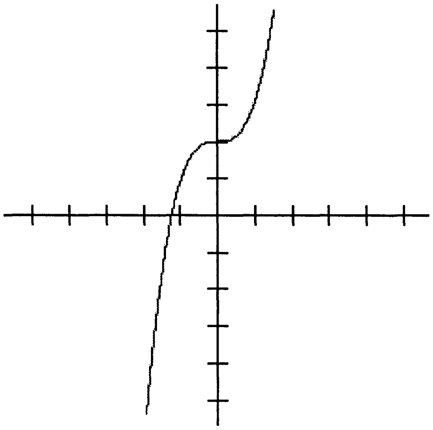
The graph of this function shifts left 3.

$$f_{(x)} = (x-3)^3$$



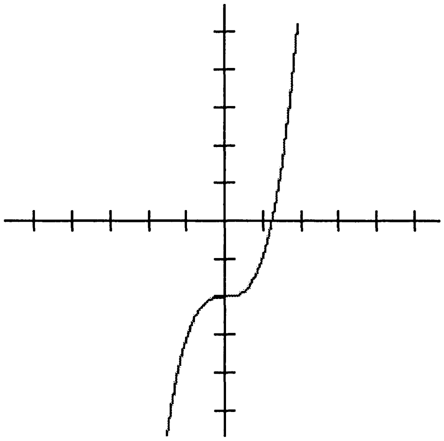
The graph of this function shifts right 3.

$$f_{(x)} = x^3 + 2$$



Graph shifts up 2.

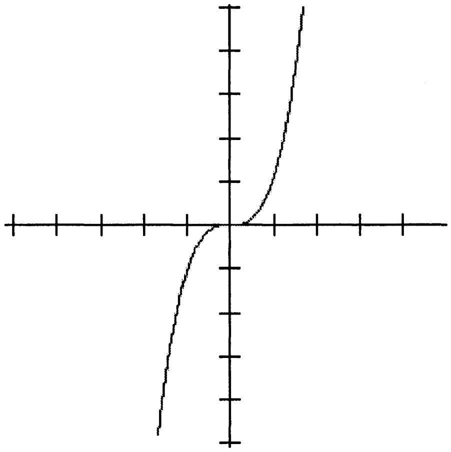
$$f_{(x)} = x^3 - 2$$



Graph shifts down 2.

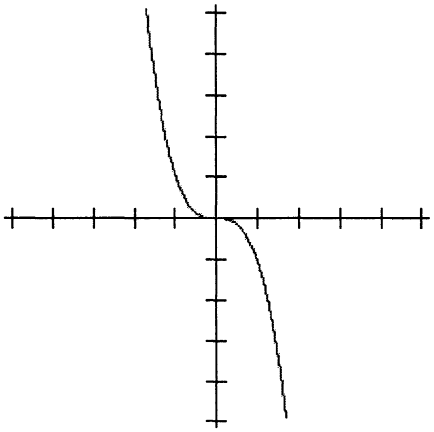
$$f_{(x)} = a(x-h)^3 + k$$

$$f_{(x)} = x^3$$



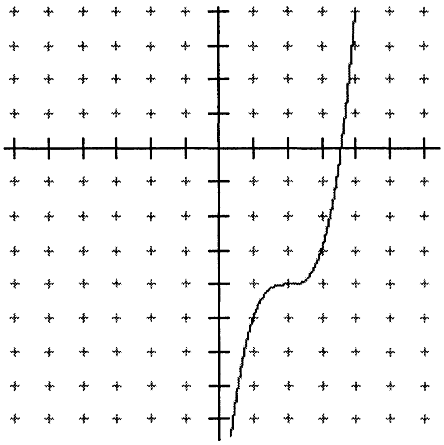
The parent function has the point of origin at (0, 0)

$$f_{(x)} = -x^3$$



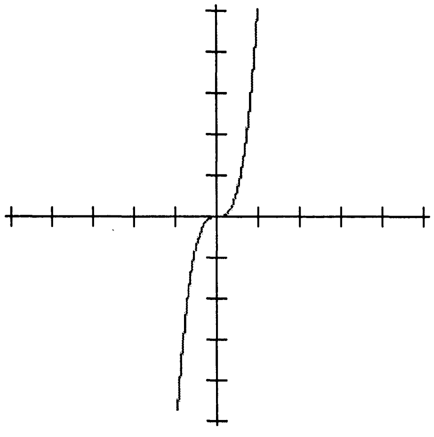
The graph of this function flips upside down.

$$f_{(x)} = (x-2)^3 - 4$$

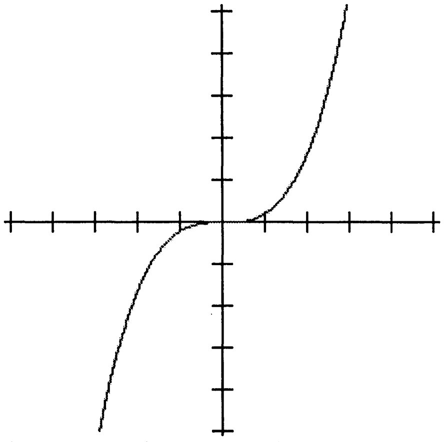


The graph of this function shifts right 2, down 4. vertex at (2,-4).

$$f_{(x)} = 5x^3$$



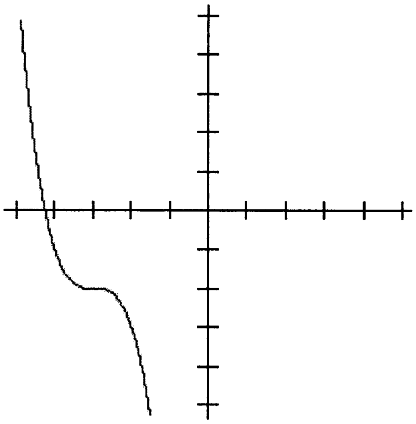
$$f_{(x)} = \frac{1}{5}x^3$$



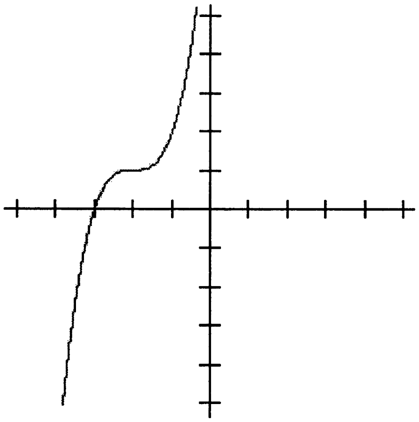
Once again, note difference the value of "a" makes in terms of the scale of the graph.

Match the appropriate graph with its equation below. Explain why each of your solutions is true.

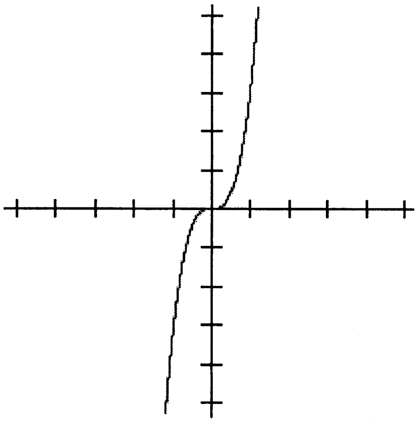
A



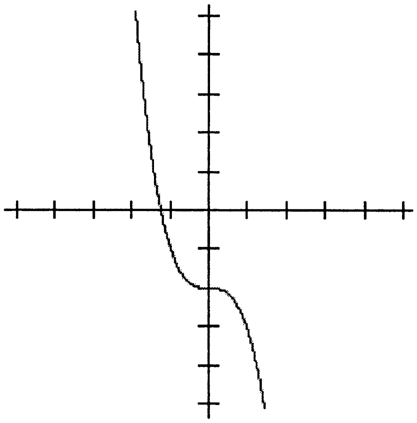
B



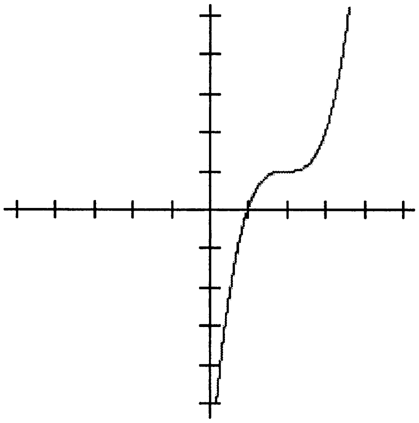
C



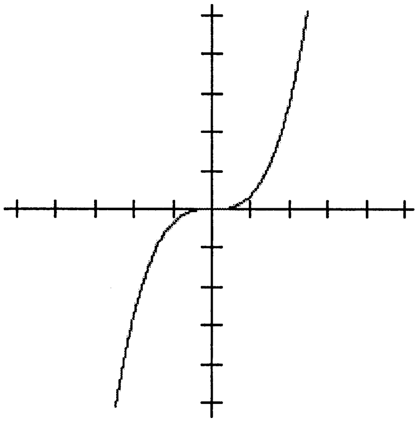
D



E



F



1) $f_{(x)} = -x^3 - 2$

D, the function is flipped right to left and shifts down 2. vertex is at (0,-2)

2) $f_{(x)} = (x-2)^3 + 1$

E, shifts up 1, right 2. vertex is at (2,1)

3) $f_{(x)} = (x+2)^3 + 1$

B, shifts left 2, up 1. vertex is at (-2,1)

4) $f_{(x)} = \frac{1}{3}x^3$

F, the scale of this function is increased by a factor of 1/3 meaning the function is wider. the vertex is at (0,0)

5) $f_{(x)} = 3x^3$

C, The scale of the function increases by a factor of 3 meaning the function appears more narrow. Vertex at (0,0)

6) $f_{(x)} = -(x+3)^3 - 2$

A, the function is flipped right to left. It also shifts left 3, down 2. Vertex is at (-3, -2)

Graph each of the following cubic functions. ***Label the vertex, find all intercepts, and the range and domain of each of the following. Don't worry about graphing the intercept if it is too far off the chart.***

A) $f_{(x)} = (x-2)^3 + 3$

y int

$f(0) = (-2)^3 + 3$

$-8 + 3$

$f(0) = -5$

x int

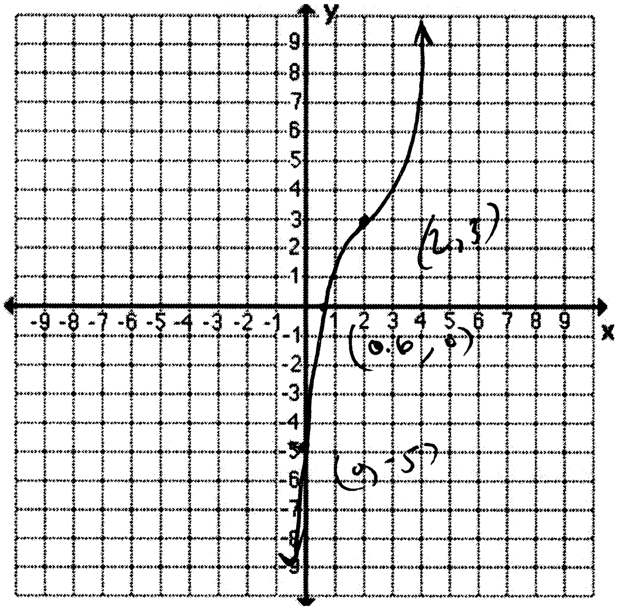
$0 = (x-2)^3 + 3$

$\sqrt[3]{(x-2)^3} = \sqrt[3]{-3}$

$x - 2 = \sqrt[3]{-3}$

$x = 2 + \sqrt[3]{-3}$

$x \simeq 0.6$



Vertex: (2,3)

Y-intercept: (0,-5)

X-intercepts: (0.6, 0)

Range: (-∞, ∞)

Domain: (-∞,∞)

B) $f_{(x)} = -(x+2)^3 - 1$

y int

$f(0) = -(0+2)^3 - 1$

$-8 - 1$

$f(0) = -9$

x int

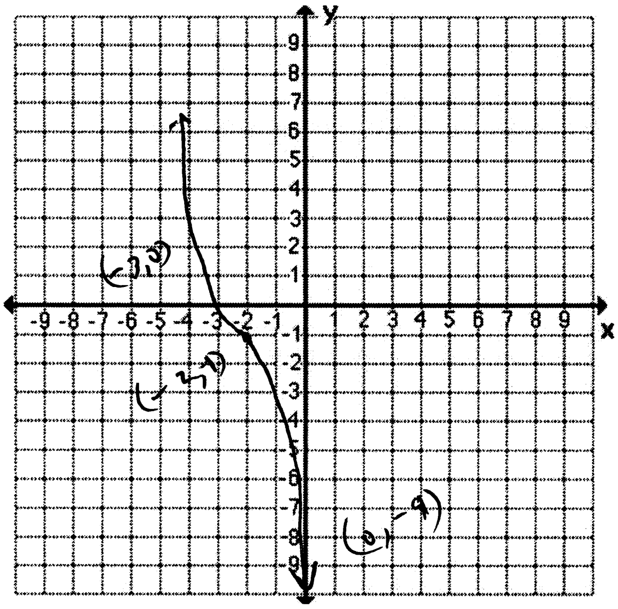
$0 = -(x+2)^3 - 1$

$\sqrt[3]{(x+2)^3} = \sqrt[3]{-1}$

$x + 2 = -1$

$-2 \quad -2$

$x = -3$



Vertex: (-2,-1)

Y-intercept: (0,-9)

X-intercepts: (-3, 0)

Range: (-∞,∞)

Domain: (-∞,∞)

C) $f_{(x)} = x^3 + 3$

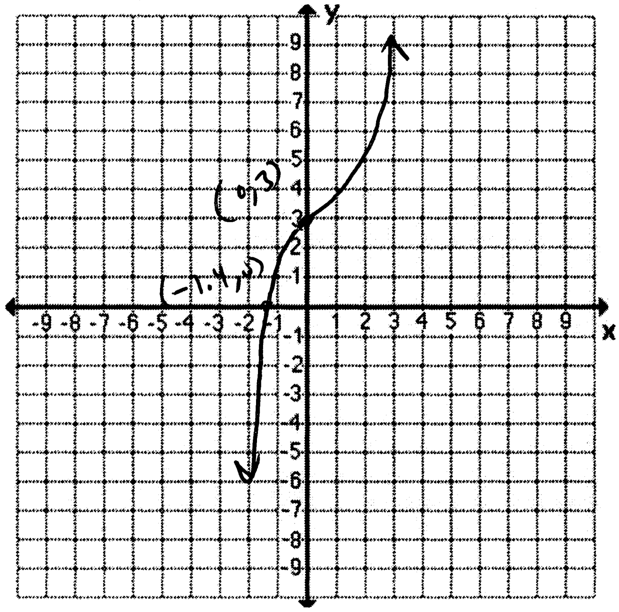
x int

$0 = x^3 + 3$

$x^3 = -3$

$x = \sqrt[3]{-3}$

$x \approx -1.4$



Vertex: (0 3)

Y-intercept: (0,3)

X-intercepts: (-1.4, 0)

Range: (-∞, ∞)

Domain: (-∞,∞)

D) $f_{(x)} = (x+3)^3 - 5$

y int

$f(0) = (3)^3 - 5$

$27 - 5$

$f(0) = 22$

x int

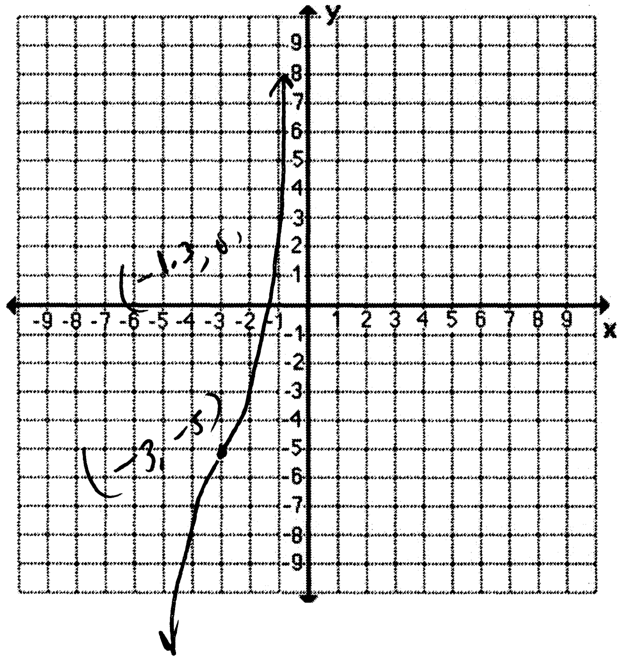
$0 = (x+3)^3 - 5$

$(x+3)^3 = 5$

$x+3 = \sqrt[3]{5}$

$x = \sqrt[3]{5} - 3$

$x \approx -1.3$



Vertex: (-3, -5)

Y-intercept: (0, 22)

X-intercepts: (-1.3, 0)

Range: (-∞, ∞)

Domain: (-∞,∞)

E) $f_{(x)} = -(x-4)^3 + 2$

y int

$f(0) = -(-4)^3 + 2$

$64 + 2$

$f(0) = 66$

x int

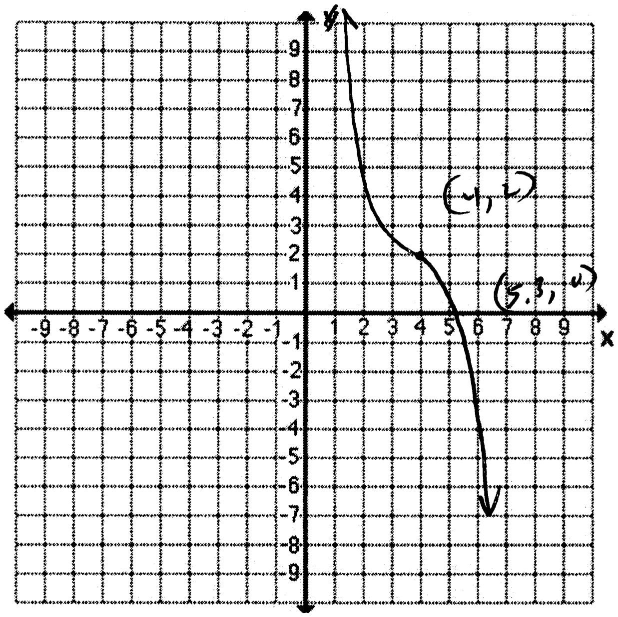
$0 = -(x-4)^3 + 2$

$(x-4)^3 = 2$

$x - 4 = \sqrt[3]{2}$

$x = 4 + \sqrt[3]{2}$

$x \approx 5.3$



Vertex: (4, 2)

Y-intercept: (0, 66)

X-intercepts: (5.3, 0)

Range: $(-\infty, \infty)$

Domain: $(-\infty, \infty)$

F) $f_{(x)} = (x-3)^3 - 5$

y int

$f(0) = (-3)^3 - 5$

$-27 - 5$

$f(0) = -32$

x int

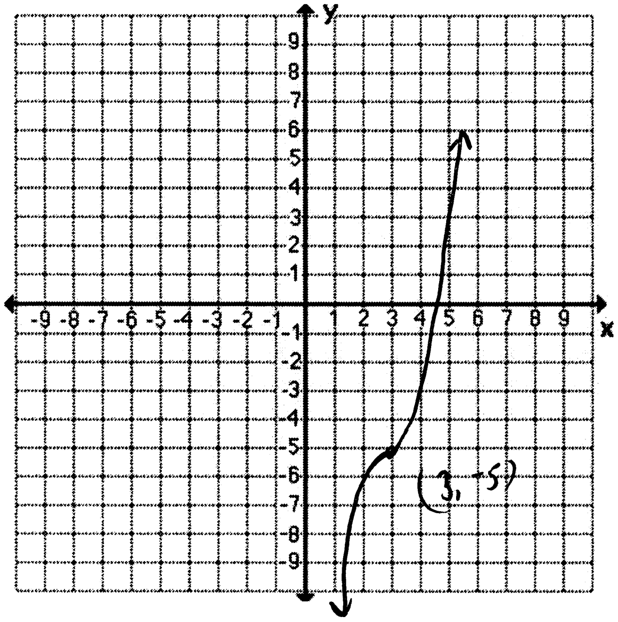
$0 = (x-3)^3 - 5$

$(x-3)^3 = 5$

$x - 3 = \sqrt[3]{5}$

$x = 3 + \sqrt[3]{5}$

$x \approx 4.7$



Vertex: (3, -5)

Y-intercept: (0, -32)

X-intercepts: (4.7, 0)

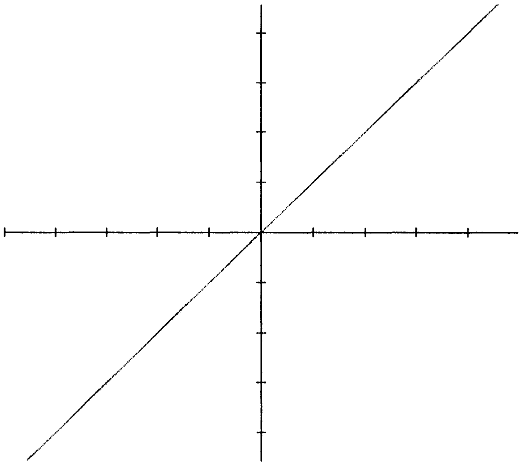
Range: $(-\infty, \infty)$

Domain: $(-\infty, \infty)$

Piece-Wise Functions

The best way to describe a piece-wise function is to look at a simple example. Consider the absolute value function.

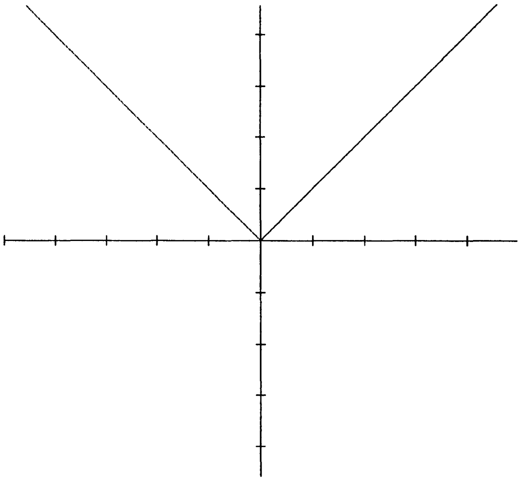
$y = x$



This is the graph of the function $y = x$. In this case, the x and y values of coordinates are identical. For example, (-3,-3). You can see the x and y values are the same.

Now, lets take a look at what happens when we want the absolute value of x.

$y = |x|$



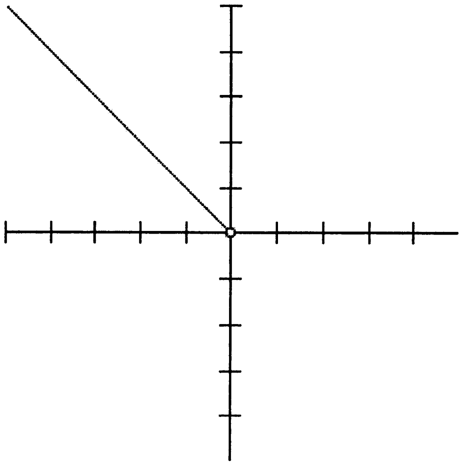
If the graph of $y = x$ above is $f_{(x)}$, the function to the left is $|f_{(x)}|$. We know that the absolute value of a number cannot be negative. If we take the absolute value of $f_{(x)}$, it would cause the left portion of the original graph to reflect above the x axis. This results in all y values of the function being positive. This is where the graph of the absolute value of x comes from.

It is possible to get this same graph if the linear function $y = x$ is graphed in the interval $[0, \infty)$, and the function $y = -x$ in the interval $(-\infty, 0)$. What happens here is a specific section of two different graphs is drawn. If these two sections are placed together on the same plane, it would result in the graph of $y = |x|$. Such an equation would appear as follows.

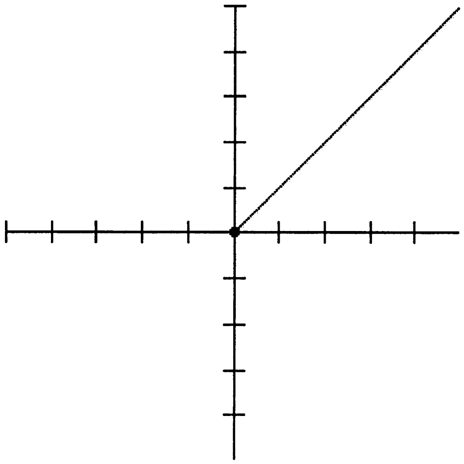
$$f_{(x)} = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

$$f_{(x)}=\begin{cases} x, & x\geq 0 \\ -x, & x<0 \end{cases}$$

$f_{(x)}=-x, \quad x<0$

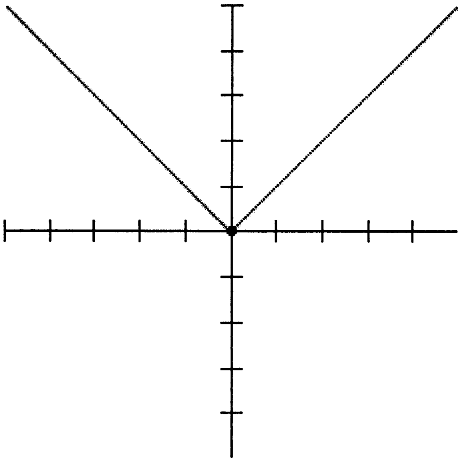


$f_{(x)}=\ x, \quad x\geq 0$



Notice the graph of the function on the left has a hole when x = 0. Since the function does not exist when x is zero, because of the domain of the function, an open circle must be used at that point. A pronounced dot is placed on the function to the right, showing the graph of the function in the interval $x \geq 0$. This is how you plot a point when you have the greater than or equal to sign in the domain of the function. If both of these functions are placed on the same plane, the result would be as follows.

$$f_{(x)}=\begin{cases} x, & x\geq 0 \\ -x, & x<0 \end{cases}$$

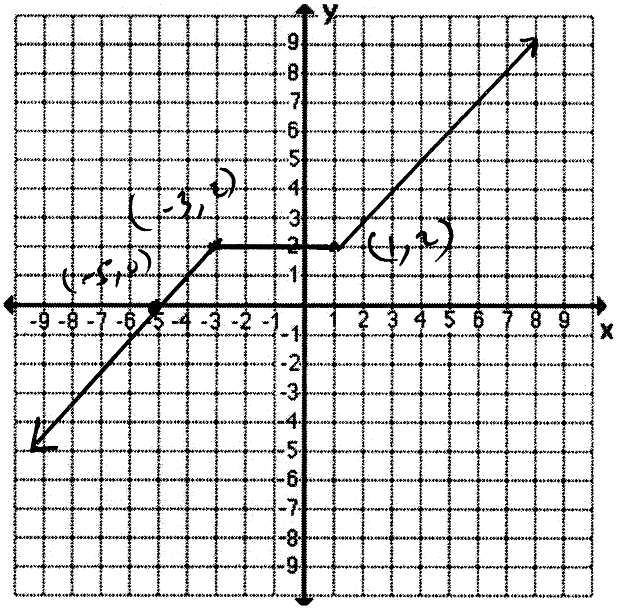


Since the open circle and solid dot are right on top of each other, the dot would "fill in" the hole on the first curve. This results in a continuous function. If the hole were still there, the function would have a removable discontinuity. If there is a complete break in the curve, from one portion of the graph to the next, it would be a discontinuous function.

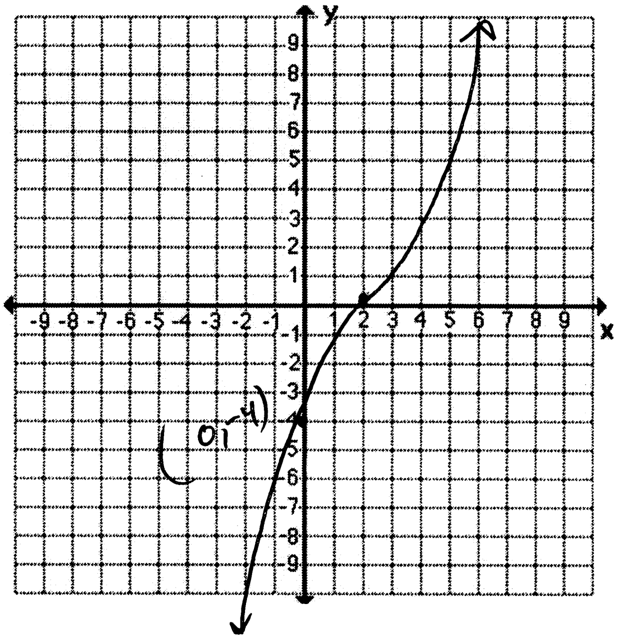
The simplest way to graph a piece-wise function is to graph the entire function and erase the portions that are not needed. This will be done for each part of the overall graph until a complete picture has been created. Very simply, a piece-wise function is just as it sounds, pieces of different functions put together to create one graph. Some examples of piecewise functions can be found in the "**Translations of Functions**" ***section of this chapter.***

Graph each of the following piece-wise functions.

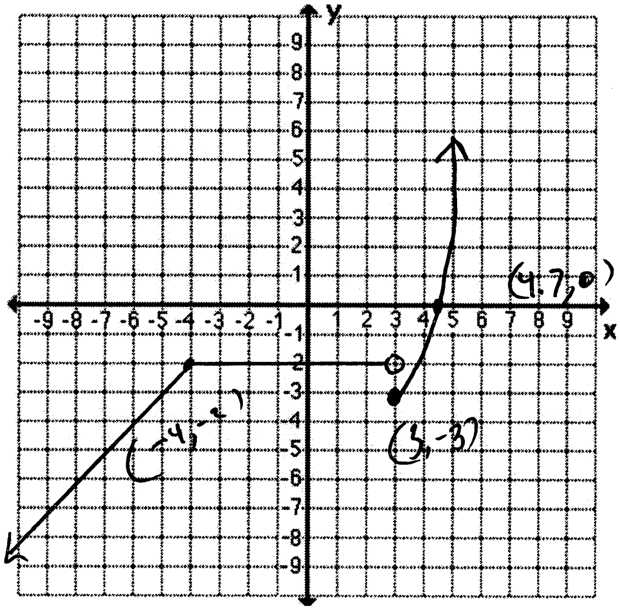
A) $f_{(x)} = \begin{cases} x+5, & x \le -3 \\ 2, & -3 < x < 1 \\ x+1, & x \ge 1 \end{cases}$



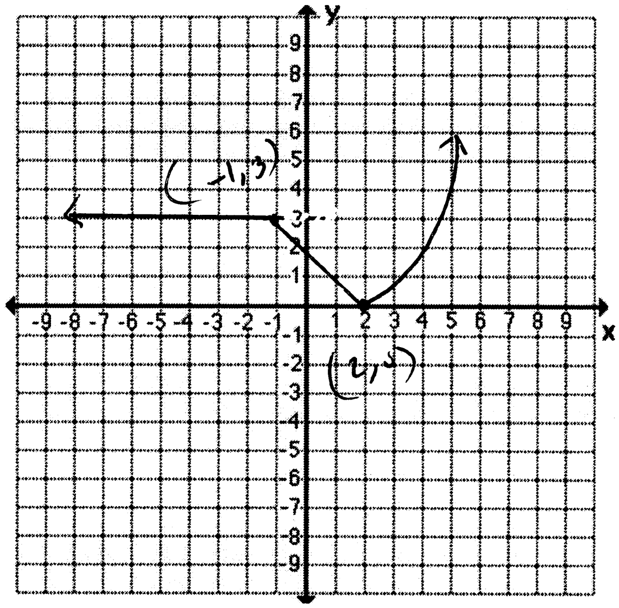
B) $f_{(x)} = \begin{cases} -(x-2)^2, & x < 2 \\ (x-2)^2, & x \ge 2 \end{cases}$



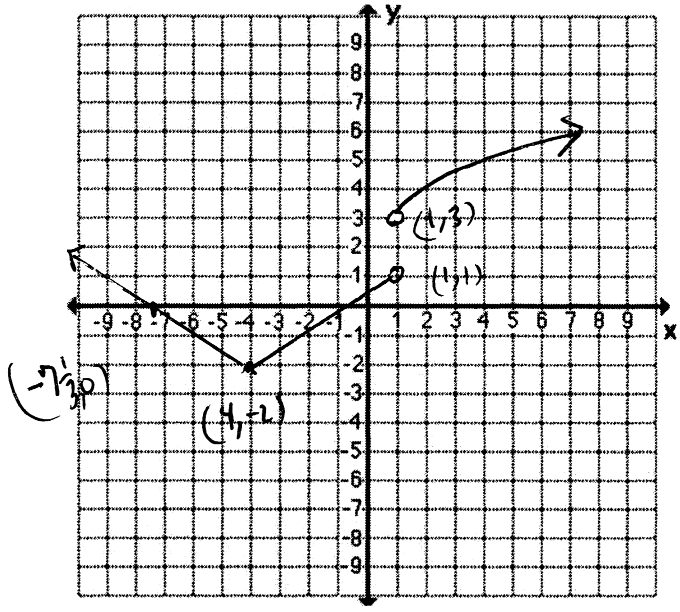
C) $f_{(x)} = \begin{cases} x+2, & x \le -4 \\ -2, & -4 < x < 3 \\ (x-3)^2 - 3, & x \ge 3 \end{cases}$



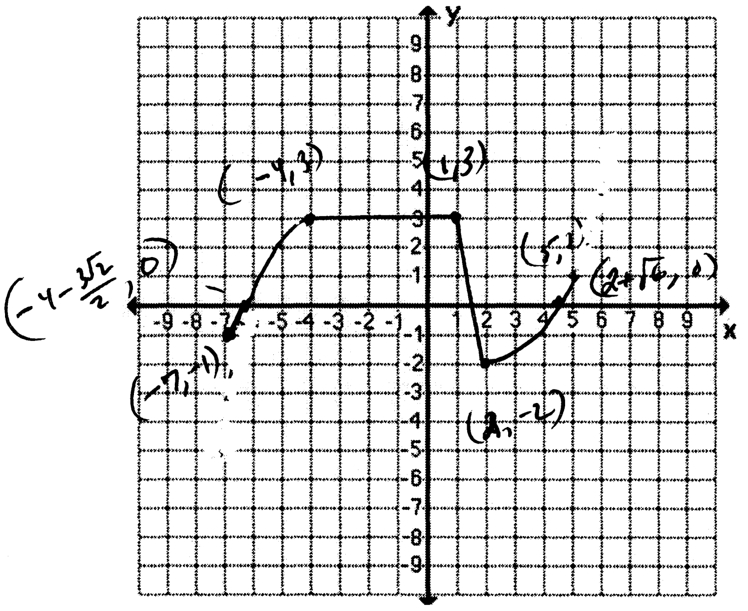
D) $f_{(x)} = \begin{cases} 3, & x < -1 \\ -x+2, & -1 \le x \le 2 \\ \frac{1}{3}(x-2)^2, & x > 2 \end{cases}$



E) $f_{(x)} = \begin{cases} \frac{3}{5}|x+4|-2, & x<1 \\ \sqrt{x-1}+3, & x>1 \end{cases}$



F) $f_{(x)} = \begin{cases} -\frac{2}{3}(x+4)^2+3, & -7 \le x < -4 \\ 3, & -4 \le x < 1 \\ -5x+8, & 1 \le x < 2 \\ \frac{1}{3}(x-2)^2-2, & 2 \le x \le 5 \end{cases}$



$f_{(-7)} = -\frac{2}{3}(-7+4)^2+3$

$(-3)^2$

$-\frac{2}{\cancel{3}}(\cancel{9})+3$

$-4+3$

-1

$f_{(5)} = \frac{1}{3}(5-2)^2-2$

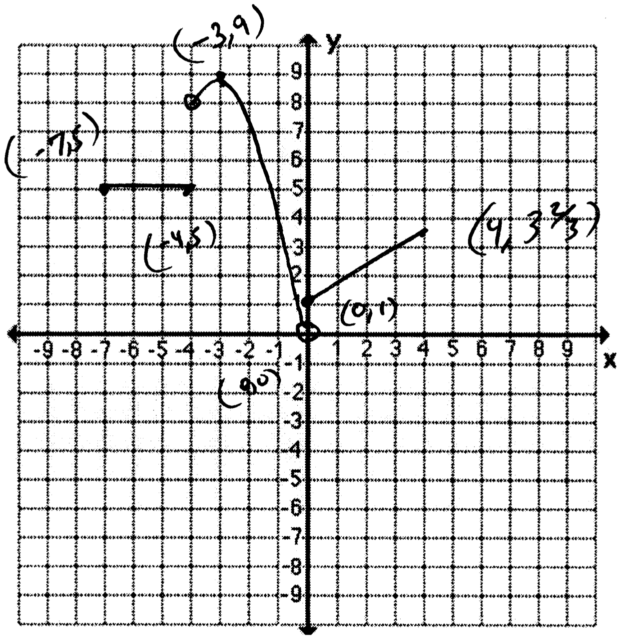
$\frac{1}{3}(9)-2$

$3-2$

$(5,1)$

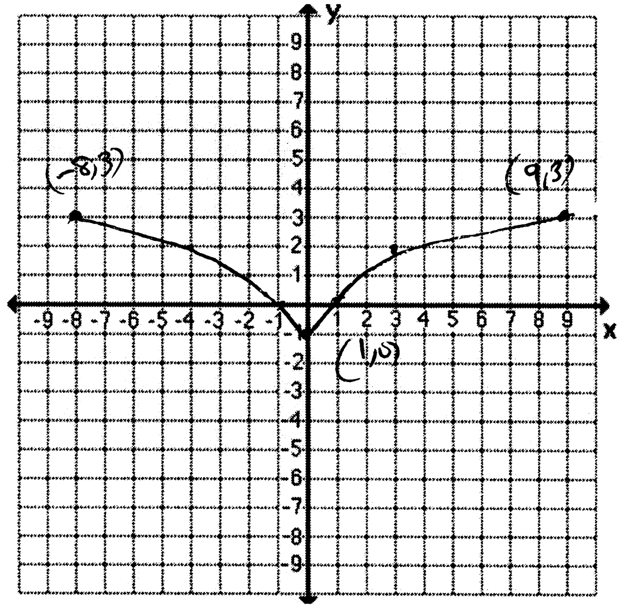
G)

$$f_{(x)} = \begin{cases} 5, & -7 \le x \le -4 \\ -(x+3)^2 + 9, & -4 < x < 0 \\ \frac{2}{3}x + 1, & 0 \le x \le 4 \end{cases}$$



H)

$$f_{(x)} = \begin{cases} \log_2(-x), & -8 \le x < -1 \\ |x| - 1, & -1 \le x \le 1 \\ \log_3 x, & 1 < x \le 9 \end{cases}$$



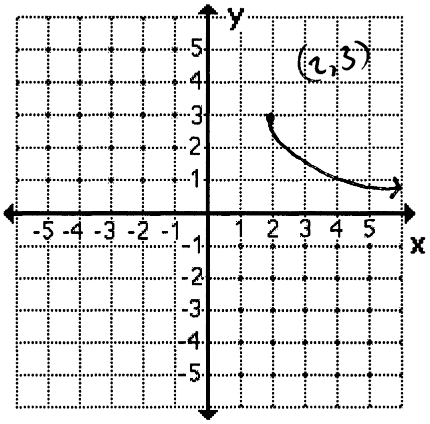
Which if any of the piece-wise functions you just graphed are discontinuous?

C and E are discontinuous

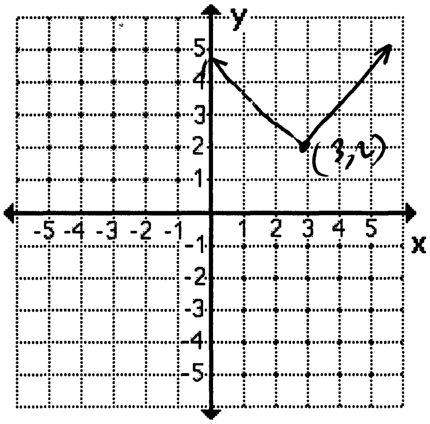
Functions Review

Graph each of the following functions. ***You need only label the key point or vertex for each. Do not worry about anything else. These problems are meant to quiz you on your knowledge of the parent functions and the translations thereof.***

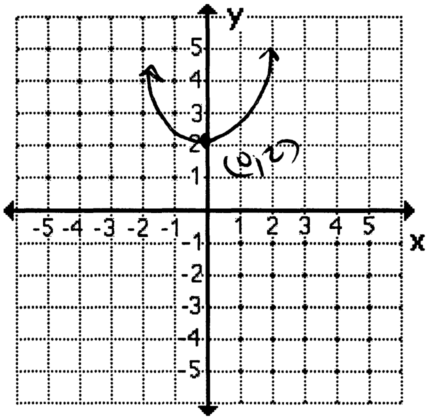
1. $f_{(x)} = -\sqrt{x-2} + 3$



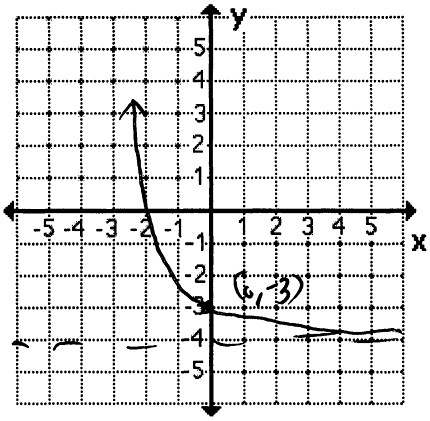
2. $f_{(x)} = |x-3| + 2$



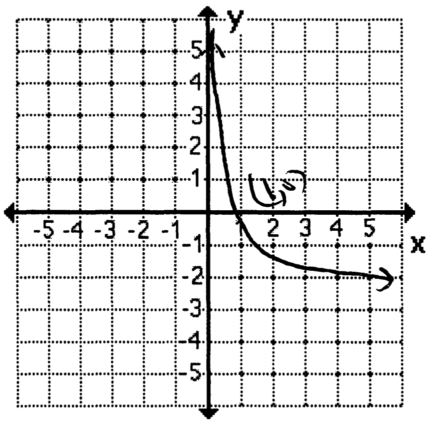
3. $f_{(x)} = x^2 + 2$



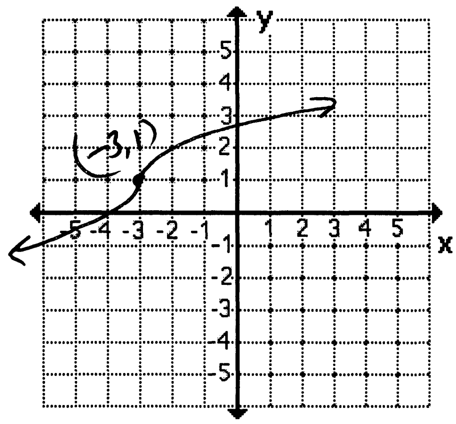
4. $f_{(x)} = \left(\frac{1}{2}\right)^x - 4$



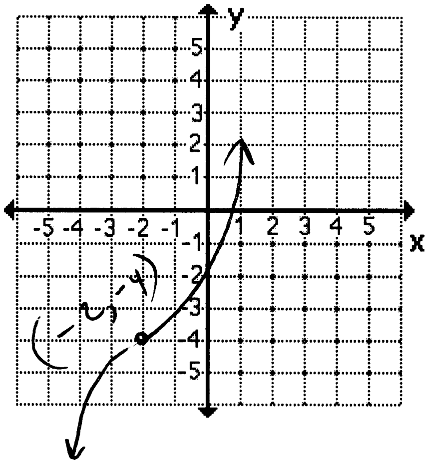
5. $f_{(x)} = -\log_3 x$



6. $f_{(x)} = \sqrt[3]{x+3} + 1$

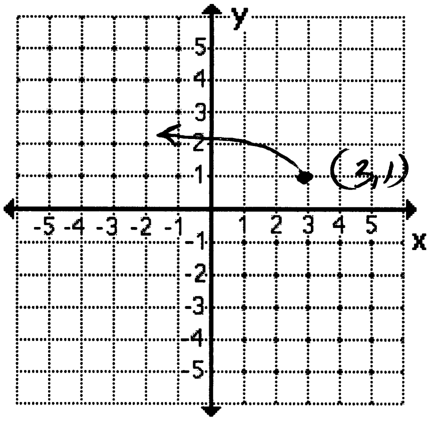


7. $f_{(x)} = (x+2)^3 - 4$

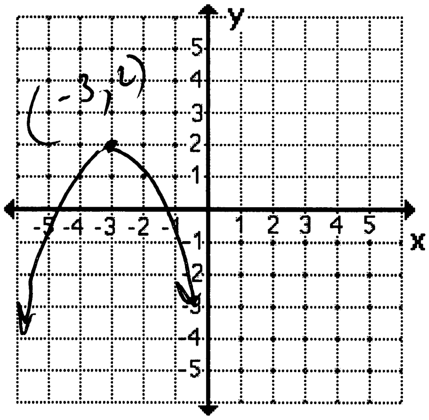


8. $f_{(x)} = \sqrt{-x+3} + 1$

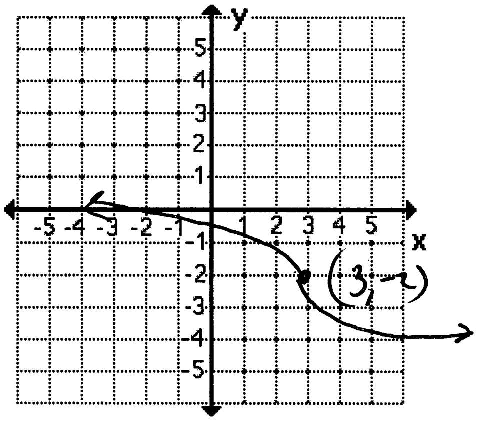
$-x+3 \geq 0$
$3 \geq x$
$x \leq 3$



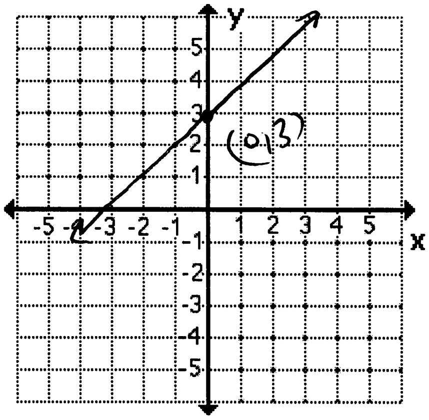
9. $f_{(x)} = -(x+3)^2 + 2$



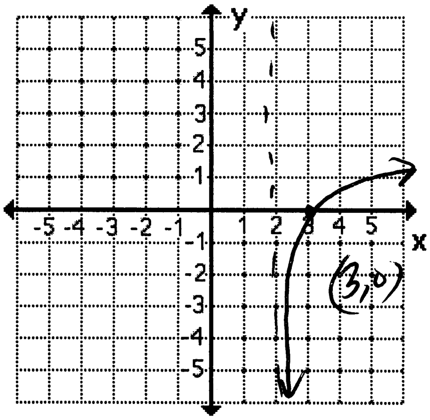
10. $f_{(x)} = -\sqrt[3]{x-3} - 2$



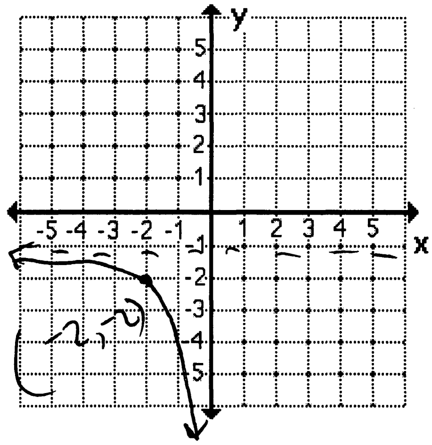
11. $f_{(x)} = x + 3$



12. $f_{(x)} = \ln(x-2)$

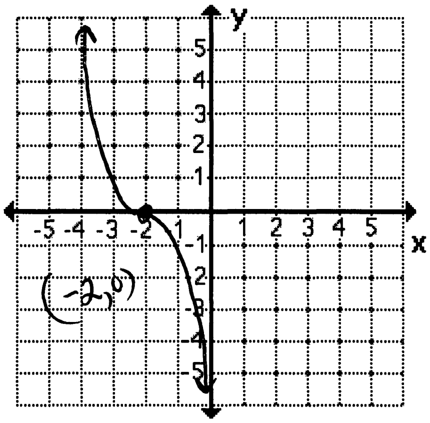


13. $f_{(x)} = -3^{x+2} - 1$

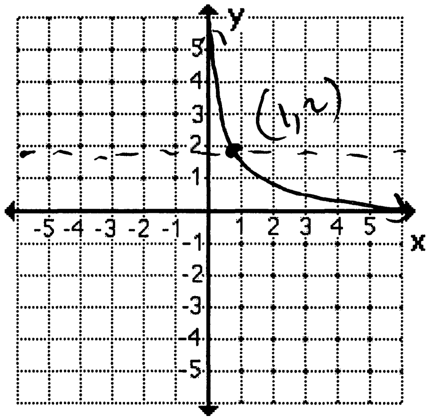


(0,-1)
-2 -1
(-2,-2)

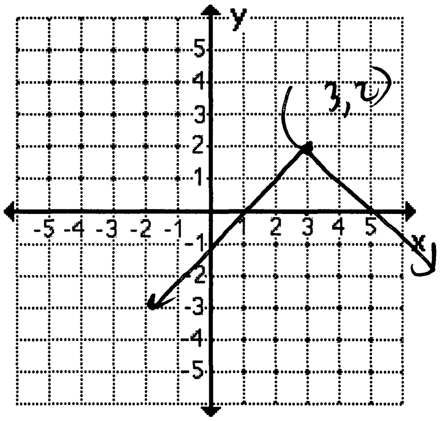
14. $f_{(x)} = -(x+2)^3$



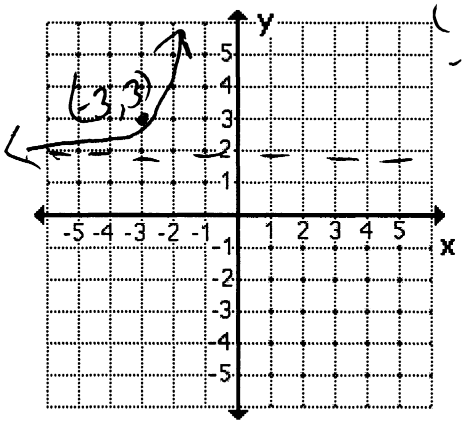
15. $f_{(x)} = -\log_2 x + 2$



16. $f_{(x)} = -|x-3| + 2$

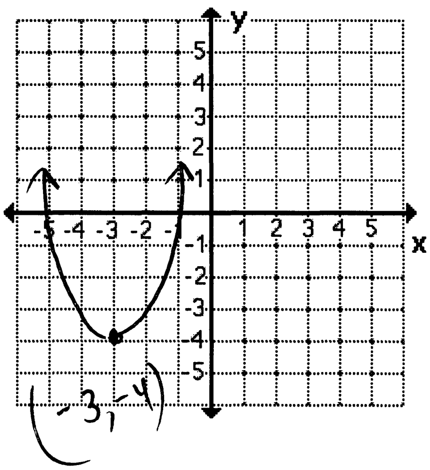


17. $f_{(x)} = e^{x+3} + 2$



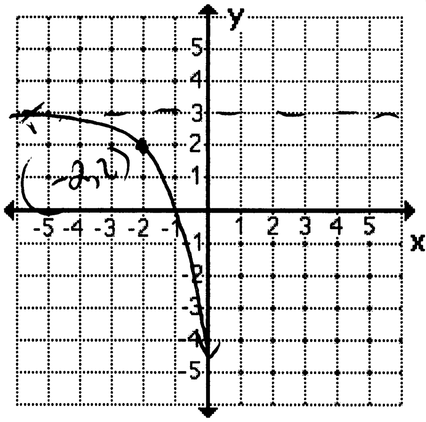
(0,1)
-3 +2
(-3,3)

18. $f_{(x)} = (x+3)^2 - 4$

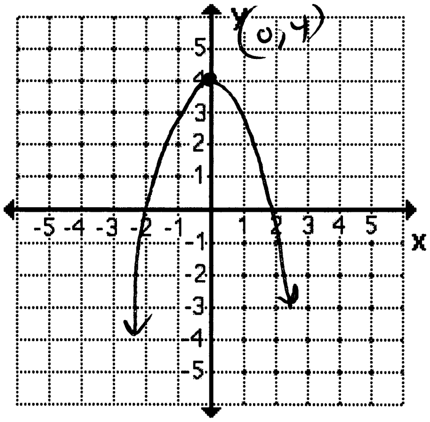


19. $f_{(x)} = -\left(\frac{1}{2}\right)^{x+2} + 3$

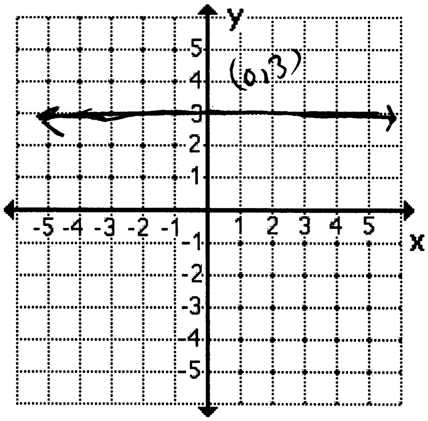
(0, -1)
-2 + 3
(-2, 2)



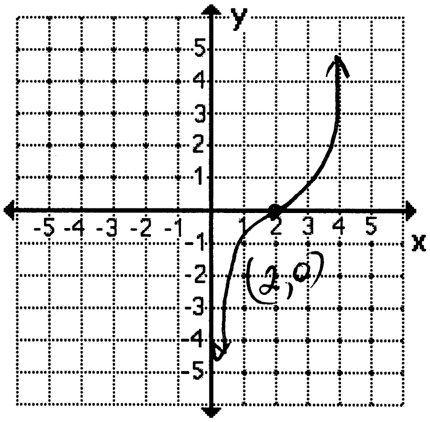
20. $f_{(x)} = -x^2 + 4$



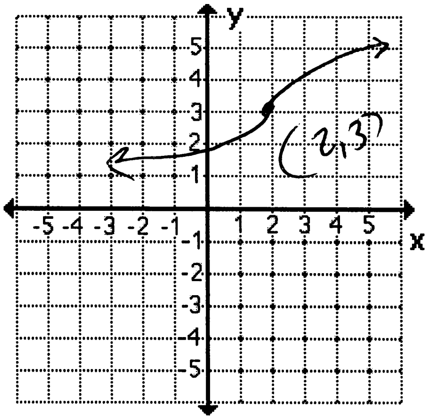
21. $f_{(x)} = 3$



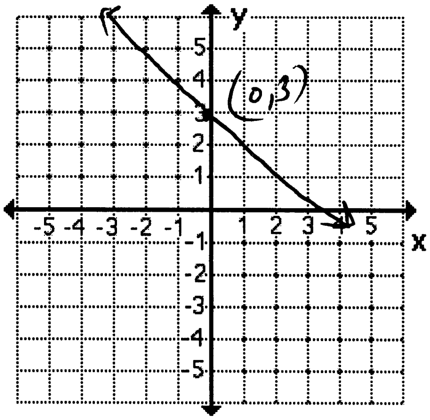
22. $f_{(x)} = (x-2)^3$



23. $f_{(x)} = \sqrt[3]{x-2} + 3$

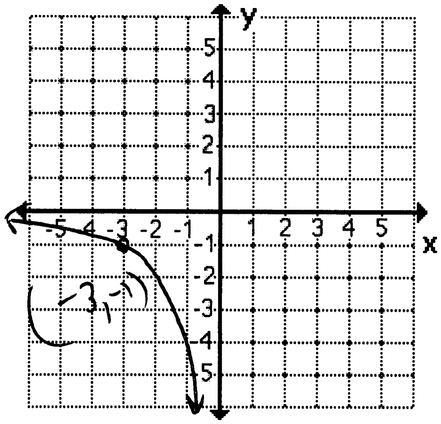


24. $f_{(x)} = -x + 3$



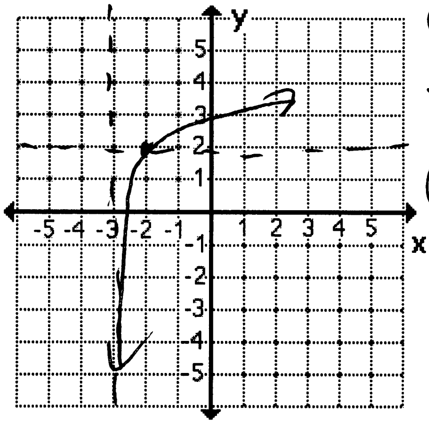
25. $f_{(x)} = -e^{x+3}$

(0,-1)
-3
(-3,-1)

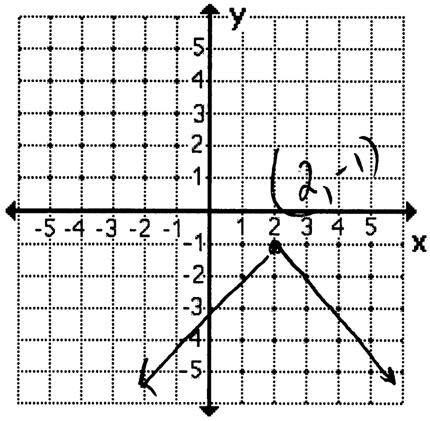


26. $f_{(x)} = \ln(x+3)+2$

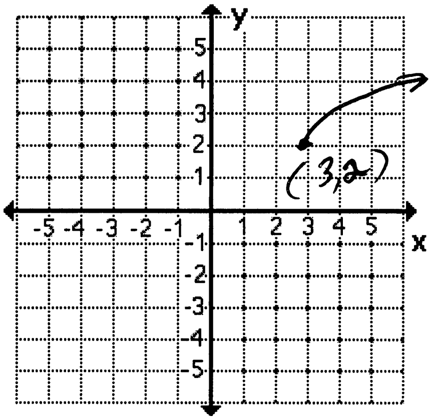
(1,0)
-3 +2
(-2,2)



27. $f_{(x)} = -|x-2|-1$

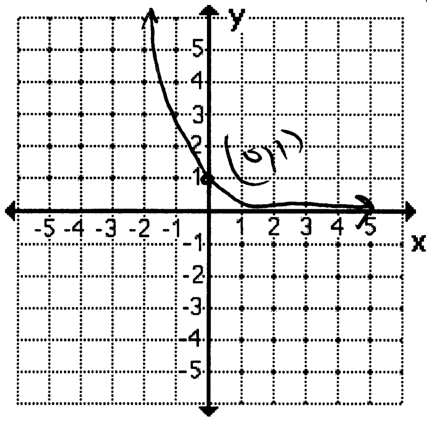


28. $f_{(x)} = \sqrt{x-3}+2$



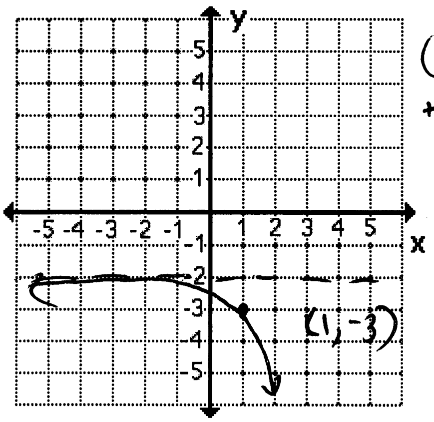
29. $f_{(x)} = 3^{-x}$

$\left(\frac{1}{3}\right)^x$



30. $f_{(x)} = -2^{x-1}-2$

(0,-1)
+1 -2
1,-3



Checking Progress

You have now completed the "Functions" section of the workbook. The following is a checklist so that you may check your progress. Check off each of the objectives you have accomplished.

The student should be able to…

QUADRATIC FUNCTIONS

- ☐ ***Determine the properties of a quadratic function in standard form.***
- ☐ ***Find the x and y intercepts of a quadratic function.***
- ☐ ***Find the range and domain of a quadratic function.***
- ☐ ***Find the vertex of a quadratic function in standard form.***
- ☐ ***Graph a quadratic function.***

ABSOLUTE VALUE FUNCTIONS

- ☐ ***Determine the properties of an absolute value function in standard form.***
- ☐ ***Find the x and y intercepts of an absolute value function.***
- ☐ ***Find the range and domain of an absolute value function.***
- ☐ ***Find the vertex of an absolute value function.***
- ☐ ***Graph an absolute value function.***

RADICAL FUNCTIONS

- ☐ ***Determine the properties of a radical function in standard form.***
- ☐ ***Find the x and y intercepts of a radical function.***
- ☐ ***Find the range and domain of a radical function.***
- ☐ ***Find the point of origin of a radical function***
- ☐ ***Graph a radical function.***

EXPONENTIAL FUNCTIONS

- ☐ ***Determine the properties of an exponential function in standard form.***
- ☐ ***Find the x and y intercepts of an exponential function.***
- ☐ ***Find the range and domain of an exponential function.***
- ☐ ***Find the key point of an exponential function.***
- ☐ ***Graph an exponential function.***

LOGARITHMIC FUNCTIONS

- ☐ ***Determine the properties of a logarithmic function in standard form.***
- ☐ ***Find the x and y intercepts of a logarithmic function.***
- ☐ ***Find the range and domain of a logarithmic function.***
- ☐ ***Find the key point of a logarithmic function.***
- ☐ ***Graph a logarithmic function.***

Checklist continued.

The student should be able to…

CUBIC FUNCTIONS

- ☐ ***Determine the properties of a cubic function in standard form.***
- ☐ ***Find the x and y intercepts of a cubic function.***
- ☐ ***Find the range and domain of a cubic function.***
- ☐ ***Find the vertex of a cubic function.***
- ☐ ***Graph a cubic function.***

PIECEWISE FUNCTIONS

- ☐ ***Shift the graph of a function without actually knowing the equation, i.e. graphing*** $f_{(x+2)}$.
- ☐ ***Graph piece-wise functions.***

POLYNOMIAL FUNCTIONS

Objectives

The following is a list of objectives for this section of the workbook.

By the time the student is finished with this section of the workbook, he/she should be able to…

- ***Find the quotient of a division problem involving polynomials using the polynomial long division method.***
- ***Find the quotient of a division problem involving polynomials using the synthetic division method.***
- ***Use the rational zero test to determine all possible rational zeros of a polynomial function.***
- ***Use the rational zero test to determine all possible roots of a polynomial equation.***
- ***Use Descarte's Rule of Signs to determine the possible number of positive or negative roots of a polynomial equation.***
- ***Find all zeros of a polynomial function.***
- ***Use the remainder theorem to evaluate the value of functions.***
- ***Write a polynomial in completely factored form.***
- ***Write a polynomial as a product of factors irreducible over the reals.***
- ***Write a polynomial as a product of factors irreducible over the rationals.***
- ***Find the equation of a polynomial function that has the given zeros.***
- ***Determine if a polynomial function is even, odd or neither.***
- ***Determine the left and right behaviors of a polynomial function without graphing.***
- ***Find the local maxima and minima of a polynomial function.***
- ***Find all x intercepts of a polynomial function.***
- ***Determine the maximum number of turns a given polynomial function may have.***
- ***Graph a polynomial function.***

Math Standards Addressed

The following state standards are addressed in this section of the workbook.

Algebra II

3.0 Students are adept at operations on polynomials, including long division.

4.0 Students factor polynomials representing the difference of squares, perfect square trinomials, and the sum and difference of two cubes.

5.0 Students demonstrate knowledge of how real and complex numbers are related both arithmetically and graphically. In particular, they can plot complex numbers as points in the plane.

6.0 Students add, subtract, multiply, and divide complex numbers.

7.0 Students add, subtract, multiply, divide, reduce, and evaluate rational expressions with monomial and polynomial denominators and simplify complicated rational expressions, including those with negative exponents in the denominator.

8.0 Students solve and graph quadratic equations by factoring, completing the square, or using the quadratic formula. Students apply these techniques in solving word problems. They also solve quadratic equations in the complex number system.

10.0 Students graph quadratic functions and determine the maxima, minima, and zeros of the function.

Mathematical Analysis

4.0 Students know the statement of, and can apply, the fundamental theorem of algebra.

Polynomial Division

There are two methods used to divide polynomials. This first is a traditional long division method, and the second is synthetic division. Using either of these methods will yield the correct answer to a division problem. There are restrictions, however, as to when each can be used.

Synthetic division can only be used if the divisor is a first degree binomial.

For the division problem $\frac{x^3+4x^2-2x+1}{2x-1}$, ***the divisor,*** $2x-1$, ***is a first degree binomial, so you may use synthetic division.***

There are no restrictions as to when polynomial long division may be used. The polynomial long division method may be used at any time. If the divisor is a polynomial greater than first degree, polynomial long division must be used.

The Division Algorithm

When working with division problems, it will sometimes be necessary to write the solution using The Division Algorithm.

The Division Algorithm: $f_{(x)} = d_{(x)} \cdot q_{(x)} + r_{(x)}$

Simply put, the function = divisor · quotient + remainder

Is 12 divisible by 4? yes

Is 18 divisible by 3? yes

Is 15 divisible by 2? no

Is 32 divisible by 8? yes

Based on your observations from the previous questions, what determines divisibility?

If there are no remainders, then it is divisble

How can you determine whether or not the polynomial x^2-3x+2 **is a factor of** x^4+10x^2-4**?**

If you divide x^2-3x+2 goes in evenly leaving no remainders, then it is a factor.

Find the quotient of each of the following. You may use synthetic or long division, but you need to know when to use each.

A) $\dfrac{6x^3-16x^2+17x-6}{3x-2}$

$3x-2=0$
$3x=2$
$x=2/3$

$$\begin{array}{r|rrrr} 2/3 & 6 & -16 & 17 & -6 \\ & 0 & 4 & -8 & 6 \\ \hline & 6 & -12 & 9 & 0 \end{array}$$

$6x^2-12x+9$

or

$2x^2-4x+3$

B) $\dfrac{3x^3-17x^2+15x-25}{x-5}$

$$\begin{array}{r|rrrr} 5 & 3 & -17 & 15 & -25 \\ & 0 & 15 & -10 & 25 \\ \hline & 3 & -2 & 5 & 0 \end{array}$$

$3x^2-2x+5$

C) $\dfrac{x^4+3x^2+1}{x^2-2x+3}$

$x^2-2x+3\overline{)x^4+0x^3+3x^2+0x+1}$
$-x^4+2x^3-3x$
$2x^3+0x^2+0x+1$
$-2x^3+4x^2-6x$
$4x^2-6x+1$
$-4x^2+8x-12$
$2x-11$

$x^2+2x+4+\dfrac{2x-11}{x^2-2x+3}$

D) $\dfrac{x^4-x^3-12x^2-2x+8}{x-4}$

$$\begin{array}{r|rrrrr} 4 & 1 & -1 & -12 & -2 & 8 \\ & 0 & 4 & 12 & 6 & -8 \\ \hline & 1 & 3 & 0 & -2 & 0 \end{array}$$

x^3+3x^2-2

E) $\dfrac{6x^3+10x^2+x+8}{2x^2+1}$

$3x+5$
$2x^2+1\overline{)6x^3+10x^2+x+8}$
$-6x^3 \quad +3x$
$10x^2-2x+8$
$-10x^2 \quad +5$
$-2x+3$

$3x+5-\dfrac{2x-3}{2x^2+1}$

F) $\dfrac{x^3-1}{x-1}$

x^3+0x^2+0x-1

$$\begin{array}{r|rrrr} 1 & 1 & 0 & 0 & -1 \\ & 0 & 1 & 1 & 1 \\ \hline & 1 & 1 & 1 & 0 \end{array}$$

x^2+x+1

G) $\dfrac{x^5-4x^4+4x^3-13x^2+3x-1}{x^2+3}$

$x^2+3\overline{)x^5-4x^4+4x^3-13x^2+3x-1}$
$-x^5 \quad +3x^3$
$-4x^4+x^3-13x^2+3x-1$
$+4x^4 \quad +12x^2$
x^3-x^2+3x-1
$-x^3 \quad +3x$
$-x^2 \quad -1$
$+x^2+3$
2

$x^3-4x^2+x-1+\dfrac{2}{x^2+3}$

H) $\dfrac{2x^3+5x^2+2x+15}{2x^2-x+5}$

$x+3$
$2x^2-x+5\overline{)2x^3+5x^2+2x+15}$
$-2x^3+x^2+5x$
$6x^2-3x+15$
$-6x^2+3x+15$
0

$x+3$

I) $\dfrac{3x^3-16x^2-72}{x-6}$

$$\begin{array}{r|rrrr} 6 & 3 & -16 & 0 & 72 \\ & 0 & 18 & 12 & 72 \\ \hline \end{array}$$

$3x^2+2x+12$

When dividing polynomials using the long division method, how do you know when you are finished?

As soon as the degree is less than the degree of the divisor, you must stop.

Is $x+2$ a factor of x^3+8?

$$\begin{array}{r|rrrr} -2 & 1 & 0 & 0 & 8 \\ & 0 & -2 & 4 & -8 \\ \hline & 1 & -2 & 4 & \boxed{0} \end{array}$$

yes

Is $x-6$ a factor of $3x^3-16x^2-72$?

$$\begin{array}{r|rrrr} 6 & 3 & -16 & 0 & -72 \\ & 0 & 18 & 12 & 72 \\ \hline & 3 & 2 & 12 & \boxed{0} \end{array}$$

yes.

Describe the manner in which you determined whether or not the given binomials above were factors of their respective polynomials.

synthetic divison or polynomial long division is used. If there is no remainders, then the division is a factor of the divided.

The Rational Zero Test

The ultimate objective for this section of the workbook is to graph polynomial functions of degree greater than 2. The first step in accomplishing this will be to find all real zeros of the function. As previously stated, the zeros of a function are the x intercepts of the graph of that function. Also, the zeros of a function are the roots of the equation that makes up that function. You should remember, the only difference between an polynomial equation and a polynomial function is that one of them has $f_{(x)}$.

You will be given a polynomial equation such as $2x^4+7x^3-4x^2-27x-18=0$, ***and be asked to find all roots of the equation.***

The Rational Zero Test states that all possible rational zeros are given by the factors of the constant over the factors of the leading coefficient.

$$\frac{factors \quad of \quad the \quad constant}{factors \quad of \quad the \quad leading \quad coefficient} = all \quad possible \quad rational \quad zeros$$

Let's find all possible rational zeros of the equation $2x^4+7x^3-4x^2-27x-18=0$.

We begin with the equation $2x^4+7x^3-4x^2-27x-18=0$.

The constant of this equation is 18, while the leading coefficient is 2. We do not care about the (-) sign in front of the 18.

Writing out all factors of the constant over the factors of the leading coefficient gives the following.

$$\frac{\pm1, \ \pm2, \ \pm3, \ \pm6, \ \pm9, \ \pm18}{\pm1, \ \pm2}$$

These are not all possible rational zeros. To actually find them, take each number on top, and write it over each number in the bottom. If one such number occurs more than once, there is no need to write them both.

$$\pm1, \ \pm2, \ \pm3, \ \pm6, \ \pm9, \ \pm18, \ \pm\frac{1}{2}, \ \pm\frac{3}{2}, \ \pm\frac{9}{2}$$

These are all possible rational zeros for this particular equation.

The order in which you write this list of numbers is not important. The rational zero test is meant to assist in the overall objective of finding all zeros to the polynomial equation $2x^4+7x^3-4x^2-27x-18=0$. ***Each of these numbers is a potential root of the equation. Therefore, each will eventually be tested.***

Using the rational zero test, list all <u>possible</u> rational zeros of the following functions.

A) $f_{(x)} = 2x^4 - 6x^2 + 5x - 15$

$$\frac{\pm 1, \pm 3, \pm 5, \pm 15}{\pm 1, \pm 2}$$

$\pm 1, \pm 3, \pm 5, \pm 15, \pm 1/2, \pm 3/2, \pm 5/2, \pm 15/2$

B) $f_{(x)} = 3x^5 - 6x^4 + 2x^2 - 6x + 12$

$$\frac{\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12}{\pm 1, \pm 3}$$

$\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12, \pm 1/3, \pm 2/3, \pm 4/3$

C) $f_{(x)} = 8x^3 - 2x + 24$

$$\frac{\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 8, \pm 12, \pm 24}{\pm 1, \pm 2, \pm 4, \pm 8}$$

$\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 8, \pm 12, \pm 24, \pm 1/2, \pm 3/2, \pm 1/4,$

$\pm 3/4, \pm 1/8, \pm 3/8$

D) $f_{(x)} = 10x^3 - 15x^2 - 16x + 12$

$$\frac{\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12}{\pm 1, \pm 2, \pm 5, \pm 10}$$

$\pm 1, \pm 2, \pm 3, \pm 5, \pm 6, \pm 10, \pm 15, \pm 30, \pm 1/2, \pm 3/2, \pm 5/2,$

$\pm 15/2, \pm 1/4, \pm 3/4, \pm 5/4, \pm 15/4$

E) $f_{(x)} = -6x^3 + 5x^2 - 2x + 18$

$$\frac{\pm 1, \pm 2, \pm 3, \pm 6, \pm 9, \pm 18}{\pm 1, \pm 2, \pm 3, \pm 6}$$

$\pm 1, \pm 2, \pm 3, \pm 6, \pm 9, \pm 18, \pm 1/2, \pm 3/2, \pm 9/2, \pm 1/3, \pm 2/3, \pm 1/6$

F) $f_{(x)} = 4x^4 - 16x^3 + 12x - 30$

$$\frac{\pm 1, \pm 2, \pm 3, \pm 5, \pm 6, \pm 10, \pm 15, \pm 30}{\pm 1, \pm 2, \pm 4}$$

$\pm 1, \pm 2, \pm 3, \pm 5, \pm 6, \pm 10, \pm 15, \pm 30, \pm 1/2, \pm 3/2, \pm 5/2, \pm 15/2,$

$\pm 1/4, \pm 3/4, \pm 5/4, \pm 15/4$

G) $f_{(x)} = 4x^4 + 3x^3 - 2x^2 + 5x - 12$

$$\frac{\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12}{\pm 1, \pm 2, \pm 4}$$

$\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12, \pm 1/2, \pm 3/2, \pm 1/4, \pm 3/4$

H) $f_{(x)} = x^5 - 6x^4 + 12x^2 - 8x + 36$

$\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 9, \pm 12, \pm 18, \pm 36$

It is important to understand, these lists of possible zeros for each of the polynomial functions above, are also lists of possible roots for the polynomial equations contained therein.

Descarte's Rule of Signs

When solving these polynomial equations use the rational zero test to find all possible rational zeros first. Synthetic division will then be used to test each one of these possible zeros, until some are found that work. When we found all possible rational zeros of the equation $2x^4 + 7x^3 - 4x^2 - 27x - 18 = 0$***, there were 18 possible solutions to this equation. It could take a very long time to test each one. Luckily there is a rule to help narrow down these choices. Descarte's Rule of Signs can help to narrow the search of possible solutions to the equation.***

Descarte's Rule of Signs

- **The number of positive zeros can be found by counting the number of sign changes in the problem. The number of positive zeros is that number, or less by an even integer.**
- **The number of negative zeros can be found by evaluating** $f_{(-x)}$**. Count the number of sign changes, and the number of negative zeros is that number, or less by an even integer.**

When using Descarte's Rule of Signs, "less by an even integer," means subtract by two until there is 1 or 0 possible zeros.

Here is an example of how to use Descarte's rule of Signs to determine the possible number of positive and negative zeros for the equation $2x^4 + 7x^3 - 4x^2 - 27x - 18 = 0$**.**

To find the number of positive roots, count the number of sign changes in $2x^4 + 7x^3 - 4x^2 - 27x - 18 = 0$.

The signs only change once in the original equation, so there is only 1 positive zero.

Evaluating $f_{(-x)}$ ***results in*** $2x^4 - 7x^3 - 4x^2 + 27x - 18 = 0$. ***To evaluate*** $f_{(-x)}$***, substitute –x for x. When this is done, only the terms were variables are being raised to odd powers change signs.***

Here, the signs changed 3 times. That means there are either 3 or 1 negative zeros.

Knowledge of complex roots will be used in conjunction with Descarte's Rule of Signs to create a table of possible combinations. Remember, COMPLEX NUMBERS ALWAYS COME IN CONJUGATE PAIRS when solving equations.

Using Descarte's Rule of Signs, state the possible number of positive zeros for each of the following functions.

A) $f_{(x)} = 3x^4 - 6x^3 + 2x^2 - x + 2$

4 sign changes 4, 2 or 0

B) $f_{(x)} = -x^5 + 2x^4 - 3x^3 - 7x + 2$

3 sign changes 3 or 1

C) $f_{(x)} = 3x^6 - 2x^5 + 7x^4 + 5x^3 - x^2 + 2x - 1$

5 sign changes 5, 3 or 1

D) $f_{(x)} = -6x^4 - 5x^2 - 8$

no sign changes

none

E) $f_{(x)} = -5x^5 + 6x^4 - 3x^2 + x - 15$

4 sign changes 4, 2 or 0

F) $f_{(x)} = \frac{1}{2}x^6 - 3x^5 + 7x^3 - 5x^2 + x$

4 sign changes 4, 2 or 0

G) $f_{(x)} = x^5 - 3x^4 + 2x^3 + 4x^2 + 5x - 12$

3 sign changes 3 or 1

H) $f_{(x)} = \frac{2}{3}x^4 - x^3 + 5x^2 - 3x + 2$

4 sign changes 4, 2 or 0

Using Descarte's Rule of Signs, state the possible number of negative zeros for each of the following functions.

A) $f_{(x)} = 3x^4 - 6x^3 + 2x^2 - x + 2$

$f(-x) = 3x^4 + 6x^3 + 2x^2 + x + 2$

none

B) $f_{(x)} = -x^5 + 2x^4 - 3x^3 - 7x + 2$

$f(-x) = x^5 + 2x^4 + 3x^3 + 7x + 2$

none

C) $f_{(x)} = 3x^6 - 2x^5 + 7x^4 + 5x^3 - x^2 + 2x - 1$

$f(-x) = 3x^6 + 2x^5 + 7x^4 - 5x^3 - x^2 - 2x - 1$

1

D) $f_{(x)} = -6x^4 - 5x^2 - 8$

$f(-x) = -6x^4 - 5x^2 - 8$

none

E) $f_{(x)} = -5x^5 + 6x^4 - 3x^2 + x - 15$

$f(-x) = 5x^5 + 6x^4 - 3x^2 - x - 15$

1

F) $f_{(x)} = \frac{1}{2}x^6 - 3x^5 + 7x^3 - 5x^2 + x$

$f(-x) = 1/2x^6 + 3x^5 - 7x^3 - 5x^2 - x$

1

G) $f_{(x)} = x^5 - 3x^4 + 2x^3 + 4x^2 + 5x - 12$

$f(-x) = -x^5 - 3x^4 - 2x^3 + 4x^2 - 5x - 12$

2 or 0

H) $f_{(x)} = \frac{2}{3}x^4 - x^3 + 5x^2 - 3x + 2$

$f(-x) = 2/3x^4 + x^3 + 5x^2 + 3x + 2$

none

The Remainder Theorem

When trying to find all zeros of a complex polynomial function, use the rational zero test to find all possible rational zeros. Each possible rational zero should then be tested using synthetic division. If one of these numbers work, there will be no remainder to the division problem. For every potential zero that works, there may be others that do not. Are these just useless? The answer is no. Every time synthetic division is attempted, we are actually evaluating the value of the function at the given x coordinate. When there is no remainder left, a zero of the function has just been found. This zero is an x intercept for the graph of the function. If the remainder is any other number, a set of coordinates on the graph has just been found. These coordinates would aid in graphing the function.

Let $P_{(x)}$ be a polynomial of positive degree n. Then for any number c,

$$P_{(x)} = Q_{(x)} \cdot (x-c) + P_{(c)},$$

Where $Q_{(x)}$ is a polynomial of degree n-1.

This simply means that if a polynomial $P_{(x)}$ is divided by $(x-c)$ using synthetic division, the resultant remainder is $P_{(c)}$.

When trying to find the zeros of the function $f_{(x)} = 2x^4 + 7x^3 - 4x^2 - 27x - 18$, first find all possible rational zeros. Then evaluate each one. Here is one particular example.

$$\begin{array}{r|rrrrr} -2 & 2 & 7 & -4 & -27 & -18 \\ & 0 & -4 & -6 & 20 & 14 \\ \hline & 2 & 3 & -10 & -7 & -4 \end{array}$$

In this example, (-2) is evaluated using synthetic division to see if it was a zero of the function. It turns out that (-2) is not a zero of the function, because there is a remainder of (-4).

Therefore, Using the Remainder Theorem, it can be stated that $f_{(-2)} = -4$.

You already saw that dividing by (-2) yields a result of (-4), giving us the statement $f_{(-2)} = -4$. This can be proven algebraically as follows.

$$f_{(-2)} = 2(-2)^4 + 7(-2)^3 - 4(-2)^2 - 27(-2) - 18$$

$$f_{(-2)} = 32 - 56 - 16 + 54 - 18$$

$$f_{(-2)} = -4$$

Given the polynomial function $f_{(x)} = 3x^3 + 2x^2 - 5x + 2$, use synthetic division to evaluate each of the following.

A) $f_{(2)} = 24$

$$\begin{array}{r|rrrr} 2 & 3 & 2 & -5 & 2 \\ & 0 & 6 & 16 & 22 \\ \hline & 3 & 8 & 11 & 24 \end{array}$$

B) $f_{(-2)} = -4$

$$\begin{array}{r|rrrr} -2 & 3 & 2 & -5 & 2 \\ & 0 & -6 & 9 & -6 \\ \hline & 3 & -4 & 3 & -4 \end{array}$$

C) $f_{(-3)} = -46$

$$\begin{array}{r|rrrr} -3 & 3 & 2 & -5 & 2 \\ & 0 & -9 & 21 & -48 \\ \hline & 3 & -7 & 16 & -46 \end{array}$$

Given the polynomial function $f_{(x)} = -2x^3 - 4x^2 - 5x + 12$, use synthetic division to evaluate each of the following.

D) $f_{(-1)} = 15$

$$\begin{array}{r|rrrr} -1 & -2 & -4 & -5 & 12 \\ & 0 & 2 & 2 & 3 \\ \hline & -2 & -2 & -3 & 15 \end{array}$$

E) $f_{(-2)} = 22$

$$\begin{array}{r|rrrr} -2 & -2 & -4 & -5 & 12 \\ & 0 & 4 & 0 & 10 \\ \hline & -2 & 0 & -5 & 22 \end{array}$$

F) $f_{(4)} = -200$

$$\begin{array}{r|rrrr} 4 & -2 & -4 & -5 & 12 \\ & 0 & -8 & -47 & -212 \\ \hline & -2 & -12 & -53 & -200 \end{array}$$

Given the polynomial function $f_{(x)} = -5x^2 - 4x + 6$, use synthetic division to evaluate each of the following.

G) $f_{(-1)} = 5$

$$\begin{array}{r|rrr} -1 & -5 & -4 & 6 \\ & 0 & 5 & -1 \\ \hline & -5 & 1 & 5 \end{array}$$

H) $f_{(-2)} = -6$

$$\begin{array}{r|rrr} -2 & -5 & -4 & 6 \\ & 0 & 10 & -12 \\ \hline & -5 & 6 & -6 \end{array}$$

I) $f_{(4)} = -90$

$$\begin{array}{r|rrr} 4 & -5 & -4 & 6 \\ & 0 & -20 & -96 \\ \hline & -5 & -24 & -90 \end{array}$$

Given the polynomial function $f_{(x)} = 3x^4 - 2x^3 + x^2 - 4x + 1$, use synthetic division to evaluate each of the following.

J) $f_{(3)} = 189$

$$\begin{array}{r|rrrrr} 3 & 3 & -2 & 1 & -4 & 1 \\ & 0 & 9 & 21 & 66 & 186 \\ \hline & 3 & 7 & 22 & 62 & 187 \end{array}$$

K) $f_{(1)} = -1$

$$\begin{array}{r|rrrrr} 1 & 3 & -2 & 1 & -4 & 1 \\ & 0 & 3 & 1 & 2 & -2 \\ \hline & 3 & 1 & 2 & -2 & -1 \end{array}$$

L) $f_{(-2)} = 77$

$$\begin{array}{r|rrrrr} -2 & 3 & -2 & 1 & -4 & 1 \\ & 0 & -6 & 16 & -34 & 76 \\ \hline & 3 & -8 & 17 & -38 & 77 \end{array}$$

Finding all Zeros of a Polynomial Function

When solving polynomial equations, use the rational zero test to find all possible rational zeros, then use Descarte's Rule of Signs to help narrow down the choices if possible. The fundamental theorem of Algebra plays a major role in this.

The Fundamental Theorem of Algebra

Every polynomial equation of degree n with complex coefficients has n roots in the complex numbers.

In other words, if you have a 5

In other words, if you have a 5^{th} degree polynomial equation, it has 5 roots.

Example: Find all zeros of the polynomial function $f_{(x)} = 2x^4 + 7x^3 - 4x^2 - 27x - 18$.

$$2x^4 + 7x^3 - 4x^2 - 27x - 18 = 0$$

Begin by setting the function equal to zero.

Find all possible rational zeros.

$$\pm 1,\ \pm 2,\ \pm 3,\ \pm 6,\ \pm 9,\ \pm 18,\ \pm\frac{1}{2},\ \pm\frac{3}{2},\ \pm\frac{9}{2}$$

Once again, there are 18 possible zeros to the function. If Descarte's Rule of Signs is used, it may or may not help narrow down the choices for synthetic division.

For this equation, there is 1 possible positive zero, and either 3 or 1 possible negative zeros.

This information was found in a previous example. Based on this, a chart may be constructed showing the possible combinations. Remember, this is a 4^{th} degree polynomial, so each row must add up to 4.

+	-	*i*
1	*3*	*0*
1	*1*	*2*

Now set up a synthetic division problem, and begin checking each zero until a root of the equation is found..

$$\begin{array}{|ccccc} 2 & 7 & -4 & -27 & -18 \\ 0 & & & & \\ \hline \end{array}$$

$$\pm 1,\ \pm 2,\ \pm 3,\ \pm 6,\ \pm 9,\ \pm 18,\ \pm\frac{1}{2},\ \pm\frac{3}{2},\ \pm\frac{9}{2}$$

$$\begin{array}{r|rrrrr} -1 & 2 & 7 & -4 & -27 & -18 \\ & 0 & -2 & -5 & 9 & 18 \\ \hline & 2 & 5 & -9 & -18 & 0 \end{array}$$

-1 works as a zero of the function. There are now 3 zeros left. We can continue to test each zero, but we need to first rewrite the new polynomial.

$2x^3 + 5x^5 - 9x - 18$

The reason this must be done is to check using the rational zero test again. Using the rational zero test again could reduce the number of choices to work with, or the new polynomial may be factorable.

$$\begin{array}{r|rrrr} -3 & 2 & 5 & -9 & -18 \\ & 0 & -6 & 3 & 18 \\ \hline & 2 & -1 & -6 & 0 \end{array}$$

Here, we found that -3 works. The reason negative numbers are being used first here is because of the chart above. The chart says there is a greater chance of one of the negatives working rather than a positive, since there are potentially 3 negative zeros here and only one positive. Notice the new equation was used for the division.

$$2x^2 - x - 6$$
$$(2x+3)(x-2) = 0$$
$$x = -3/2 \quad and \quad x = 2$$

This is now a factorable polynomial. Solve by factoring.

We now have all zeros of the polynomial function. They are $-3/2,\ -1,\ -3$ *and* 2

Be aware, the remaining polynomial may not be factorable. In that case, it will be necessary to either use the quadratic formula, or complete the square.

Find all real zeros of the following functions (no complex numbers). Remember, if there is no constant with which to use the rational zero test, factor out a zero first, then proceed.

A) $f_{(x)} = x^3 - 6x^2 + 11x - 6$

$x^3-6x^2+11x-6=0$

$\pm1, \pm2, \pm3, \pm6$

$x=3$

$$\begin{array}{r|rrrr} 3 & 1 & -6 & 11 & -6 \\ & 0 & 3 & -9 & 6 \\ \hline & 1 & -3 & 2 & 0 \end{array}$$

$x^2-3x+2=0$

$(x-1)(x-2)=0$

$x=1 \quad x=2$

$\{1,2,3\}$

B) $f_{(x)} = x^3 - 9x^2 + 27x - 27$

$\pm1, \pm3, \pm9, \pm27$

$x^3-9x^2+27x-27=0$

$x=3$

$$\begin{array}{r|rrrr} 3 & 1 & -9 & 27 & -27 \\ & 0 & 3 & -18 & 27 \\ \hline & 1 & -6 & 9 & 0 \end{array}$$

$x^2-6x+9=0$

$(x-3)^2=0$

$x=3$

3 triple zero

C) $f_{(x)} = x^3 - 9x^2 + 20x - 12$

$\pm1, \pm2, \pm3, \pm4, \pm6, \pm12$

$x^3-9x^2+20x-12=0$

$x=6$

$$\begin{array}{r|rrrr} 6 & 1 & -9 & 20 & -12 \\ & 0 & 6 & -18 & 12 \\ \hline & 1 & -3 & 2 & 0 \end{array}$$

$x^2-3x+2=0$

$(x-1)(x-2)=0$

$x=1 \quad x=2$

$\{1,2,6\}$

D) $f_{(x)} = x^4 - 7x^2 + 12$

$(x^2-3)(x^2-4)=0$

$x^2-3=0 \quad x^2-4=0$

$\sqrt{x^2}=\sqrt{3} \quad \sqrt{x^2}=\sqrt{4}$

$x=\pm\sqrt{3} \quad x=\pm2$

$\{\pm\sqrt{3}, \pm2\}$

E) $f_{(x)} = x^5 - 7x^4 + 10x^3 + 14x^2 - 24x$

$\pm1, \pm2, \pm3, \pm4, \pm6, \pm8, \pm12, \pm24$

$x^5-7x^4+10x^3+14x^2-24x=0$

$x(x^4-7x^3+10x^2+14x-24)=0$

$x=0$

$x=4$

$$\begin{array}{r|rrrrr} 4 & 1 & -7 & 10 & 14 & -24 \\ & 0 & 4 & -12 & -8 & 24 \\ \hline & 1 & -3 & -2 & 6 & 0 \end{array}$$

$x^3-3x^2-2x+6=0$

$x^2(x-3)-2(x-3)=0$

$(x^2-2)(x-3)=0$

$\sqrt{x^2}=\sqrt{2} \quad x=3$

$x=\pm\sqrt{2}$

$\{0, \pm\sqrt{2}, 3, 4\}$

F) $f_{(x)} = x^4 - 13x^2 - 12x$

$x(x^3-13x-12)=0$

$\pm1, \pm2, \pm3, \pm4, \pm6, \pm12$

$x=0$

$x=4$

$$\begin{array}{r|rrrr} 4 & 1 & 0 & -13 & -12 \\ & 0 & 4 & 16 & 12 \\ \hline & 1 & 4 & 3 & 0 \end{array}$$

$x^2+4x+3=0$

$(x+3)(x+1)=0$

$x=-3 \quad x=-1$

$\{-3,-1,0,4\}$

There is a difference between the questions: "find all real zeros" and "find all zeros." Be careful to pay attention to which is being asked.

The following questions, denoted by *, show up in the "**Writing a Polynomial Function in Factored Form*****" topic of the workbook. Since these functions show up later, the zeros of these functions only need to be found once. As we move on to that topic, on page 326, the answers found below will be used to rewrite the polynomial in the manner indicated.***

***Find all zeros of the following functions. (Include any complex solutions)**

A) $f_{(x)} = x^4 - 81$

$(x^2+9)(x^2-9)=0$

$(x^2+9)(x+3)(x-3)=0$

$\sqrt{x^2}=\sqrt{-9}$ $x=-3$ $x=3$

$x=\pm 3i$ $\{\pm 3, \pm 3i\}$

B) $f_{(x)} = x^4 - 7x^2 + 12$

$(x^2-4)(x^2-3)=0$

$\sqrt{x^2}=\sqrt{4}$ $\sqrt{x^2}=\sqrt{3}$

$x=\pm 2$ $x=\pm\sqrt{3}$

$\{\pm\sqrt{3}, \pm 2\}$

C) $f_{(x)} = x^3 - x + 6$

$x^3-x+6=0$

$\pm 1, \pm 2, \pm 3, \pm 6$

$x=-2$

$$\begin{array}{r|rrrr} -2 & 1 & 0 & -1 & 6 \\ & 0 & -2 & 4 & -6 \\ \hline & 1 & -2 & 3 & 0 \end{array}$$

$\{-2, 1\pm\sqrt{2}i\}$

$x^2-2x+3=0$

$x^2-2x+1=-3+1$

$(\frac{b}{2})^2=(\frac{-2}{2})^2=(-1)^2=1$

$\sqrt{(x-2)^2}=\sqrt{-2}$

$x=1\pm\sqrt{2}i$

D) $f_{(x)} = x^6 + 4x^4 - 41x^2 + 36$

$\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 9, \pm 12, \pm 18, \pm 36$

$$\begin{array}{r|rrrrrrr} 2 & 1 & 0 & 4 & 0 & -41 & 0 & 36 \\ & 0 & 2 & 4 & 16 & 32 & -18 & -36 \\ \hline 1 & 1 & 2 & 8 & 16 & -9 & -18 & 0 \\ & 0 & 1 & 3 & 11 & 27 & 18 & \\ \hline -1 & 1 & 3 & 11 & 27 & 18 & 0 & \\ & 0 & -1 & -2 & -9 & -18 & & \\ \hline -2 & 1 & 2 & 9 & 18 & 0 & & \\ & 0 & -2 & 0 & -18 & & & \\ \hline & 1 & 0 & 9 & 0 & & & \end{array}$$

$\boxed{x=2}$ $\boxed{x=1}$ $\boxed{x=-1}$ $\boxed{x=-2}$

$x^2+9=0$

$\sqrt{x^2}=\sqrt{-9}$ $x=\pm 3i$

$\{\pm 1, \pm 2, \pm 3i\}$

E) $f_{(x)} = x^4 + 10x^2 + 9$

$(x^2+9)(x^2+1)=0$

$\sqrt{x^2}=\sqrt{-9}$ $\sqrt{x^2}=\sqrt{-1}$

$x=\pm 3i$ $x=\pm i$

$\{\pm 3i, \pm i\}$

F) $f_{(x)} = x^4 - x^3 + 25x^2 - 25x$

$x^3(x-1)+25x(x-1)=0$

$(x^3+25x)(x-1)=0$

$x(x^2+25)(x-1)=0$

$x=0$ $x^2=-25$ $\boxed{x=1}$

$x=\pm 5i$

$\{0, 1, \pm 5i\}$

G) $f_{(x)} = x^4 - x^3 - 2x^2 - 4x - 24$

$\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 8, \pm 12, \pm 24$

$\boxed{x=3}$

$$\begin{array}{r|rrrrr} 3 & 1 & -1 & -2 & -4 & -24 \\ & 0 & 3 & 6 & 12 & 24 \\ \hline & 1 & 2 & 4 & 8 & 0 \end{array}$$

$x^3+2x^2+4x+8=0$

$x^2(x+2)+4(x+2)=0$

$(x^2+4)(x+2)=0$

$\sqrt{x^2}=\sqrt{-4}$ $\boxed{x=-2}$

$\boxed{x=\pm 2i}$

$\{\pm 2i, -2, 3\}$

H) $f_{(x)} = x^4 - x^3 - 29x^2 - x - 30$

$\pm 1, \pm 2, \pm 3, \pm 5, \pm 6, \pm 10, \pm 15, \pm 30$

$\boxed{x=6}$

$$\begin{array}{r|rrrrr} 6 & 1 & -1 & -29 & -1 & -30 \\ & 0 & 6 & 30 & 6 & 30 \\ \hline & 1 & 5 & 1 & 5 & 0 \end{array}$$

$x^3+5x^2+x+5=0$

$x^2(x+5)+1(x+5)=0$

$(x^2+1)(x+5)=0$

$\sqrt{x^2}=\sqrt{-1}$ $\boxed{x=-5}$

$\boxed{x=\pm i}$

$\{\pm i, -5, 6\}$

I) $f_{(x)} = x^3 - x^2 - 3x + 3$

$x^3-x^2-3x+3=0$

$x^2(x-1)-3(x-1)=0$

$(x^2-3)(x-1)=0$

$\sqrt{x^2}=\sqrt{3}$ $\boxed{x=1}$

$\boxed{x=\pm\sqrt{3}}$

$\{\pm\sqrt{3}, 1\}$

J) $f_{(x)} = x^4 - 7x^2 + 10$

$x^4-7x^2+10=0$

$(x^2-2)(x^2-5)=0$

$x^2=2$ $x^2=5$

$x=\pm\sqrt{2}$ $x=\pm\sqrt{5}$

$\{\pm\sqrt{2}, \pm\sqrt{5}\}$

K) $f_{(x)} = x^3 - 6x^2 + 13x - 10$

$\pm 1, \pm 2, \pm 5, \pm 10$

$x^3-6x^2+13x-10=0$

$\boxed{x=2}$

$$\begin{array}{r|rrrr} 2 & 1 & -6 & 13 & -10 \\ & 0 & 2 & -8 & 10 \\ \hline & 1 & -4 & 5 & 0 \end{array}$$

$x^2-4x+5=0$

$x^2-4x+4=-5+4$

$(\frac{b}{2})^2=(-\frac{4}{2})^2=(-2)^2=4$

$\sqrt{(x-2)^2}=\sqrt{-1}$

$x-2=\pm i$

$x=2\pm i$

$\{2, 2\pm i\}$

L) $f_{(x)} = x^5 + 15x^3 - 16x$

$x(x^4+15x^2-16)=0$

$x(x^2+16)(x^2-1)=0$

$x=0$ $x^2=-16$ $x^2=1$

$x=0$ $x=\pm 4i$ $x=\pm 1$

$\{\pm 4i, 0, \pm 1\}$

Writing a Polynomial Function in Factored Form

Once all zeros of a polynomial function are found, the function can be rewritten in one of several different ways.

A polynomial function may be written in one of the following ways.

- ***As a product of factors that are irreducible over the rationals.***
  This means only rational numbers may be used in the factors.
- ***As a product of factors that are irreducible over the reals.***
  Irrational numbers may be used as long as they are real, i.e. $\left(x+\sqrt{3}\right)\left(x-\sqrt{3}\right)$.
- ***In completely factored form. This may also be written as a product of linear factors.***
  Complex numbers may be used in the factors, i.e. $(x+2i)(x-2i)$.

This section involves writing polynomials in one of the factored forms illustrated above. These are the same problems that were solved on the previous page, so there is no need to solve them again. Use the solutions previously found, to write the polynomial in the desired form.

For example, a polynomial function that has zeros of 3 ***and*** $2\pm\sqrt{3}$ ***would look like the following; in completely factored form.***

$$f_{(x)}=(x-3)\left(x-2+\sqrt{3}\right)\left(x-2-\sqrt{3}\right)$$

Notice each variable x is to the first power, so these are linear factors.

When polynomial functions are written like this, it is obvious where the x intercepts lie.

***Write the polynomial function as a product of factors that are irreducible over the reals.**

A) $f_{(x)} = x^4 - 81$

$\pm 3, \pm 3i$

$f(x) = (x^2+9)(x+3)(x-3)$

B) $f_{(x)} = x^4 - 7x^2 + 12$

$\pm\sqrt{3}, \pm 2$

$f(x) = (x+\sqrt{3})(x-\sqrt{3})(x+2)(x-2)$

C) $f_{(x)} = x^3 - x + 6$

$1 \pm \sqrt{2}i, -2$

$f(x) = (x^2-2x+3)(x+2)$

D) $f_{(x)} = x^6 + 4x^4 - 41x^2 + 36$

$\pm 1, \pm 2, \pm 3i$

$f(x) = (x+1)(x-1)(x+2)(x-2)(x^2+9)$

E) $f_{(x)} = x^4 + 10x^2 + 9$

$\pm 3i, \pm i$

$f(x) = (x^2+9)(x^2+1)$

F) $f_{(x)} = x^4 - x^3 + 25x^2 - 25x$

$0, 1, \pm 5i$

$f(x) = x(x^2+25)(x-1)$

G) $f_{(x)} = x^4 - x^3 - 2x^2 - 4x - 24$

$\pm 2i, -2, 3$

$f(x) = (x^2+4)(x+2)(x-3)$

H) $f_{(x)} = x^4 - x^3 - 29x^2 - x - 30$

$\pm i, -5, 6$

$f(x) = (x^2+1)(x+5)(x-6)$

I) $f_{(x)} = x^3 - x^2 - 3x + 3$

$\pm\sqrt{3}, 1$

$f(x) = (x+\sqrt{3})(x-\sqrt{3})(x-1)$

J) $f_{(x)} = x^4 - 7x^2 + 10$

$\pm\sqrt{2}, \pm\sqrt{5}$

$f(x) = (x+\sqrt{2})(x-\sqrt{2})(x+\sqrt{5})(x-\sqrt{5})$

K) $f_{(x)} = x^3 - 6x^2 + 13x - 10$

$2 \pm i, 2$

$f(x) = (x^2-4x+5)(x-2)$

L) $f_{(x)} = x^5 + 15x^3 - 16x$

$0, \pm 4i, \pm 1$

$f(x) = x(x^2+16)(x+1)(x-1)$

***Write the polynomial function as a product of factors that are irreducible over the rationals.**

A) $f_{(x)} = x^4 - 81$

$\pm 3, \pm 3i$

$f(x) = (x^2+9)(x+3)(x-3)$

B) $f_{(x)} = x^4 - 7x^2 + 12$

$\pm\sqrt{3}, \pm 2$

$f(x) = (x^2-3)(x+2)(x-2)$

C) $f_{(x)} = x^3 - x + 6$

$1 \pm \sqrt{2}i, -2$

$f(x) = (x^2-2x+3)(x+2)$

D) $f_{(x)} = x^6 + 4x^4 - 41x^2 + 36$

$\pm 1, \pm 2, \pm 30$

$f(x) = (x^2+9)(x+2)(x-2)(x+1)(x-1)$

E) $f_{(x)} = x^4 + 10x^2 + 9$

$\pm 3i, \pm i$

$f(x) = (x^2+9)(x^2+1)$

F) $f_{(x)} = x^4 - x^3 + 25x^2 - 25x$

$\pm 5i, 0, 1$

$f(x) = x(x^2+25)(x-1)$

G) $f_{(x)} = x^4 - x^3 - 2x^2 - 4x - 24$

$\pm 2i, -2, 3$

$f(x) = (x^2+4)(x+2)(x-3)$

H) $f_{(x)} = x^4 - x^3 - 29x^2 - x - 30$

$\pm i, -5, 6$

$f(x) = (x^2+1)(x+5)(x-6)$

I) $f_{(x)} = x^3 - x^2 - 3x + 3$

$\pm\sqrt{3}, 1$

$f(x) = (x^2-3)(x-1)$

J) $f_{(x)} = x^4 - 7x^2 + 10$

$\pm\sqrt{2}, \pm\sqrt{5}$

$f(x) = (x^2-2)(x^2-5)$

K) $f_{(x)} = x^3 - 6x^2 + 13x - 10$

$2 \pm i, 2$

$f(x) = (x^2-4x+5)(x-2)$

L) $f_{(x)} = x^5 + 15x^3 - 16x$

$0, \pm 4i, \pm 1$

$f(x) = x(x^2+16)(x+1)(x-1)$

***Write the polynomial functions in completely factored form. (Remember, this can also be asked in the form "Write polynomial as a product of linear factors.")**

A) $f_{(x)} = x^4 - 81$

$\pm 3, \pm 3i$

$f(x) = (x+3)(x-3)(x+3i)(x-3i)$

B) $f_{(x)} = x^4 - 7x^2 + 12$

$\pm\sqrt{3}, \pm 2$

$f(x) = (x+\sqrt{3})(x-\sqrt{3})(x+2)(x-2)$

C) $f_{(x)} = x^3 - x + 6$

$1 \pm \sqrt{2}i, -2$

$f(x) = (x-1+\sqrt{2}i)(x-1-\sqrt{2}i)(x+2)$

D) $f_{(x)} = x^6 + 4x^4 - 41x^2 + 36$

$\pm 1, \pm 2, \pm 3i$

$f(x) = (x+1)(x-1)(x+2)(x-2)(x+3i)(x-3i)$

E) $f_{(x)} = x^4 + 10x^2 + 9$

$\pm 3i, \pm i$

$f(x) = (x+3i)(x-3i)(x+i)(x-i)$

F) $f_{(x)} = x^4 - x^3 + 25x^2 - 25x$

$0, 1, \pm 5i$

$f(x) = x(x-1)(x+5i)(x-5i)$

G) $f_{(x)} = x^4 - x^3 - 2x^2 - 4x - 24$

$\pm 2i, -2, 3$

$f(x) = (x+2i)(x-2i)(x+2)(x-3)$

H) $f_{(x)} = x^4 - x^3 - 29x^2 - x - 30$

$\pm i, -5, 6$

$f(x) = (x+i)(x-i)(x+5)(x-6)$

I) $f_{(x)} = x^3 - x^2 - 3x + 3$

$\pm\sqrt{3}, 1$

$f(x) = (x+\sqrt{3})(x-\sqrt{3})(x-1)$

J) $f_{(x)} = x^4 - 7x^2 + 10$

$\pm\sqrt{2}, \pm\sqrt{5}$

$f(x) = (x+\sqrt{2})(x-\sqrt{2})(x+\sqrt{5})(x-\sqrt{5})$

K) $f_{(x)} = x^3 - 6x^2 + 13x - 10$

$2 \pm i, 2$

$f(x) = (x-2+i)(x-2-i)(x-2)$

L) $f_{(x)} = x^5 + 15x^3 - 16x$

$0, \pm 4i, \pm 1$

$f(x) = x(x+4i)(x-4i)(x+1)(x-1)$

Write $x^4 - 81$ as a product of linear factors.

$(x^2+9)(x^2-9)$

$(x+3i)(x-3i)(x+3)(x-3)$

Write $x^4 - 16$ as a product of linear factors.

$(x^2+4)(x^2-4)$

$(x+2i)(x-2i)(x+2)(x-2)$

How can you identify linear factors?

each factor is to the first degree. (no exponents)

Finding the Equation of a Polynomial Function

In this section we will work backwards with the roots of polynomial equations or zeros of polynomial functions. As we did with quadratics, so we will do with polynomials greater than second degree. Given the roots of an equation, work backwards to find the polynomial equation or function from whence they came. Recall the following example.

Find the equation of a parabola that has x intercepts of $(-3,0)$ *and* $(2,0)$.

$(-3,0)$ *and* $(2,0)$.	***Given x intercepts of -3 and 2***
$x=-3 \quad x=2$	***If the x intercepts are -3 and 2, then the roots of the equation are -3 and 2. Set each root equal to zero.***
$(x+3)\ (x-2)$	***For the first root, add 3 to both sides of the equal sign.*** ***For the second root, subtract 2 to both sides of the equal sign.***
x^2+x-6	***Multiply the results together to find a quadratic expression.***
$y=x^2+x-6$	***Set the expression equal to y, or*** $f_{(x)}$***, to write as the equation of a parabola.***

The exercises in this section will result in polynomials greater than second degree. <u>Be aware, you may not be given all roots with which to work.</u>

Consider the following example:

Find a polynomial function that has zeros of $0,\ 3$ *and* $2+3i$. ***Although only three zeros are given here, there are actually four. Since complex numbers always come in conjugate pairs,*** $2-3i$ ***must also be a zero. Using the fundamental theorem of algebra, it can be determined that his is a 4th degree polynomial function.***

Take the zeros of $0,\ 3,\ 2\pm 3i$, ***and work backwards to find the original function.***

$x=0$	$x=3$	$x=2\pm 3i$
		$x = 2 \pm 3i$
	$x=\ 3$	$x-2=\pm 3i$
$x=0$	$-3 \quad -3$	$x^2-4x+4=9i^2$
	$x-3=0$	$x^2-4x+4=-9$
		$x^2-4x+13=0$
x	$(x-3)$	$(x^2-4x+13)$

The polynomial function with zeros of $0,\ 3,\ 2\pm 3i$, ***is equal to*** $f_{(x)}=x(x-3)(x^2-4x+13)$. ***Multiplying this out will yield the following.***

$$f_{(x)}=x^4-7x^3+25x^2-39x$$

Find a polynomial function that has the following zeros.

A) -3, 2, 1

$x=-3 \quad x=2 \quad x=1$

$(x+3)(x-2)(x-1)$

$(x+3)(x^2-3x+2)$

x^3-3x^2+2x

$3x^2-9x+6$

$f(x)=x^3-7x+6$

B) -4, 0, 1, 2

$x=0 \quad x=-4 \quad x=1 \quad x=2$

$x(x+4)(x-1)(x-2)$

$x(x+4)(x^2-3x+2)$

x^3-3x^2+2x

$4x^2-12x+8$

$x(x3+x^2-10x+8)$

$f(x)=x^4+x^3-10x^2+8x$

C) $\pm 1, \pm\sqrt{2}$

$(x)^2=(\pm 1)^2 \quad (x)^2=(\pm\sqrt{2})^2$

$x^2=1 \quad x^2=2$

$(x^2-1)(x^2-2)$

$f(x)=x^4-3x^2+2$

D) 0, 2, 5

$x=0 \quad x=2 \quad x=5$

$x(x-2)(x-5)$

$x(x^2-7x+10)$

$f(x)=x^3-7x^2+10x$

E) $2, 1\pm\sqrt{3}$

$x=2 \quad x=1\pm\sqrt{3}$

$(x-2) \quad (x-1)^2=(\pm\sqrt{3})^2$

$x^2-2x+1=3$

$(x-2)(x^2-2x-2)$

x^3-2x^2-2x

$-2x^2+4x+4$

$f(x)=x^3-4x^2+2x+4$

F) $\pm 4, 0, \pm\sqrt{2}$

$x=0 \quad x=\pm 4 \quad x=\pm\sqrt{2}$

$x=0 \quad (x)^2=(\pm 4)^2 \quad (x)^2=(\pm\sqrt{2})^2$

$x^2=16 \quad x^2=2$

$x(x^2-16)(x^2-2)$

$x(x^4-18x^2+32)$

$f(x)=x^5-18x^3+32x$

G) -2, -1, 0, 1, 2

$x=0 \quad x=\pm 2 \quad x=\pm 1$

$(x)^2=(\pm 2)^2 \quad (x)^2=(\pm)^2$

$x^2=4 \quad x^2=1$

$x(x^2-4)(x^2-1)$

$x(x^4-5x^2+4)$

$f(x)=x^5-5x^3+4x$

H) $1\pm\sqrt{2}, \pm\sqrt{3}$

$x=\pm\sqrt{3} \quad x=1\pm\sqrt{2}$

$(x)^2=(\pm\sqrt{3})^2 \quad (x-1)^2=(\pm\sqrt{2})^2$

$x^2=3 \quad x^2-2x+1=2$

$(x^2-3)(x^2-2x-1)$

$x^4-2x^3-x^2$

$-3x^2+6x+3$

$f(x)=x^4-2x^3-4x+6x+3$

I) 0, -3

$x=0 \quad x=-3$

$x(x+3)$

$f(x)=x^2+3x$

Here is a little practice with complex numbers.

*complex #s come in
conjugate pairs

Find a polynomial function that has the given zeros.

A) $0, 3, \pm 2i$

$x=0 \quad x=3 \quad x=\pm 2i$

$x^2=\pm 2i)^2$

$(x-3) \quad x^2=4i^2$

$x^2=-4$

$x(x-3)(x^2+4)$

$x(x^3-3x^2+4x-12)$

B) $\pm 2i, \ \pm 3i, \ 4$

$x=\pm 2 \quad x=\pm 3 \quad x=4$

$(x)^2=(\pm 2)^2, \quad x)^2=(\pm 3)^2 \quad (x-4)$

$x^2=4 \quad x^2=9$

$(x^2-4)(x^2-9)(x-4)$

$(x-4)(x^4-13x^2+36)$

x^5-13x^3+36x

$4x^4+5x^2 \quad -144$

$f(x)=x^5+4x^4-13x^3+52x^2+36x-144$

C) $-2, \ 3, \ 3i$

$x=-2 \quad x=3 \quad x=\pm 3i$

$x^2=(\pm 3i)^2$

$x^2=9i^2$

$x^2=-9$

$(x+2)(x-3(x^2+9)$

$(x^2+9)(x^2-x-6)$

$x^4-x^3-6x^2$

$9x^2-9x-54$

$f(x)=x^4-x^3+3x^2-9x-54$

D) $-i, \ 2i, \ -3i$

$x=\pm i \quad x=\pm 2i \quad x=\pm 3i$

$x^2=(\pm i)2 \quad x^2=(\pm 2i)^2 \quad (x)^2=(\pm 3i)^2$

$x^2=i^2 \quad x^2=4i^2 \quad x^2=9i^2$

$x^2=-1 \quad x^2=4 \quad x^2=-9$

$(x^2+1)(x^2+4)(x^2+9)$

$(x^2+1)(x^4+13x^2+36)$

$x^6+13x^4 \quad +36x^2$

$x^4 \quad +13x^2+36$

$f(x)=x^6+14x^4+49x^2+36$

E) $-4, \ 1\pm 2i$

$x=-4 \quad x=1\pm 2i$

$(x-1)^2=(\pm 2i)^2$

$x^2-2x+1=4i^2$

$x^2-2x+1=-4$

$(x+4)(x^2-2x+5)$

x^3-2x^2+5x

$4x^2-8x+20$

$f(x)=x^3+2x^2-3x+20$

F) $1-i, \ 1+3i, \ 0$

$x=1\pm i \quad x=1\pm 3i \quad x=0$

$(x-1)^2=(\pm i)^2 \quad (x-1)^2=(\pm 3i)^2$

$x^2-2x+1=i^2 \quad x^2-2x+1=9i^2$

$x^2-2x+1=-1 \quad x^2-2x+1=-9$

$(x^2-2x+2)(x^2-2x+10)$

$x^4-2x^3+10x^2$

$-2x^3+4x^2-20x$

$2x^2-4x+20$

$x(x^4-4x^3+16x^2-24x+20)$

Complex solutions always come in Conjugate pairs**.**

What is the standard form of a complex number?

a+bi

Even vs. Odd Functions

One of your many tasks in future mathematics courses will be to determine whether a function is even, odd or neither. This is very simple to do.

A function is even if $f_{(-x)} = f_{(x)}$

This means if a (-x) is substituted into the problem, and no signs change, the function is even.

A function is odd if $f_{(-x)} = -f_{(x)}$

In this case, a (-x) is substituted into the problem, and all signs change. If all signs change, this is an odd function.

If only some of the signs change, the function is neither even nor odd.

Even functions are symmetrical to the y axis.
Odd functions are symmetrical to the origin.

Based on this statement, can we conclude that all parabolas are even functions? Explain your answer.

no, if the parabola shifts away from the y-axis, it will not be an even function.

ex.) $f(x) = x^2+2x-1$ ★only one sign changed, so this function
$f(-x) = x^2-2x-1$ is neither even nor odd

Determine whether each of the following functions is even, odd or neither.

A) $f_{(x)} = 2x^4 - 6x^2 + 5x - 15$

$f(-x) = 2x^4 - 6x^2 - 5x - 15$

neither

B) $f_{(x)} = -4x^7 + 3x^5 + 2x^3 - 6x$

$f(-x) = 4x^7 - 3x^5 - 2x^3 + 6x$

odd

C) $f_{(x)} = 8x^4 - 2x^2 + 24$

$f(-x) = 8x^4 - 2x^2 + 24$

even

D) $f_{(x)} = 10x^3 - 15x^2 - 16x + 12$

$f(-x) = -10x^3 - 15x^2 + 16x + 12$

neither

E) $f_{(x)} = -6x^3 + 5x^2 - 2x + 18$

$f(-x) = 6x^3 + 5x^2 + 2x + 18$

neither

F) $f_{(x)} = 4x^5 - 16x^3 + 12x$

$f(-x) = -4x^5 + 16x^3 - 12x$

odd

G) $f_{(x)} = 4x^6 + 3x^4 - 2x^2 - 12$

$f(-x) = 4x^6 + 3x^4 - 2x^2 - 12$

even

H) $f_{(x)} = x^5 - 6x^3 - 8x$

$f(-x) = -x^5 + 6x^3 + 8x$

odd

Left and Right Behaviors of Polynomial Functions

As we graphed various functions, you should have noticed something about the graph of a polynomial function of an even degree versus the graph of a polynomial function of an odd degree. Think of a parabola versus a cubic function. The left and right behaviors of polynomial functions are pretty simple to memorize.

If the degree of the polynomial is even, the graph of the function will have either "both sides up", or "both sides down."

If the degree of the polynomial is odd, the graph of the function will have one side up and one side down.

As to which side is up and which is down, that all depends on the leading coefficient.

Refer to the following.

	Even Degree	*Odd Degree*
+ leading coefficient	↑↑	↓↑
- leading coefficient	↓↓	↑↓

Therefore, a 7th degree polynomial function having a leading coefficient that is negative, will rise on the left, and fall to the right.

In contrast, if the 7th degree polynomial has a positive leading coefficient, the graph of the function will fall on the left, and rise on the right hand side.

These rules are for polynomial functions in a single variable only!

When we begin to graph these polynomial functions, the first step will be to find all zeros of the function. The x intercepts have been found, plot them on the x axis, and refer to the two intercepts on the ends. At this point, use the rules for left and right behaviors of functions to draw a portion of the graph.

Describe the left and right behaviors of the following polynomial functions.

A) $f_{(x)} = 3x^3 + 6x^2 - 5$

down on left
up on right

B) $f_{(x)} = x^6 + 5x^3 + 7x^2 - x - 2$

up on left and right

C) $f_{(x)} = x(x+2)^3(x-3)(x+1)$

6th degree

up on left and right

D) $f_{(x)} = -x^4 + 5x - 6$

down on left and right

E) $f_{(x)} = -\frac{1}{2}x^3 + 2x^2 - 3x + 5$

up on left, down on right

F) $f_{(x)} = 7x^8 - 6x^6 + 2x^4 - 8$

up on left and right

G) $f_{(x)} = 2x^5 + 3x^2 - x + 5$

down on left, up on right

H) $f_{(x)} = 14x^2 + 7x^3$

$f(x) = 7x^3 + 14x^2$

down on left, up on right

I) $f_{(x)} = -(x+3)^3(x-4)$

4th degree

down on left and right

J) $f_{(x)} = -0.2x^5 + 6x^3 - x^2$

up on left, down on right

K) $f_{(x)} = -x^3 + 3x^2 + 12x^6 - 8$

$f(x) = 12x^6 - x^3 + 3x^2 - 8$

up on left and right

L) $f_{(x)} = (x+2)^2(x-5)^5$

7th degree

down on left, up on right

Graphing Polynomial Functions

When graphing a polynomial function, there are a series of steps to follow.

1. ***Find all zeros of the function. This will give the x intercepts of the function***
2. ***Plot all x intercepts for the function on the x axis.***
3. ***Using the properties of polynomial functions, determine the left and right behaviors of the function, and draw those segments.***
4. ***Substitute zero for x, and find the y intercept of the function.***
5. ***Using the graphing calculator, find the maxima and minima between each x intercept.***
6. ***Draw the rest of the function, making sure the maxima and minima are in their appropriate locations.***

The first step is the most time consuming, as the rational zero test and other methods are used to find all real zeros of the function. Depending on the degree of the function, there may be quite a few x intercepts to find. Imaginary solutions to the equation are not x intercepts. They will not be on the graph of the function.

The two most important properties of polynomial used to graph, are the left and right behaviors of a polynomial function, and the rule regarding the number of turns of a polynomial function.

	Even Degree	***Odd Degree***
+ leading coefficient	↑↑	↓↑
- leading coefficient	↓↓	↑↓

Any polynomial function to the nth degree, has <u>at most</u> n-1 turns.

This means a 5th degree polynomial function will have at most 4 turns. It does not have to have that many, but it can have no more than 4 turns.

Sketch the graph of each of the following polynomial functions. (Label all x intercepts, y intercepts, maxima, minima, and identify the range and domain.) Scale the graphs as needed.

A) $f_{(x)} = x^4 - 10x^2 + 24$

$x^4 - 10x^2 + 24 = 0$

$(x^2-4)(x^2-6)=0$

$(x+2)(x-2)(x^2-6)=0$

$x^2 = 6$

$x=-2$ $x=2$ $x=\pm\sqrt{6}$

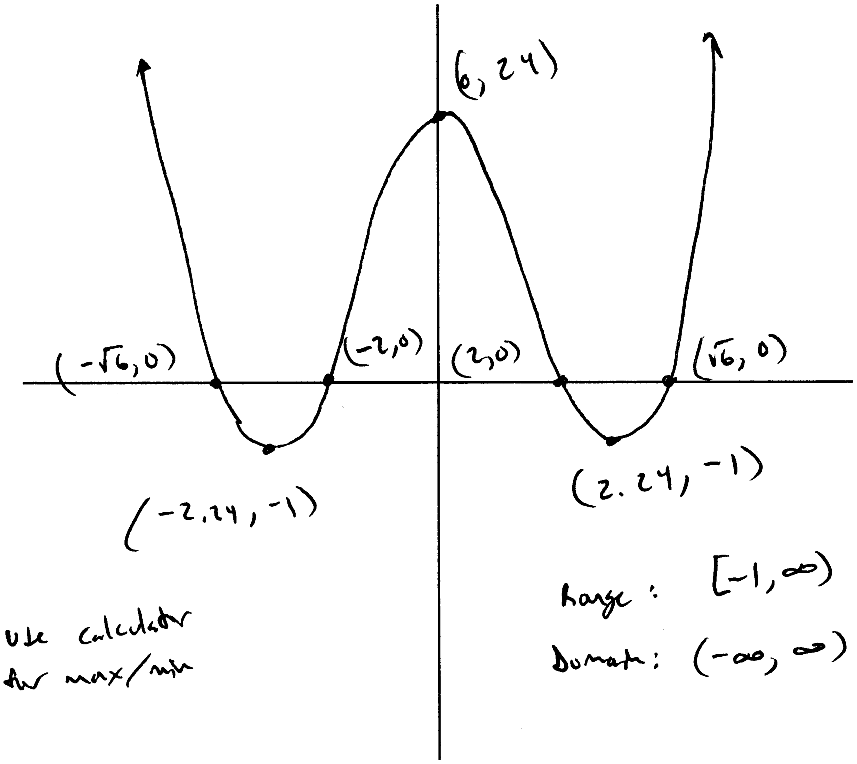
x int: ± 2, $\pm\sqrt{6}$

y int

$f(0) = (0)^4 - 10(0)^2 + 24$

$f(0) = 24$

y int: $(0, 24)$



use calculator for max/min

Range: $[-1, \infty)$

Domain: $(-\infty, \infty)$

B) $f_{(x)} = \frac{1}{2}\left(x^2 - 2x + 15\right)$

not factorable

$2f(x) = x^2 - 2x + 15$

$2f(x) - 15 + 1 = x^2 - 2x + 1$

$\left(\frac{b}{2}\right)^2 = \left(-\frac{2}{2}\right)^2 = (-1)^2 = 1$

$2f(x) - 14 = (x-1)^2$

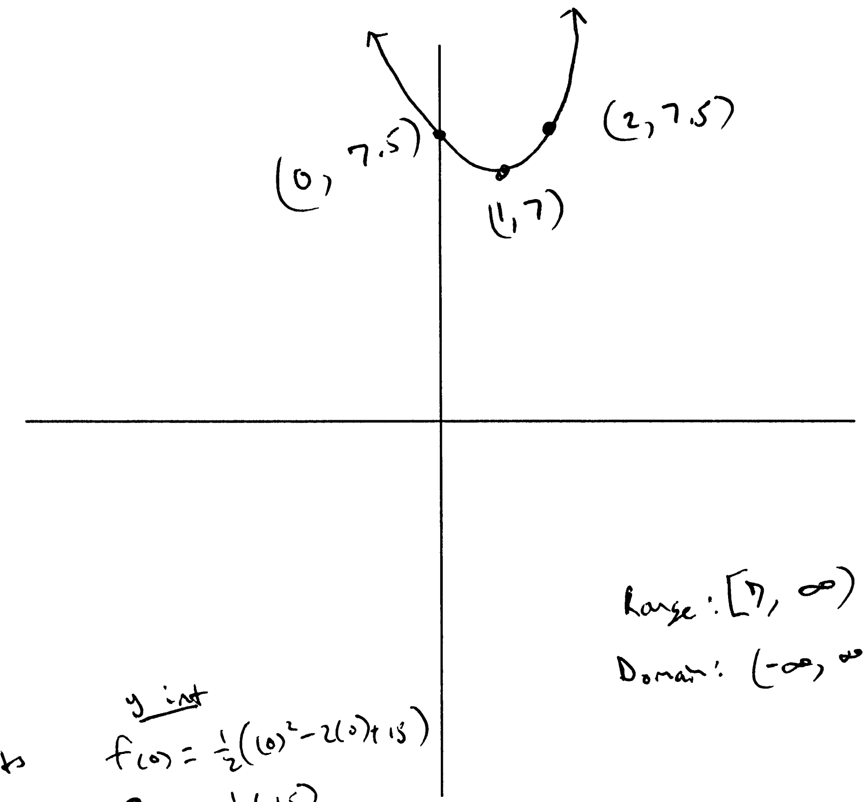
$2f(x) = (x-1)^2 + 14$

$f(x) = \frac{1}{2}(x-1)^2 + 7)$

vertex: $(1, 7)$

opens up

no x intercepts



Range: $[7, \infty)$

Domain: $(-\infty, \infty)$

y int

$f(0) = \frac{1}{2}((0)^2 - 2(0) + 15)$

$f(0) = \frac{1}{2}(15)$

$f(0) = 7.5$

3rd degree leading coefficient (-) so... ↑ ↓

C) $f_{(x)} = -(x-1)^2(x+4)$

x int

$-(x-1)^2(x+4)=0$

$x=1 \quad x=-4$

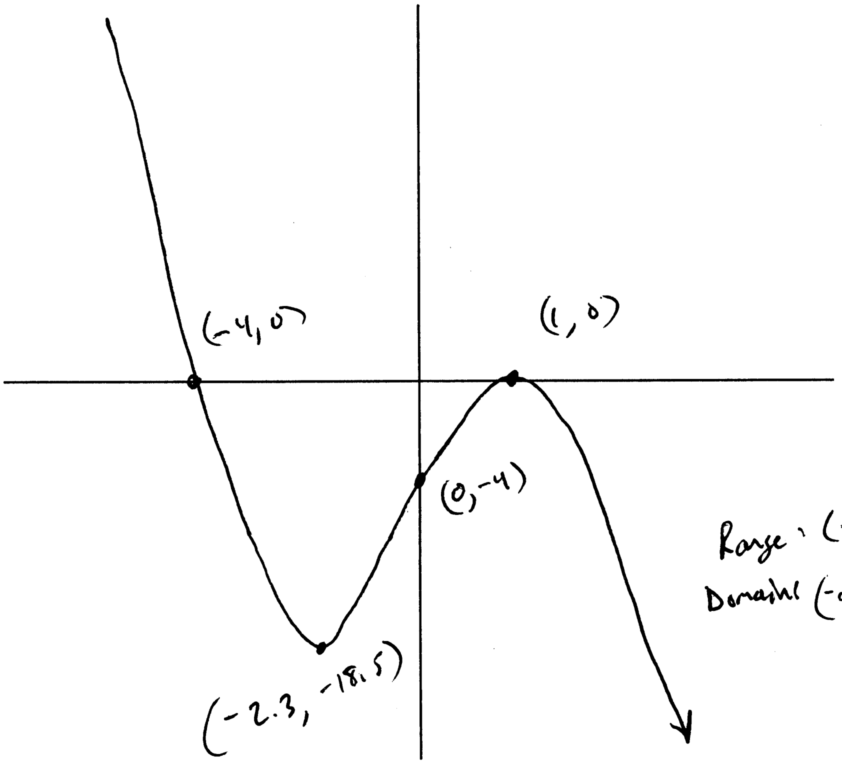
y int

$f(0) = -(0-1)^2(0+4)$

$f(0) = -(-1)^2(4)$

$f(0) = -4$

$(0,-4)$



Range: $(-\infty,\infty)$

Domain: $(-\infty,\infty)$

D) $f_{(x)} = \frac{1}{2}x(x-4)(x+7)$ 3rd degree leading coefficient (+) so... ↓ ↑

x int

$\frac{1}{2}x(x-4)(x+7)=0$

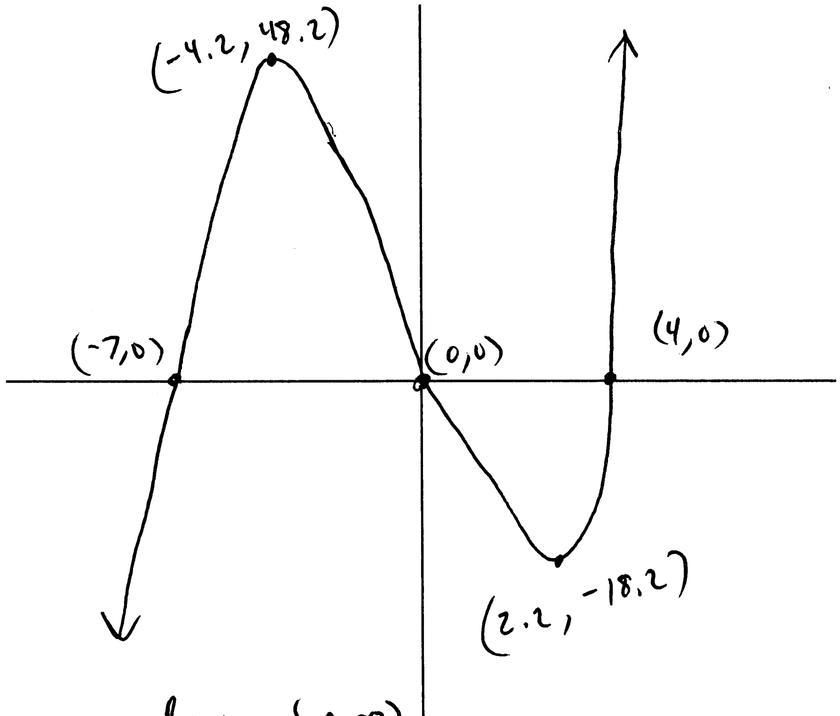
$x=0 \quad x=4 \quad x=-7$

y int

$f(0) = \frac{1}{2}(0)(0-4)(0+7)$

$f(0) = 0$

$(0,0)$



Range: $(-\infty,\infty)$

Domain: $(-\infty,\infty)$

E) $f_{(x)} = -(x+3)^3(x-1)(x-5)$

5th degree Leading coefficient (−) . . . ↑↓

x int

$-(x+3)^3(x-1)(x-5)=0$

$x=-3$ $x=1$ $x=5$

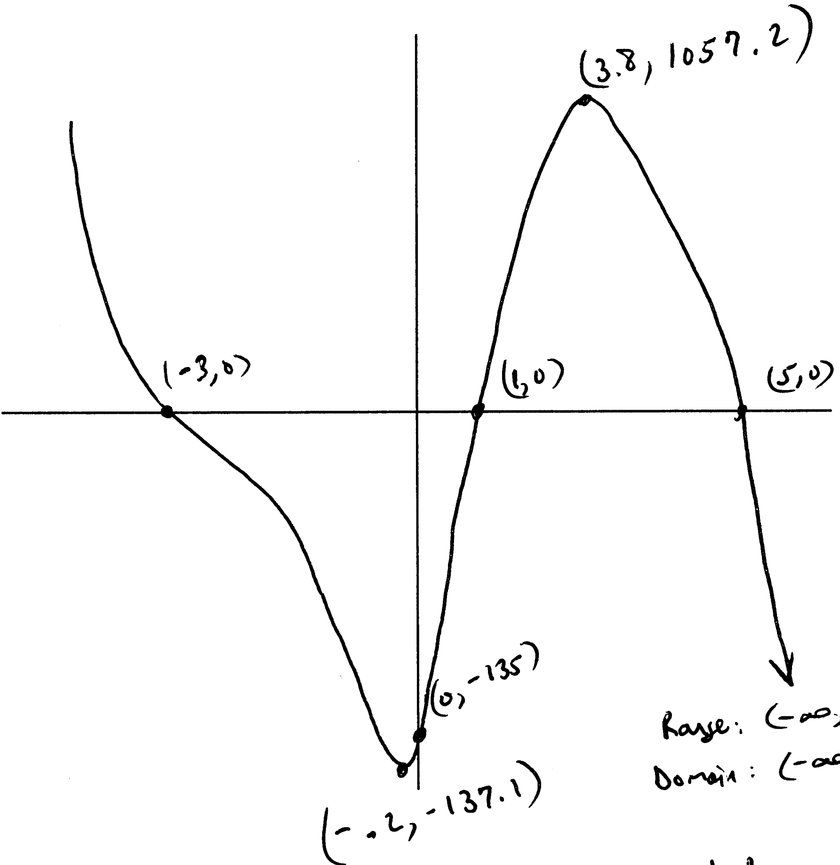
y int

$f(0) = -(0+3)^3(0-1)(0-5)$

$= -(27)(-1)(-5)$

$f(0) = -135$

$(0, -135)$



Range: $(-\infty, \infty)$

Domain: $(-\infty, \infty)$

F) $f_{(x)} = -3x(x-2)^2(x+5)$

4th degree - Leading coefficient (−) . . . ↓↓

x int

$-3x(x-2)^2(x+5)=0$

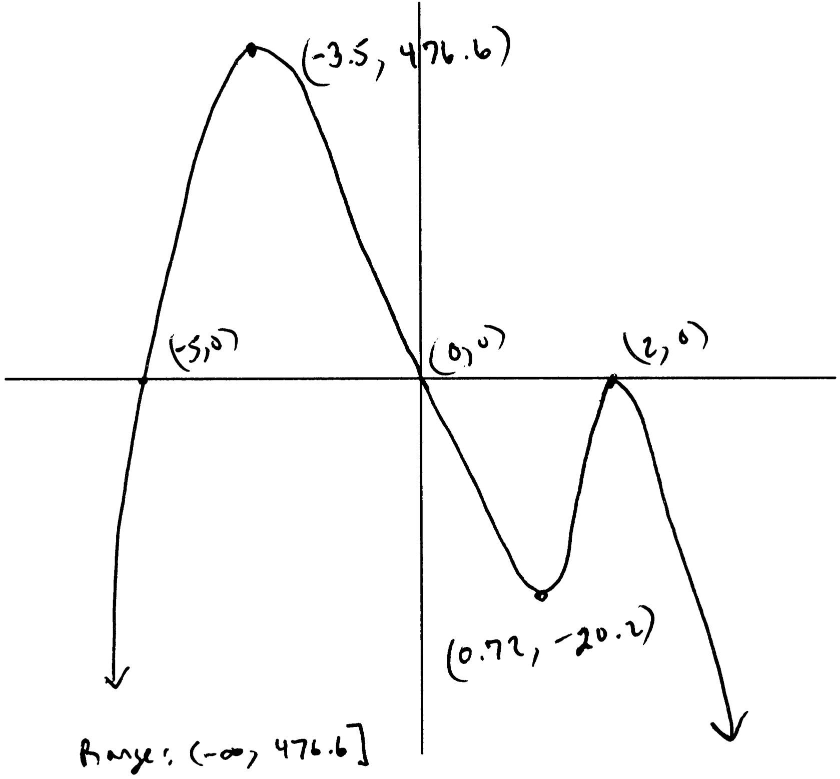
$x=0$ $x=2$ $x=-5$

y int

$f(0) = -3(0)(0-2)^2(0+5)$

$f(0) = 0$

$(0, 0)$



Range: $(-\infty, 476.6]$

Domain: $(-\infty, \infty)$

5th degree - Leading coefficient (+) ... ↓↑

G) $f_{(x)} = \frac{1}{4}(x+5)^3(x+1)(x-3)$

x int

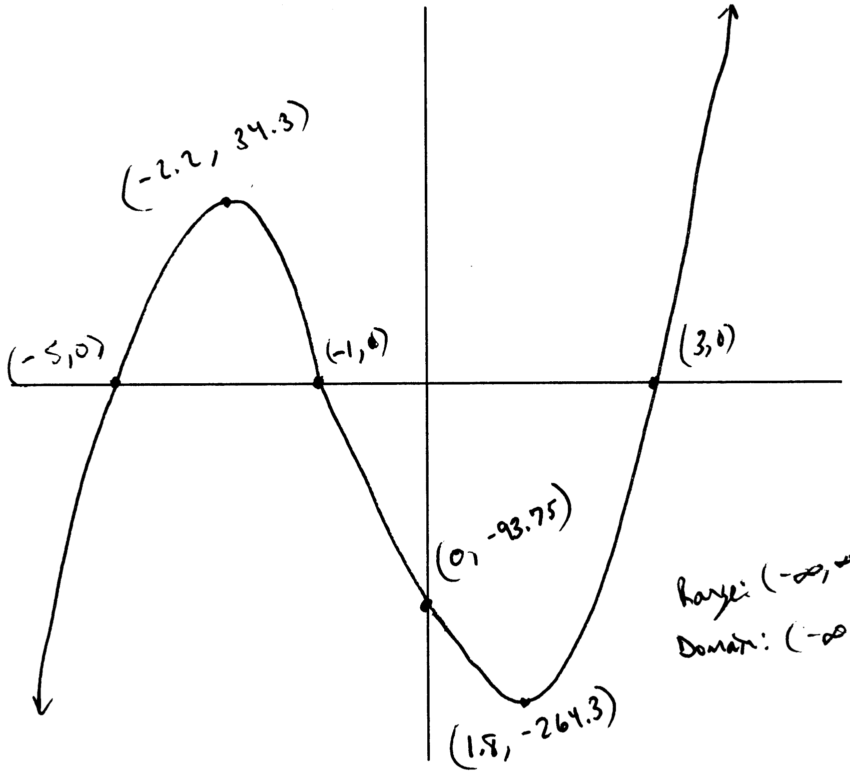
$\frac{1}{4}(x+5)^3(x+1)(x-3)=0$

$x=-5$ $x=-1$ $x=3$

y int

$f(5) = \frac{1}{4}(5)^3(1)(-3)$

(0, -93.75)



Range: (-∞,∞)
Domain: (-∞,∞)

H) $f_{(x)} = -x^2(x+3)(x-2)(x-6)^2$ 6th Degree - Leading coefficient (-) ... ↓↓

x int

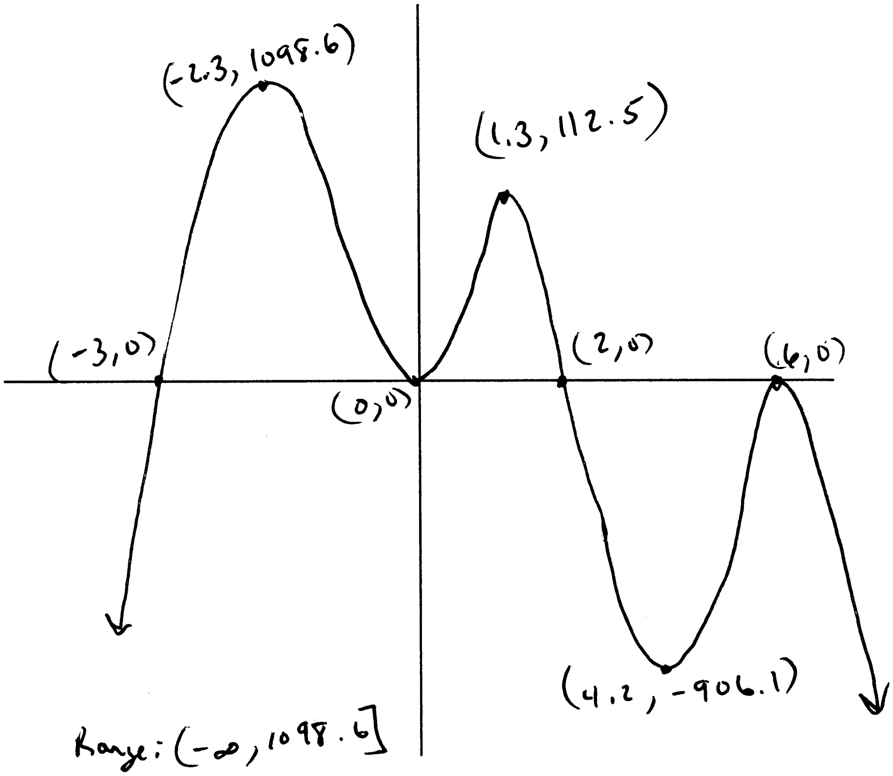
$-x^2(x+3)(x-2)(x-6)^2=0$

$x=0$ $x=-3$ $x=2$ $x=6$

y int

$f(0) = -(0)^2(0+3)(0-2)(0-6)^2$

(0,0)



Range: (-∞, 1098.6]
Domain: (-∞,∞)

5th degree - Leading coefficient (-) ... ↑↓

I) $f_{(x)} = -(x+3)^3(x-5)^2$

x int

$-(x+3)^3(x-5)^2=0$

$-1\neq 0 \quad (x+3)^3=0 \quad (x-5)^2=0$

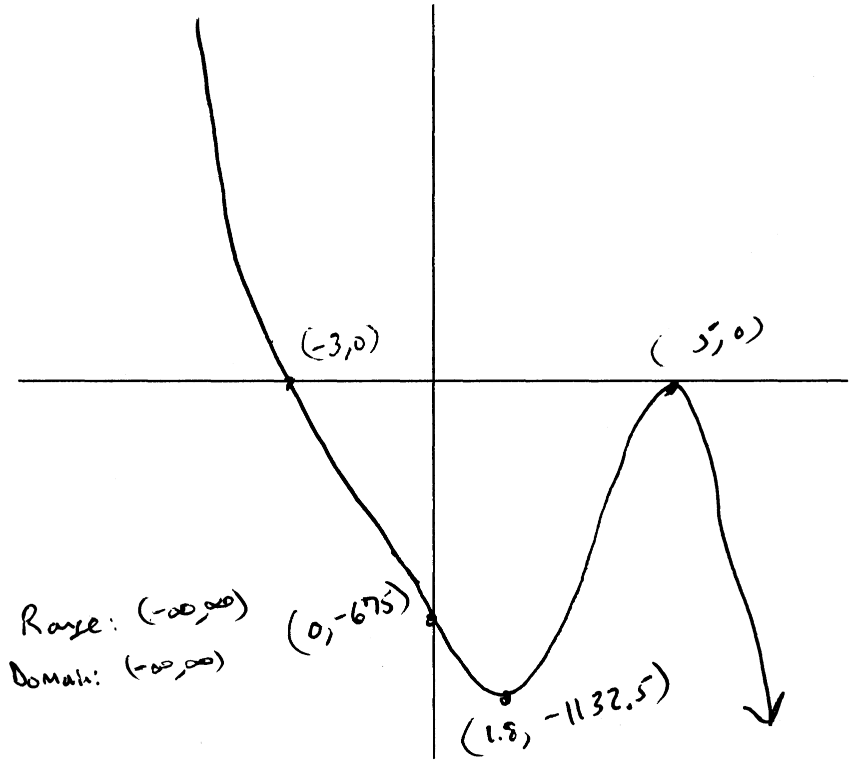
$x=-3 \quad x=5$

y int

$f(0) = -(3)^3(-5)^2$

$f(0) = -(27)(25)$

$(0, -675)$



Range: (-∞,∞)

Domain: (-∞,∞)

3rd degree ... Leading coefficient (+) ... ↓↑

J) $f_{(x)} = x^3 - 6x^2 - 9x + 54$

x int

$x^2(x-6) - 9(x-6) = 0$

$(x^2-9)(x-6) = 0$

$(x+3)(x-3)(x-6) = 0$

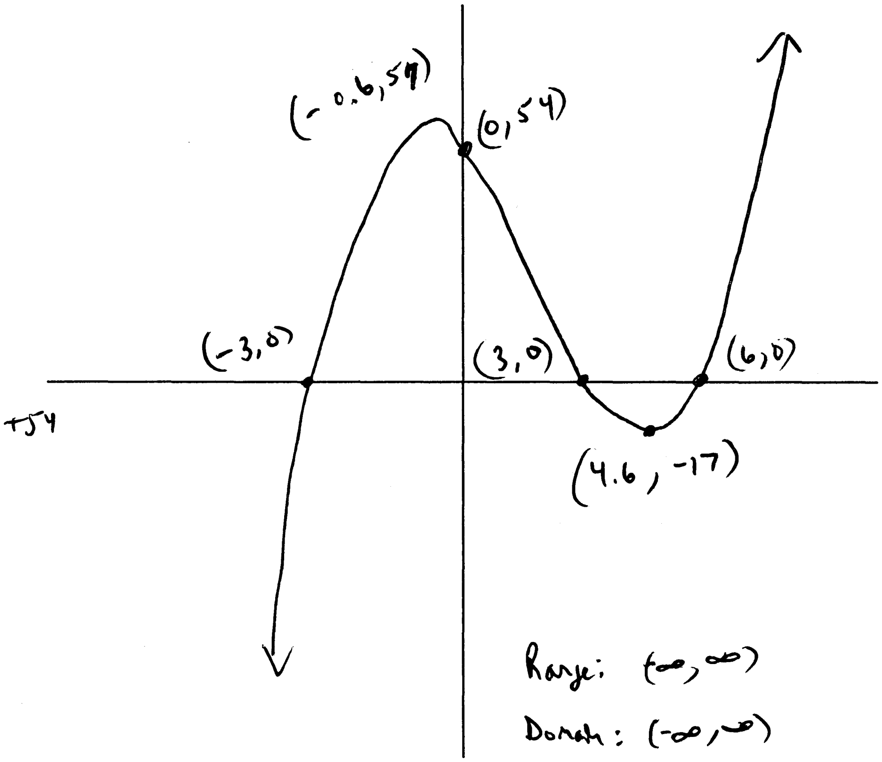
$x=-3 \quad x=3 \quad x=6$

y int

$f(0) = (0)^3 - 6(0)^2 - 9(0) + 54$

$f(0) = 54$

$(0, 54)$



Range: (-∞,∞)

Domain: (-∞,∞)

4th degree .. Leading coefficient (-) ---- ↓↓

K) $f_{(x)} = -x^4 + 37x^2 - 24x - 180$

x int

$$\begin{array}{r|rrrrr} 5 & -1 & 0 & 37 & -24 & -180 \\ & 0 & -5 & -25 & 60 & 180 \\ \hline 3 & -1 & -5 & 12 & 36 & \boxed{0} \\ & 0 & -3 & -24 & -36 & \\ \hline & -1 & -8 & -12 & \boxed{0} & \end{array}$$

x=5

x=3

$-x^2 - 8x - 12 = 0$

$-(x^2 + 8x + 12) = 0$

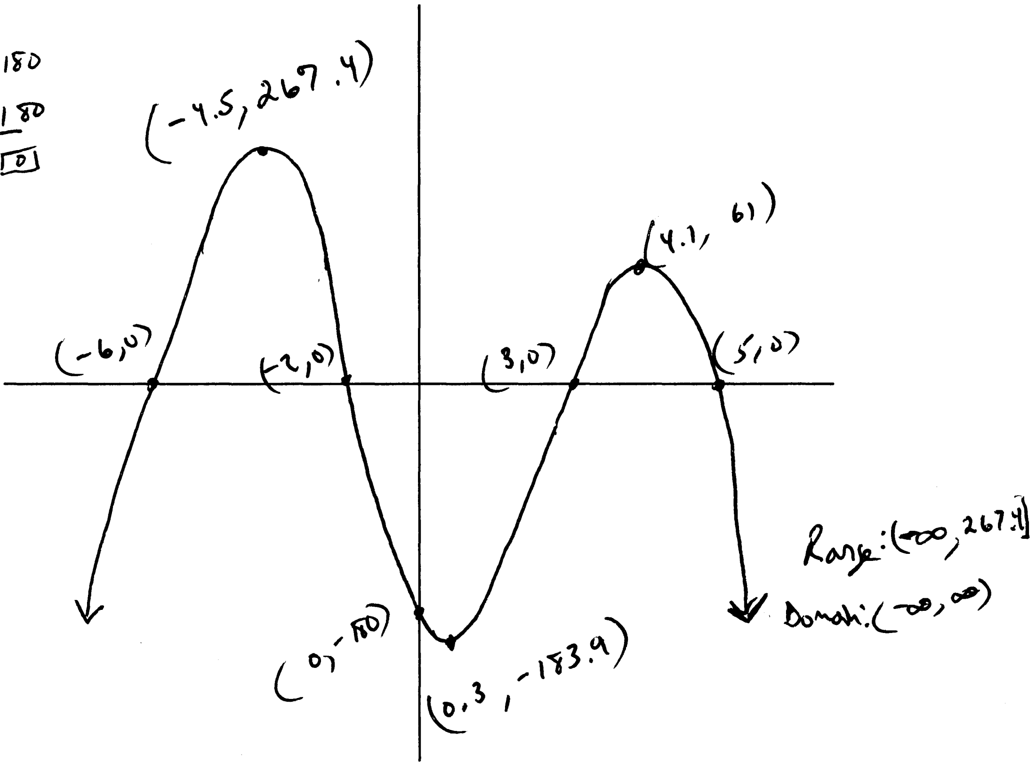
$-(x+2)(x+6) = 0$

$x = -2 \quad x = -6$

y int

$f(0) = -180$

$(0, -180)$



Range: (-∞, 267.4]

Domain: (-∞, ∞)

6th degree ... Leading Coefficient (+) .. ↑ ↑

L) $f_{(x)} = x^6 - 25x^4 - 16x^2 + 400$

x int

$x^4(x^2 - 25) - 16(x^2 - 25) = 0$

$(x^4 - 16)(x^2 - 25) = 0$

$(x^2 + 4)(x^2 - 4)(x^2 - 25) = 0$

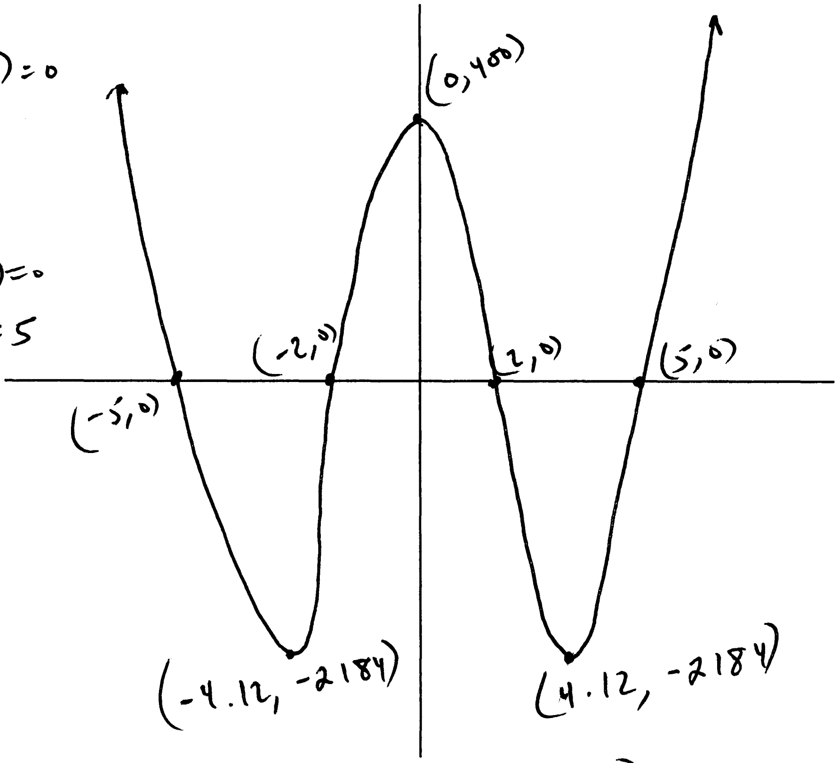
$(x^2 + 4)(x+2)(x-2)(x+5)(x-5) = 0$

not real $\quad x = -2 \quad x = 2 \quad x = -5 \quad x = 5$

y-int

$f(0) = 400$

$(0, 400)$

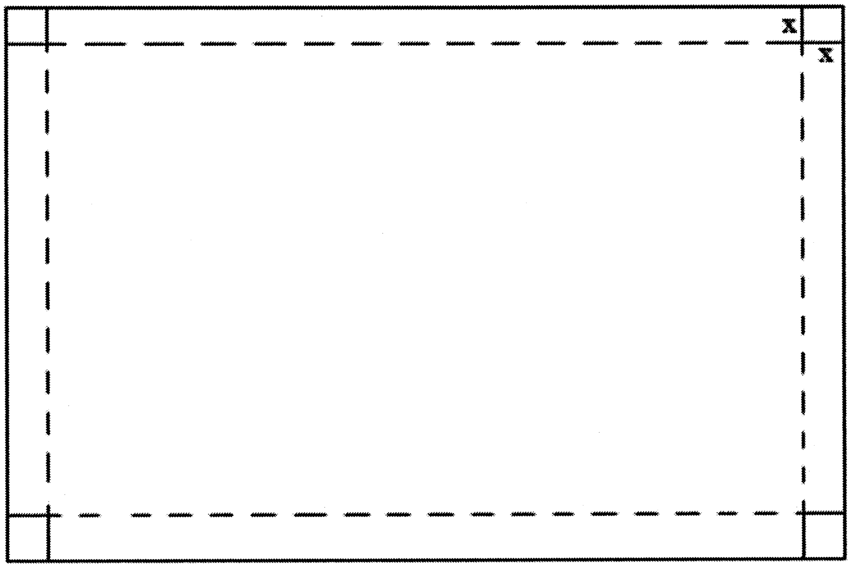


Range: (-2184, ∞)

Domain: (-∞, ∞)

Word Problems

Refer to the following diagram for problems A-D.



A) You have a rectangular piece of wood that is 16 in by 8 in. A small length, x, is cut from each of the 4 sides yielding a surface area of 48 in^2. Find the value of x.

$A = L \cdot w$

$(16 - 2x)(8 - 2x) = 48$

$128 - 32x - 16x + 4x^2 = 48$

$4x^2 - 48x + 128 = 48$

$4x^2 - 48x + 80 = 0$

$4(x^2 - 12x + 20) = 0$

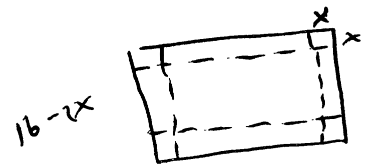
$4(x - 2)(x - 10) = 0$

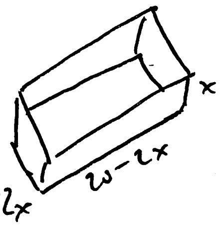
$4 \neq 0 \quad x = 2 \quad x = 10$

x is 2 inches

x cannot be 10 inches because the original lengths are too small to accomodate that.

B) You have a rectangular piece of steel whose dimensions are 20 inches by 16 inches. You are required to cut out the four corners of the rectangle so that you may fold up the sides to create a box. Write the function you would use to find the volume of the box if x represents the length of the cuts.





$v = L \cdot w \cdot h$

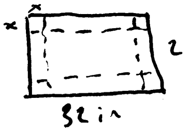
$v = x(20 - 2x)(16 - 2x)$

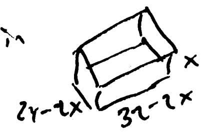
$v = x(2x - 20)(2x - 16) \leftarrow$ factor out a (−1) from each to change the signs and order

$v = x(4x^2 - 32x - 40x + 320)$

$V = 4x^3 - 72x^2 + 320x$

C) You are given a rectangular sheet of metal that is 32 inches by 24 inches. You are required to cut a length from each corner of the sheet so that you may fold up the ends and create a box. What is the domain of the function you will use to find the volume of the box? Explain your answer.



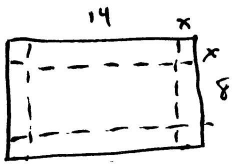


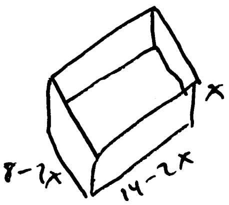
Domain

$0 < x < 12$

The domain is all real #'s greater than zero and less than 12. This must be the domain, because if x equalled zero, you cannot fold up the sides to create a box. Therefore, x must be greater than zero. However, since the shorter side of the sheet of metal is 24 inches, the value of x must be less than 12. If 12 inches were cut from each corner, there would be nothing left because 12 is a zero of the function. Therefore, the domain is $0<x<12$

D) You are given a 14 inch by 8 inch rectangular sheet of metal from which you are to construct a box. You are to cut a length, x, from each corner of the sheet of metal so that you can fold up the sides creating a box. Find the value of x that will yield the maximum volume of the box. Round your solution to 3 significant digits.





$v = x(2x-14)(2x-8)$

$v = x(4x^2-16x-28x+112)$

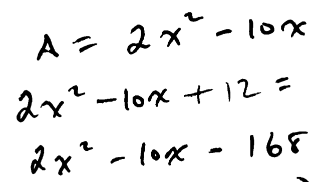
$v = 4x^3-44x^2+112x$

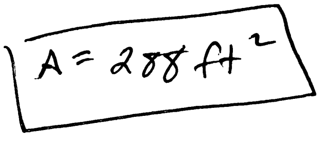
Domain $0<x<4$

using the graphing calculator, find the local maximum within the domain $(0,4)$

$\boxed{x \approx 1.333 \text{ inches}}$

E) A rectangular field is twice as long as it is wide. If 3 feet are taken from the width, and 4 feet taken from the length, the resultant are of the field is 180 ft^2. Find the area of the original field.





$A = L \cdot W$

$A = (2x-4)(x-3)$

$A = 2x^2 - 10x + 12$

$2x^2 - 10x + 12 = 180$

$2x^2 - 10x - 168 = 0$

$2(x^2 - 5x - 84) = 0$

$2(x+7)(x-12) = 0$

$2 \neq 0 \quad x \neq -7 \quad x = 12$

$A = (24)(12)$

$\boxed{A = 288 \text{ ft}^2}$

Checking Progress

You have now completed the "Polynomial Functions" section of the workbook. The following is a checklist so that you may check your progress. Check off each of the objectives you have accomplished.

The student should be able to…

- ☐ ***Find the quotient of a division problem involving polynomials using the polynomial long division method.***
- ☐ ***Find the quotient of a division problem involving polynomials using the synthetic division method.***
- ☐ ***Use the rational zero test to determine all possible rational zeros of a polynomial function.***
- ☐ ***Use the rational zero test to determine all possible roots of a polynomial equation.***
- ☐ ***Use Descarte's Rule of Signs to determine the possible number of positive or negative roots of a polynomial equation.***
- ☐ ***Find all zeros of a polynomial function.***
- ☐ ***Use the remainder theorem to evaluate the value of functions.***
- ☐ ***Write a polynomial in completely factored form.***
- ☐ ***Write a polynomial as a product of factors irreducible over the reals.***
- ☐ ***Write a polynomial as a product of factors irreducible over the rationals.***
- ☐ ***Find the equation of a polynomial function that has the given zeros.***
- ☐ ***Determine if a polynomial function is even, odd or neither.***
- ☐ ***Determine the left and right behaviors of a polynomial function without graphing.***
- ☐ ***Find the local maxima and minima of a polynomial function.***
- ☐ ***Find all x intercepts of a polynomial function.***
- ☐ ***Determine the maximum number of turns a given polynomial function may have.***
- ☐ ***Graph a polynomial function.***

RATIONAL FUNCTIONS

Objectives

The following is a list of objectives for this section of the workbook.

By the time the student is finished with this section of the workbook, he/she should be able to...

- ***Find the vertical asymptotes of a rational function.***
- ***Determine if the function has a horizontal or oblique asymptote.***
- ***Find the horizontal asymptote of a rational function if it exists.***
- ***Find the oblique asymptote of a rational function if it exists.***
- ***Find the domain of a rational function.***
- ***Find the x intercepts of a rational function.***
- ***Find the y intercept of a rational function.***
- ***Graph a rational function.***

Math Standards Addressed

The following state standards are addressed in this section of the workbook.

Algebra II

3.0 Students are adept at operations on polynomials, including long division.

4.0 Students factor polynomials representing the difference of squares, perfect square trinomials, and the sum and difference of two cubes.

7.0 Students add, subtract, multiply, divide, reduce, and evaluate rational expressions with monomial and polynomial denominators and simplify complicated rational expressions, including those with negative exponents in the denominator.

8.0 Students solve and graph quadratic equations by factoring, completing the square, or using the quadratic formula. Students apply these techniques in solving word problems. They also solve quadratic equations in the complex number system.

15.0 Students determine whether a specific algebraic statement involving rational expressions, radical expressions, or logarithmic or exponential functions is sometimes true, always true, or never true.

Mathematical Analysis

6.0 Students find the roots and poles of a rational function and can graph the function and locate its asymptotes.

Finding Asymptotes

Rational functions have various asymptotes. The following will aid in finding all asymptotes of a rational function. The first step to working with rational functions is to completely factor the polynomials. Once in factored form, find all zeros.

Vertical Asymptotes

- **The Vertical Asymptotes of a rational function are found using the zeros of the denominator.**

For Horizontal Asymptotes use the following guidelines.

- **If the degree of the numerator is greater than the degree of the denominator by more than one, the graph has no horizontal asymptote.(none)**
- **If the degree of the numerator is equal to the degree of the denominator, the horizontal asymptote is the ratio of the two leading coefficients.(y = #)**
- **If the degree of the numerator is less than the degree of the denominator, the horizontal asymptote is zero. (y = 0)**

Oblique Asymptotes

- **If the degree of the numerator is greater than the degree of the denominator by one, there is an oblique asymptote. The asymptote is the quotient numerator divided by the denominator.**

An asymptote is like an imaginary line that cannot be crossed. All rational functions have vertical asymptotes. A rational function may also have either a horizontal or oblique asymptote. A rational function will never have both a horizontal and oblique asymptote. It is either one or the other. Horizontal asymptotes are the only asymptotes that may be crossed. The vertical asymptotes come from zeroes of the denominator.

$$f_{(x)} = \frac{x}{(x+2)(x-3)}$$

Here is a rational function in completely factored form.

$$x = -2 \quad and \quad x = 3$$

The zeros of the denominator are -2 and 3. Therefore, these are the vertical asymptotes of the function.

Since an x value of -2 or 3 would create a zero in the denominator, the function would be undefined at that location. As a result, these are the vertical asymptotes for this function.

In this same function, the degree of the numerator is less than the degree of the denominator, therefore, the horizontal asymptote is $y=0$.

When finding the oblique asymptote, find the quotient of the numerator and denominator. If there are any remainders, disregard them. You only need the quotient. The graph of the function can have a either a horizontal asymptote, or an oblique asymptote. You can not have one of each. This particular function does not have an oblique asymptote.

Here is an example with an oblique asymptote.

Find the oblique asymptote of the rational function $f_{(x)}=\dfrac{x^2+8x-20}{x-1}$.

$$\begin{array}{r} x+9 \\ x-1\overline{)\,x^2+8x-20} \\ \underline{-x^2+x} \\ 9x-20 \\ \underline{-9x+9} \\ -11 \end{array}$$

Dividing the polynomials, the quotient x+9 is found.

$$y=x+9$$

This is the equation for the oblique asymptote of the function. Notice the remainder of the division problem is disregarded. It plays no part in the equation for the oblique asymptote.

Finally, let us look at a rational function where the degree of the numerator is equal to the degree of the denominator.

Find the horizontal asymptote for the rational function $f_{(x)}=\dfrac{2x^2-4x+8}{3x^2-27}$.

$$f_{(x)}=\frac{2x^2-4x+8}{3x^2-27}$$

Notice the degree of the numerator is the same as the degree of the denominator.

$$y=\frac{2}{3}$$

Since the degree of the numerator equals that of the denominator, the equation for the horizontal asymptote is the ratio of the two leading coefficients.

Find all asymptotes of the following functions. **(Do not graph these)**

A) $f_{(x)} = \dfrac{x-7}{x+5}$

V.A.:

$x+5=0$

$x=-5$

H.A.

$y=\frac{1}{1}$

$y=1$

V.A.: $x=-5$

H.A.: $y=1$

B) $f_{(x)} = \dfrac{3}{x^2-2}$

V.A.

$x^2-2=0$

$x^2=2$

$x=\pm\sqrt{2}$

H.A.

$y=0$

V.A.: $x=-\sqrt{2}$

$x=\sqrt{2}$

HA: $y=0$

C) $f_{(x)} = \dfrac{x^2}{x-5}$

VA

$x-5=0$

$x=5$

O.A.

$$\begin{array}{r} x-5 \\ x-5 \overline{\big) x^2 + 0x } \\ x^2 - 5x \\ \hline -5x \\ +5x \mp 25 \\ \hline -25 \end{array}$$

VA: $x=5$

OA: $y=x-5$

D) $f_{(x)} = \dfrac{2x^2-5x+3}{x-1}$

VA

$x-1=0$

$x=1$

OA

$$\begin{array}{r} 2x-3 \\ x-1 \overline{\big) 2x^2-5x+3} \\ -2x^2+2x \\ \hline -3x+3 \\ +3x\mp 3 \\ \hline 0 \end{array}$$

VA: $x=1$

OA: $y=2x-3$

E) $f_{(x)} = \dfrac{7x^2+5x-2}{2x^2-18}$

VA

$2x^2-18=0$

$2(x+3)(x-3)=0$

$2\neq 0$ $x=-3$ $x=3$

HA

$y=\frac{7}{2}$

VA: $x=-3$

$x=3$

HA: $y=\frac{7}{2}$

F) $f_{(x)} = \dfrac{2x^2-5x+5}{x-2}$

VA

$x-2=0$

$x=2$

OA

$$\begin{array}{r} 2x-1 \\ x-2 \overline{\big) 2x^2-5x+5} \\ -2x^2\mp 4x \\ \hline -x+5 \\ +x\mp 2 \\ \hline 3 \end{array}$$

VA: $x=2$

OA: $y=2x-1$

G) $f_{(x)} = \dfrac{1}{3-x}$

VA

$3-x=0$

$+x \quad +x$

$3=x$

HA

$y=0$

VA: $x=3$

HA: $y=0$

H) $f_{(x)} = \dfrac{x^2-4}{x^4-81}$

VA

$(x^2+9)(x^2-9)=0$

$(x^2+9)(x+3)(x-3)=0$

not real $\quad x=-3 \quad x=3$

HA

$y=0$

VA: $x=-3$

$x=3$

HA: $y=0$

I) $f_{(x)} = \dfrac{x^3-2x^2+5}{x^2}$

VA

$x^2=0$

OA

$$\begin{array}{r} x-2 \\ x^2 \overline{\big) x^3-2x^2+0x+5} \\ -x^3 \\ \hline -2x^2 \\ +2x^2 \\ \hline 5 \end{array}$$

VA: $x=0$

OA: $y=x-2$

The Domain

The domain of a rational function is found using only the vertical asymptotes. As previously noted, rational functions are undefined at vertical asymptotes. The rational function will be defined at all other x values of the domain.

$$f_{(x)}=\frac{x}{(x+2)(x-3)}$$

Here is a rational function in completely factored form.

$$x=-2 \quad and \quad x=3$$

Since the zeros of the denominator are -2 and 3, these are the vertical asymptotes of the function.

Therefore, the domain of this function is $(-\infty,-2)\cup(-2,3)\cup(3,\infty)$. ***Notice there are two vertical asymptotes, and the domain is split into three parts. This pattern will repeat. If there are 4 vertical asymptotes, the domain of that function will be split into 5 parts.***

Find the domain of each of the following rational functions.

A) $f_{(x)}=\frac{x-7}{x+5}$

VA

$x+5=0$

$x=-5$

Domain : $(-\infty,-5)\cup(-5\infty)$

B) $f_{(x)}=\frac{3}{x^2-4}$

VA

$(x+2)(x-2)=0$

$x=-2 \quad x=2$

Domain : $(-\infty,-2)\cup(-2,2)\cup(2,\infty)$

C) $f_{(x)}=\frac{x^2}{x-5}$

VA

$x-5=0$

$x=5$

Domain : $(-\infty,5)\cup(5,\infty)$

D) $f_{(x)}=\frac{2x^2-5x+3}{x-1}$

VA

$x-1=0$

$x=1$

Domain: $(-\infty,1)\cup(1,\infty)$

E) $f_{(x)}=\frac{x-8}{x^3-x^2-12x}$

VA

$x(x^2-x-12)=0$

$x(x-4)(x+3)=0$

$x=0 \quad x=4 \quad x=-3$

Domain: $(-\infty,-3)\cup(-3,0)\cup(0,4)\cup(4,\infty)$

F) $f_{(x)}=\frac{x^3}{x^2-7x+12}$

VA

$(x-3)(x-4)=0$

$x=3 \quad x=4$

Domain : $(-\infty,3)\cup(3,4)\cup(4,\infty)$

G) $f_{(x)}=\frac{1}{3-x}$

VA $3-x=0$

$3=x$

Domain: $(-\infty,3)\cup(3,\infty)$

H) $f_{(x)}=\frac{x^2-4}{x^4-81}$

VA

$(x^2+9)(x^2-9)$

$(x^2+9)(x+3)(x-3)=0$

$x=-3 \quad x=3$

Domain : $(-\infty,-3)\cup(-3,3)\cup(3,\infty)$

I) $f_{(x)}=\frac{x^3-2x^2+5}{x^2}$

VA

$x^2=0$

$x=0$

Domain : $(-\infty,0)\cup(0,\infty)$

Finding Intercepts

We have found that the zeros of the denominator of a rational function are the vertical asymptotes of the function. The zeros of the numerator on the other hand, are the x intercepts of the function.

Find all x and y intercepts of the function $f_{(x)}=\dfrac{x^2-9}{x-1}$.

$$f_{(x)}=\frac{(x+3)(x-3)}{x-1}$$

Write out the function in completely factored form.

Now, find the zeros of the numerator

$x=-3 \quad and \quad x=3$

These are the x intercepts of the function.

Look at the original function.

$$f_{(x)}=\frac{x^2-9}{x-1}$$

From here, substitute zero for x, and find the y intercept, which in this case will be the ratio of the two constants.

$y=9$

This is the y intercept of the function. In this case, it is the ratio of the two remaining constants once zero is substituted in for x. If there is no constant in the denominator, then there will be no y intercept as x=0 is a vertical asymptote and the graph is undefined at the y axis.

The x intercepts are $(-3,0)$ ***and*** $(3,0)$

The y intercept is $(0,9)$

As demonstrated above, the y intercept of a rational function is the ratio of the two constants. Like always, substitute zero for x, and solve for y to find the y intercept.

Find the x and y intercepts of each rational function.

A) $f_{(x)}=\dfrac{x-7}{x+5}$

x int

$x-7=0$

$x=7$

$(7,0)$

y int

$f(0)=\frac{0-7}{0+5}$

$f(0)=-7/5$

$(0,-\frac{7}{5})$

B) $f_{(x)}=\dfrac{3}{x^2-4}$

x int

$3\neq 0$

none

y int

$f(0)=\frac{3}{-4}$

$(0,-3/4)$

C) $f_{(x)}=\dfrac{x^2}{x-5}$

x int

$x-5=0$

$x=5$

$(5,0)$

y int

$f(0)=\frac{0}{-5}$

$f(0)=0$

$(0,0)$

D) $f_{(x)}=\dfrac{2x^2-5x+3}{x-1}$

x int

$(2x+1)(x-3)=0$

$x=-1/2 \quad x=3$

$(-\frac{1}{2},0) \quad (3,0)$

y int

$f(0)=\frac{3}{-1}$

$f(0)=-3$

$(0,-3)$

E) $f_{(x)}=\dfrac{x-8}{x^3-x^2-12x}$

x int

$x-8=0$

$x=8$

$(8,0)$

y int

$f(0)=\frac{-8}{0}$

none

F) $f_{(x)}=\dfrac{x^3}{x^2-7x+12}$

x int

$x^3=0$

$x=0$

$(0,0)$

y int

$f(0)=\frac{0}{12}$

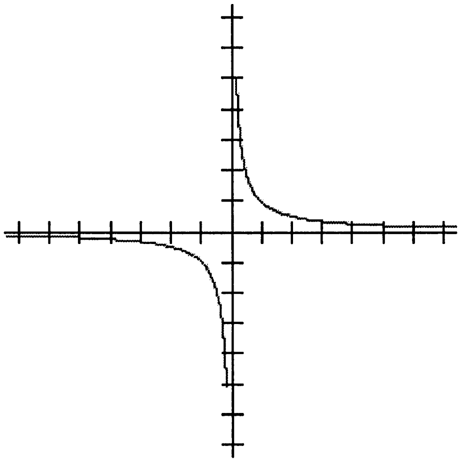
$f(0)=0$

$(0,0)$

Graphing Rational Functions

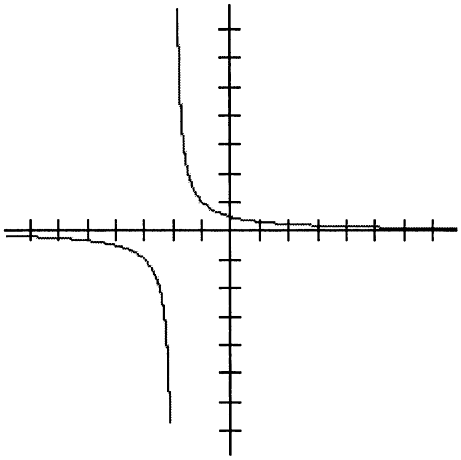
We really have no standard form of a rational function to look at, so we will concentrate on the parent function of $f_{(x)} = \frac{1}{x}$. The following pages illustrate the effects of the denominator, as well as the behavior of $-f_{(x)}$. A graphing calculator may be used to help get the overall shape of these functions. <u>DO NOT</u>, however, just copy the picture the calculator gives you.

$$f_{(x)} = \frac{1}{x}$$



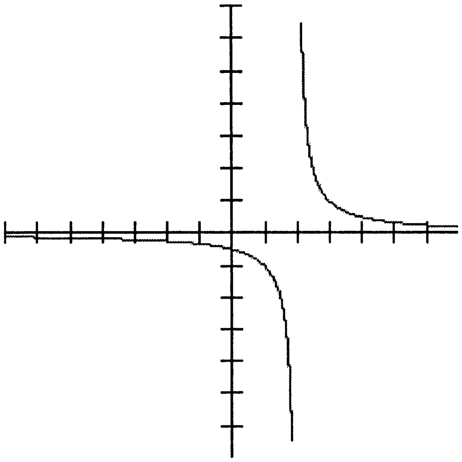
Here, the vertical asymptote is at x=0, and the horizontal asymptote is y=0.

$$f_{(x)} = \frac{1}{x+2}$$



The graph of this function shifts left 2.

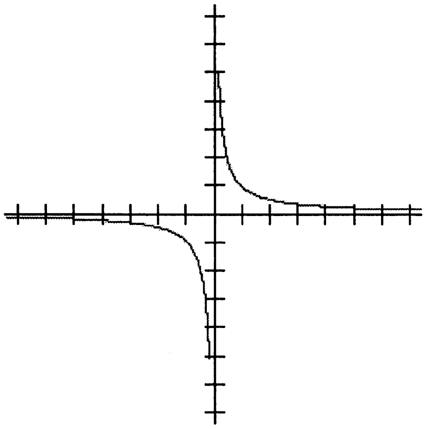
$$f_{(x)} = \frac{1}{x-2}$$



The graph of this function shifts right 2.

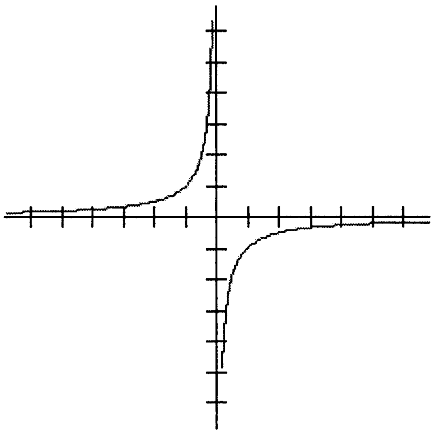
The range for each of these functions is $(-\infty, 0) \cup (0, \infty)$. There is no way to tell what the range of a rational function will be until it is graphedt. Remember, the curve may cross the horizontal axis.

$$f_{(x)} = \frac{1}{x}$$



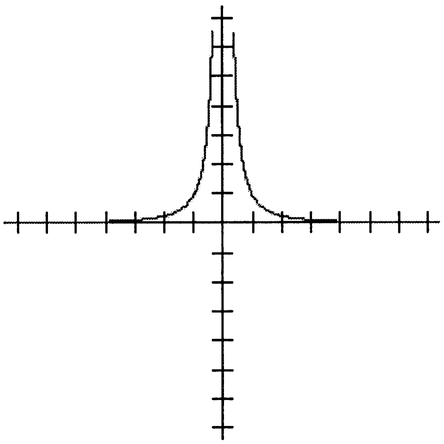
Here, the vertical asymptote is at x=0, and the horizontal asymptote is y=0.

$$f_{(x)} = -\frac{1}{x}$$



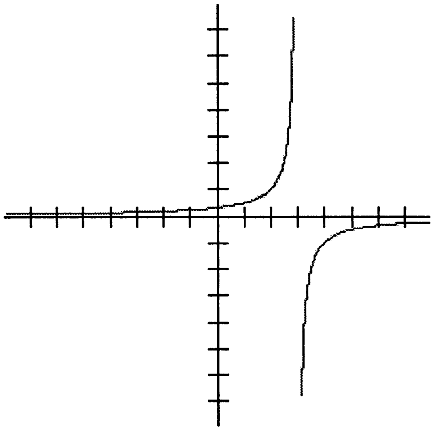
The graph of this function is a reflection of the parent function.

$$f_{(x)} = \frac{1}{x^2}$$



Notice how x^2 affects the function. Normally, one side of the function would go up, and the other would go down. Since there is no way to get a negative number in the denominator, both sides are going in the same direction.

$$f_{(x)} = \frac{1}{3-x}$$



The graph of the function to the left flips upside down, similar to $f_{(x)} = -\frac{1}{x}$, and shifts right 3. What happens here is a -1 is factored out of the denominator, changing the function to the following.

$$f_{(x)} = \frac{1}{3-x} \Rightarrow \frac{1}{-(x-3)} \Rightarrow -\frac{1}{x-3}$$

As a result, this graph is a combination of shifting the graph and reflecting it about the horizontal asymptote.

Example

Graph the function $f_{(x)} = \dfrac{x-3}{x^2-x-12}$. Be sure to find all asymptotes, x and y intercepts, and the range and domain.

$$f_{(x)} = \frac{x-3}{(x+3)(x-4)}$$

The first step is to completely factor the rational function.

Vertical Asymptotes are $x=-3$ and $x=4$

The zeros of the denominator are the vertical asymptotes of the function.

Looking at the original function, the horizontal asymptote is $y=0$.

If the degree of the numerator is less than the degree of the denominator, the horizontal asymptote is $y=0$.

The x intercept is $(3,0)$

The zero of the numerator is the x intercept of the function.

The y intercept is $\left(0,\frac{1}{4}\right)$.

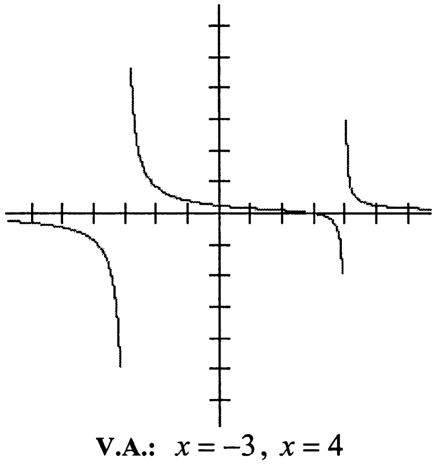
Substituting zero for x and evaluating the ratio of the two constants, -3 and -12. Yields a y intercept of $\frac{1}{4}$.

The domain of the function is
$(-\infty,-3)\cup(-3,4)\cup(4,\infty)$

The domain is found using the vertical asymptotes. The domain here is all real numbers except -3 and 4.

The range should not be found without first graphing the function.

$$f_{(x)} = \frac{x-3}{x^2-x-12}$$



Due to limitations in the graphing software, the graph to the left is incomplete. Each of these lines is continuous. Before attempting to graph the functions, graph <u>all</u> asymptotes using broken lines. This will ensure that a vertical asymptote is not crossed. To get the general shape of the equation, use a combination of the x and y intercepts that were found, and plug in values for x close to the vertical asymptote. Based on the properties of all rational functions, it should be obvious how these curves behave on the outer intervals of these functions. They will always ride along the asymptotes in these areas.

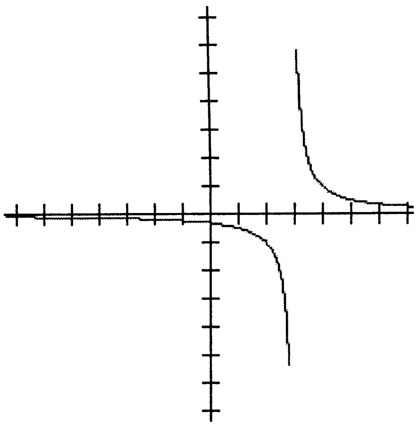
V.A.: $x=-3$, $x=4$
H.A.: $y=0$
x-int: $(3,0)$
y-int: $\left(0,\frac{1}{4}\right)$
Range: $(-\infty,\infty)$
Domain: $(-\infty,-3)\cup(-3,4)\cup(4,\infty)$

Here is a list of all required information needed for each rational function. Since the graph of the function crossed the horizontal asymptote in the interval $(-3,4)$, the range of this function is all real numbers.

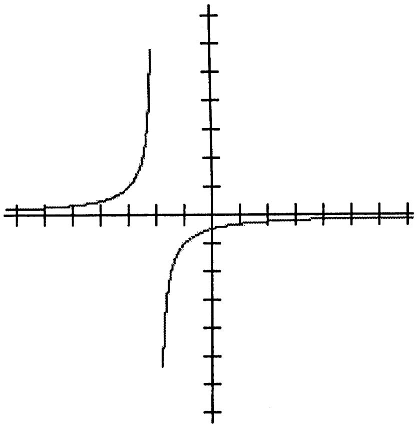
<u>These procedures must be used when graphing any rational function.</u>

Match the appropriate graph with its equation below. Explain why each of the solutions is true.

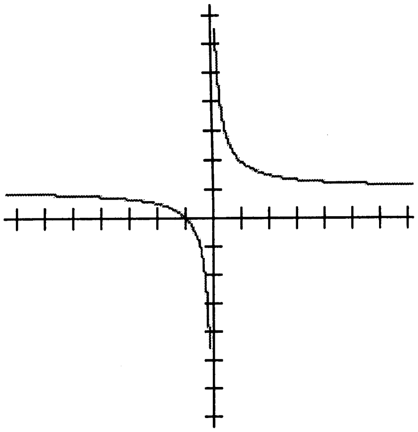
A



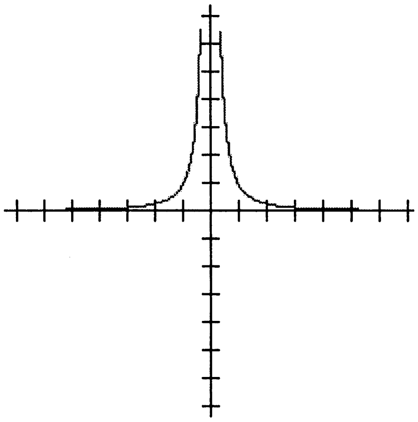
B



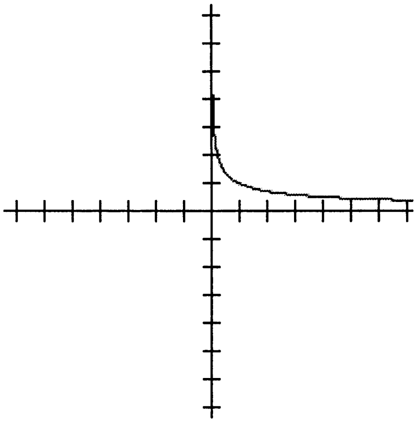
C



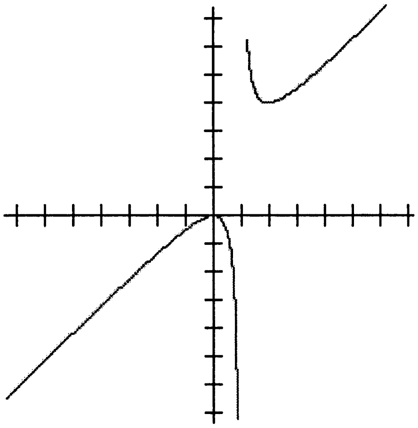
D



E



F



1) $f_{(x)}=\frac{1}{x^2}$

D, the vertical asymptote is at $x=0$. Since the only variable is squared, all values of the function will be (+) meaning it is all above the x axis.

2) $f_{(x)}=\frac{1}{x-3}$

A, the vertical asymptote is at $x=3$ ad the horizontal asymptote is at $y=0$

3) $f_{(x)}=\frac{1}{\sqrt{x}}$

E, since there is a radical in the denominator the domain is $(0,\infty)$ which means the graph will only exist on the positive side of the y-axis.

4) $f_{(x)}=-\frac{1}{x+2}$

B, the vertical asymptote is at $x=-2$, and the function is above the x axis on the left, and below on the right

5) $f_{(x)}=\frac{x^2}{x-1}$

F, this function has an oblique asymptote.

6) $f_{(x)}=\frac{1}{x}+1$

C, This is the graph of $\frac{1}{x}$, shifted up 1.

A graphing calculator may be used to help get a picture of the curve that will be created, but simply copying the picture shown in the calculator is unwise.

What is the problem with the picture of rational functions in graphing calculators?

On a graphing calculator the asymptotes sometimes show up as solid lines leading one to believe they are part of the graph of the function.

Sketch the graph of each of the following functions. Be sure to find all asymptotes, x and y intercepts, and the range and domain of each of the following.

A) $f_{(x)} = \dfrac{1}{x-4}$

VA

$x-4=0$

$x=4$

HA

$y=0$

x int

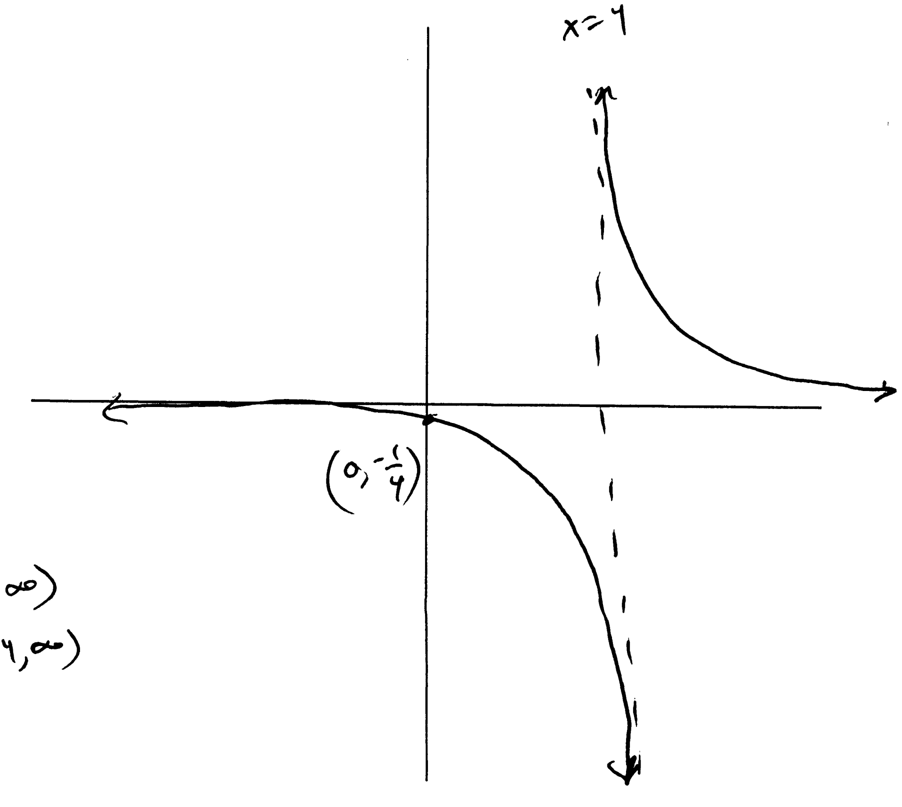
$1 \neq 0$

none

y int

$f_{(0)} = \frac{1}{-4}$

$\left(0, -\frac{1}{4}\right)$



Range: $(-\infty, 0) \cup (0, \infty)$

Domain: $(-\infty, 4) \cup (4, \infty)$

B) $f_{(x)} = -\frac{1}{x+2}$

VA

$x+2=0$

$x=-2$

HA

$y=0$

x int

$-1 \neq 0$

none

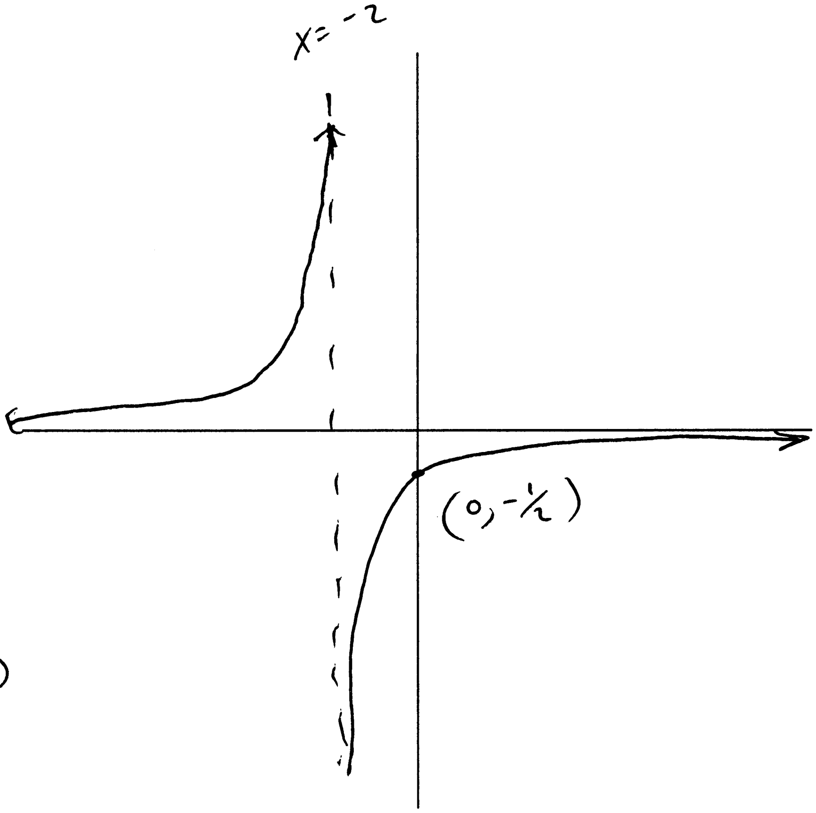
y int

$f(0) = -\frac{1}{2}$

$(0, -\frac{1}{2})$

Range: $(-\infty, 0) \cup (0, \infty)$

Domain: $(-\infty, -2) \cup (-2, \infty)$



C) $f_{(x)} = \frac{x-3}{x+2}$

VA

$x+2=0$

$x=-2$

HA

$y=1$

x int

$x-3=0$

$x=3$

$(3,0)$

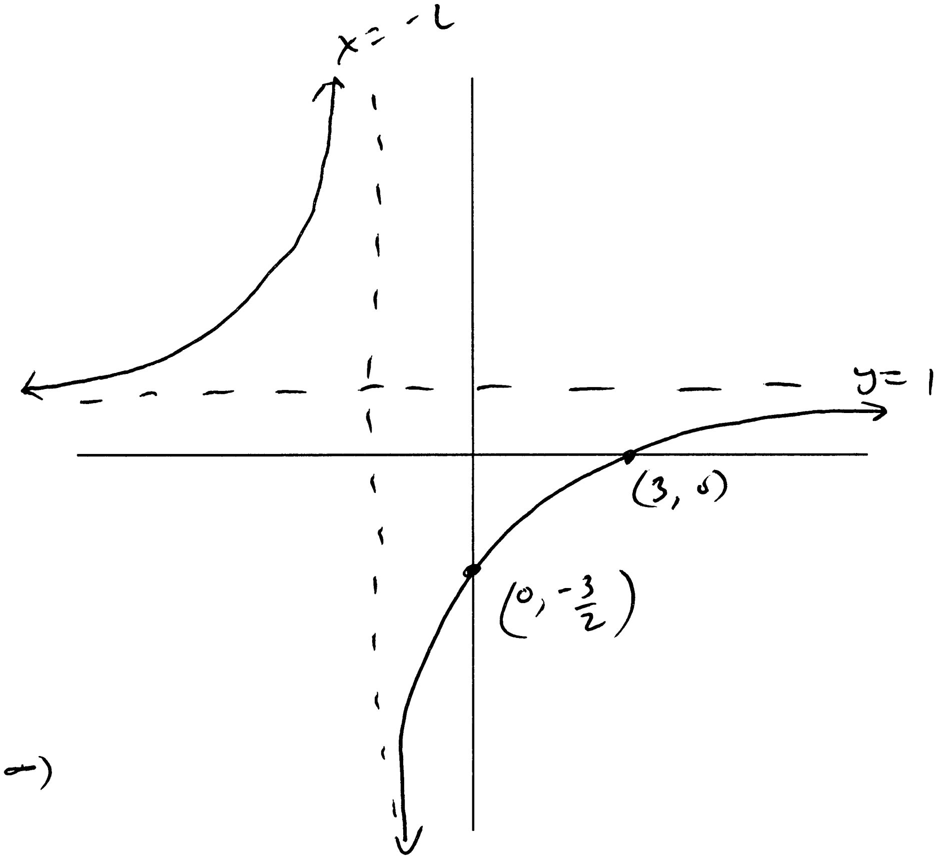
y int

$f(0) = -\frac{3}{2}$

$(0, -\frac{3}{2})$

Range: $(-\infty, 1) \cup (1, \infty)$

Domain: $(-\infty, -2) \cup (-2, \infty)$



D) $f_{(x)} = \frac{x+2}{3x-9}$

VA

$3x-9=0$

$3x=9$

$x=3$

HA

$y = \frac{1}{3}$

x int

$x+2=0$

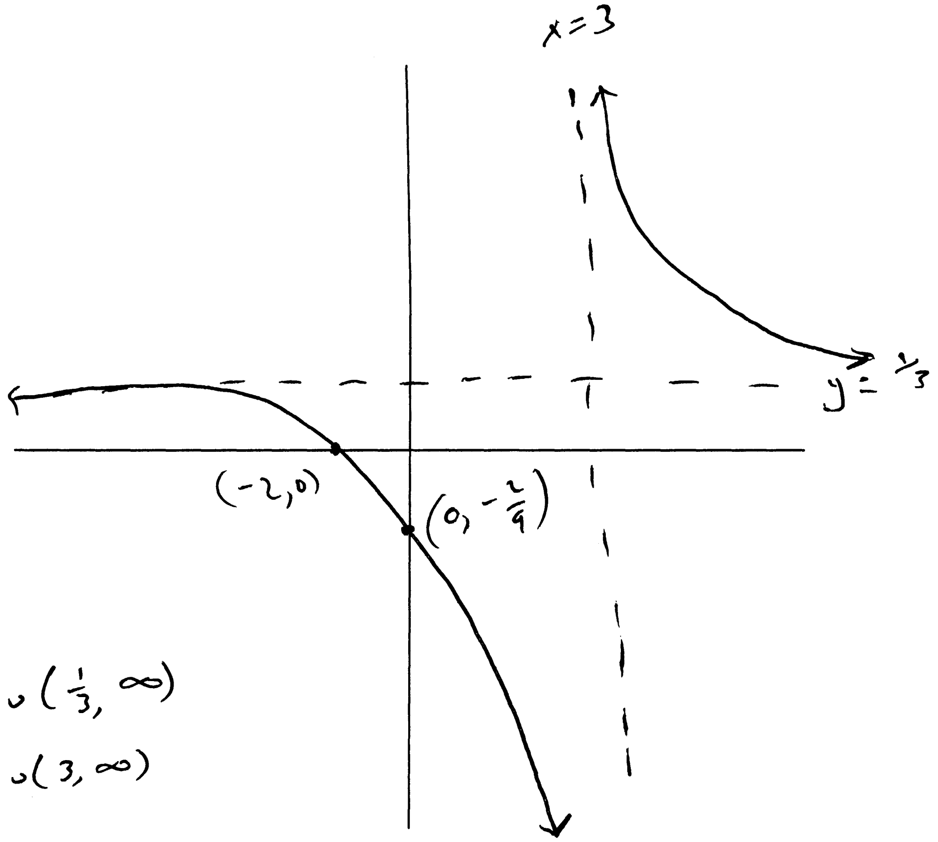
$x=-2$

$(-2,0)$

y int

$f_{(0)} = \frac{2}{-9}$

$(0, -\frac{2}{9})$



Range: $(-\infty, \frac{1}{3}) \cup (\frac{1}{3}, \infty)$

Domain: $(-\infty, 3) \cup (3, \infty)$

E) $f_{(x)} = \frac{2x^2+1}{x}$

VA

$x=0$

OA

$$\begin{array}{r} 2x \\ x\,\overline{)\,2x^2+0x+1} \\ \underline{-2x^2 } \\ 0 +1 \end{array}$$

$y = 2x$

x int

$2x^2+1=0$

$2x^2=-1$

$\sqrt{x^2} = \sqrt{-\frac{1}{2}}$

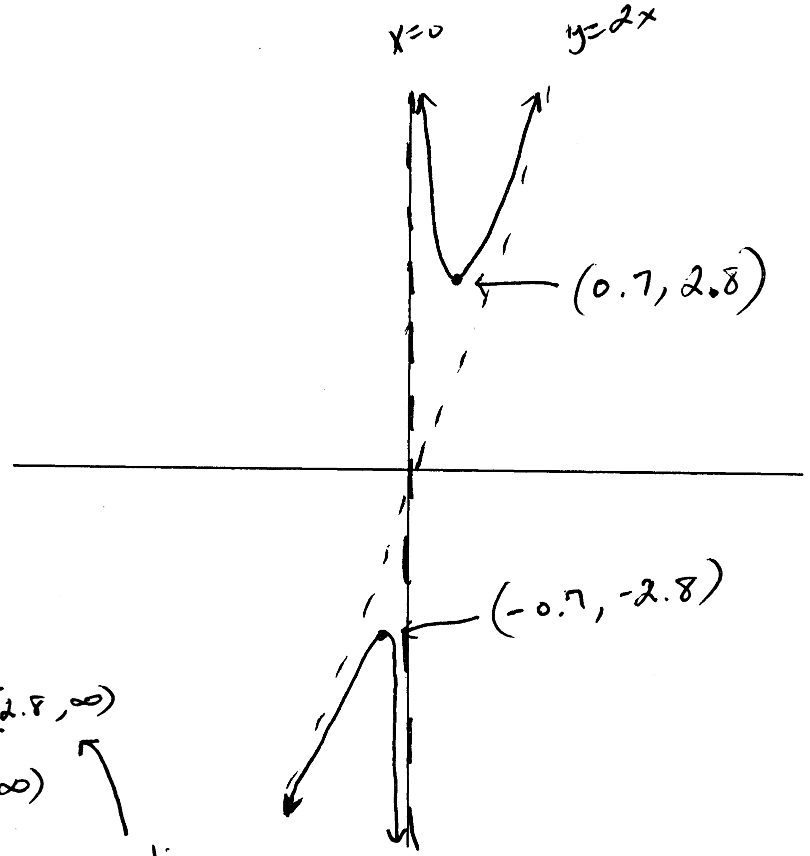
none

y int

$f_{(0)} = \frac{1}{0}$

undefined

none



Range: $(-\infty, -2.8] \cup [2.8, \infty)$

Domain: $(-\infty, 0) \cup (0, \infty)$

use graphing calculator to find max and min needed for the range

F) $f_{(x)} = \frac{x+2}{x^2-9}$

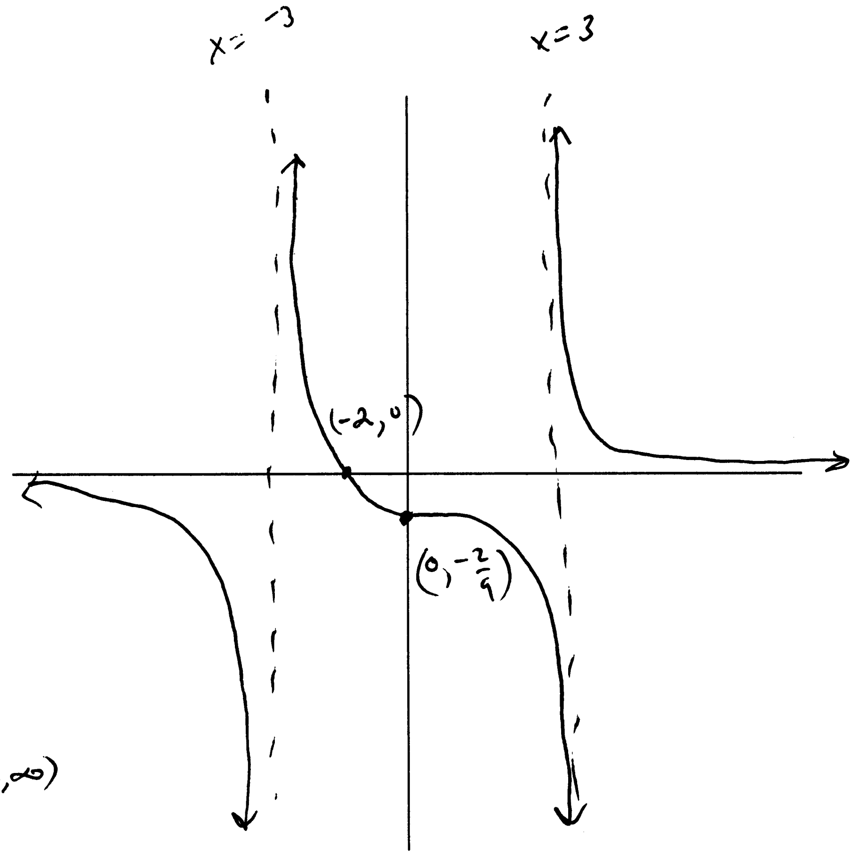
VA	HA
$x^2-9=0$	$y=0$
$x^2=9$	
$x=\pm 3$	

$x=-3, x=3$

x int	y int
$x+2=0$	$f(0)=\frac{2}{-9}$
$x=-2$	$(0,-\frac{2}{9})$
$(-2,0)$	

Range: $(-\infty, \infty)$

Domain: $(-\infty,-3)\cup(-3,3)\cup(3,\infty)$



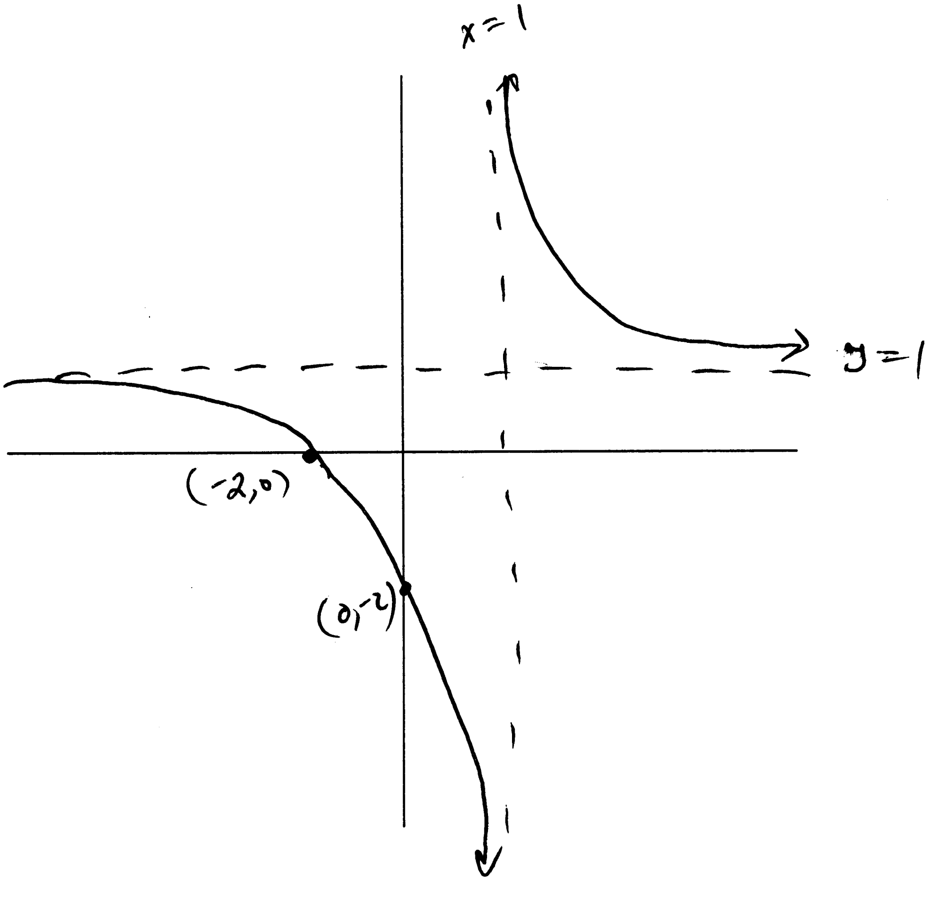
G) $f_{(x)} = \frac{x+2}{x-1}$

VA	HA
$x-1=0$	$y=1$
$x=1$	

x int	y int
$x+2=0$	$f(0)=\frac{2}{-1}$
$x=-2$	$(0,-2)$
$(-2,0)$	

Range : $(-\infty,1)\cup(1,\infty)$

Domain: $(-\infty,1)\cup(1,\infty)$



H) $f_{(x)} = \frac{2x^2}{x^2-4}$

VA

$x^2-4=0$

$(x+2)(x-2)=0$

$x=-2,\ x=2$

HA

$y=2$

x int

$2x^2=0$

$x^2=0$

$x=0$

$(0,0)$

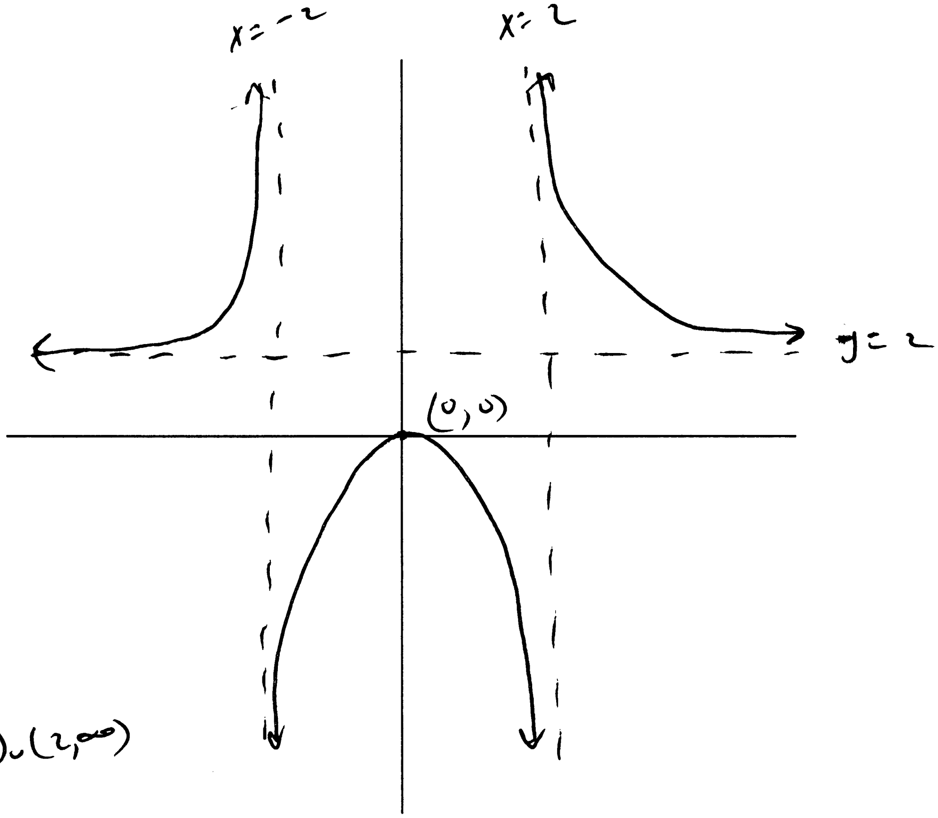
y int

$f(0)=\frac{0}{-4}$

$(0,0)$

Range: $(-\infty,0]\cup(2,\infty)$

Domain: $(-\infty,-2)\cup(-2,2)\cup(2,\infty)$



I) $f_{(x)} = -\frac{x^3}{x^2-9}$

VA

$x^2-9=0$

$(x+3)(x-3)=0$

$x=-3 \quad x=3$

OA

$$\begin{array}{r} -x \\ x^2-9\ \overline{)\,-x^3+0x^2+0x} \\ +x^3 \qquad \mp 9x \\ \hline -9x \end{array}$$

$y=-x$

x int

$x^3=0$

$(0,0)$

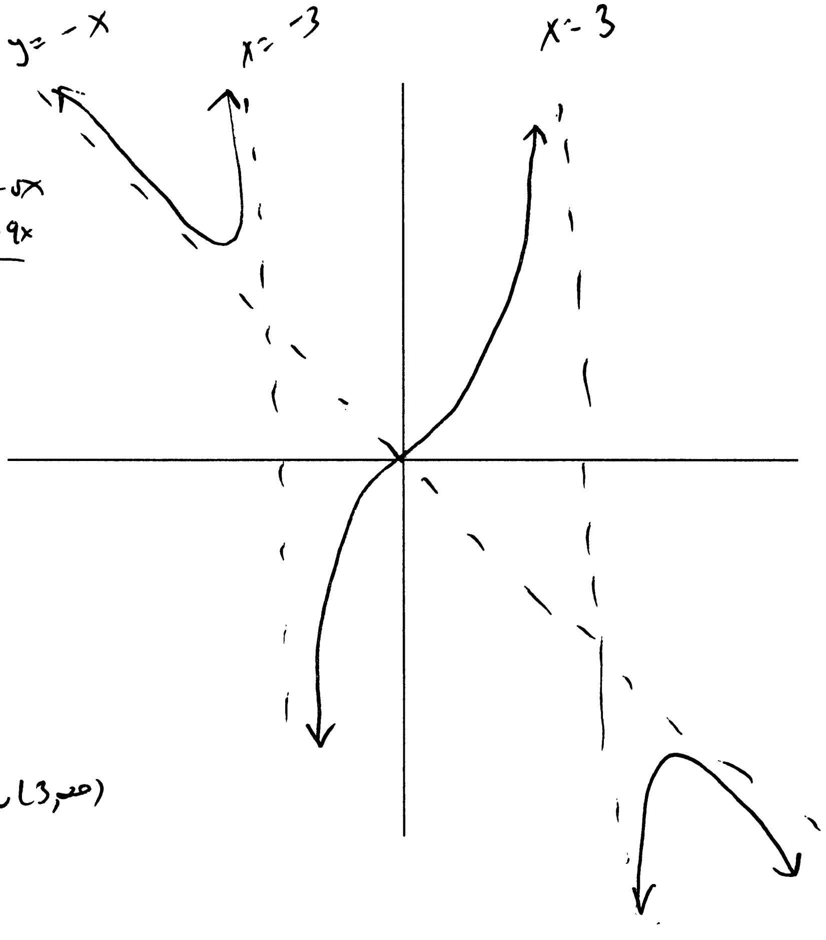
y int

$f(0)=-\frac{0}{-9}$

$(0,0)$

Range: $(-\infty,\infty)$

Domain: $(-\infty,-3)\cup(-3,3)\cup(3,\infty)$



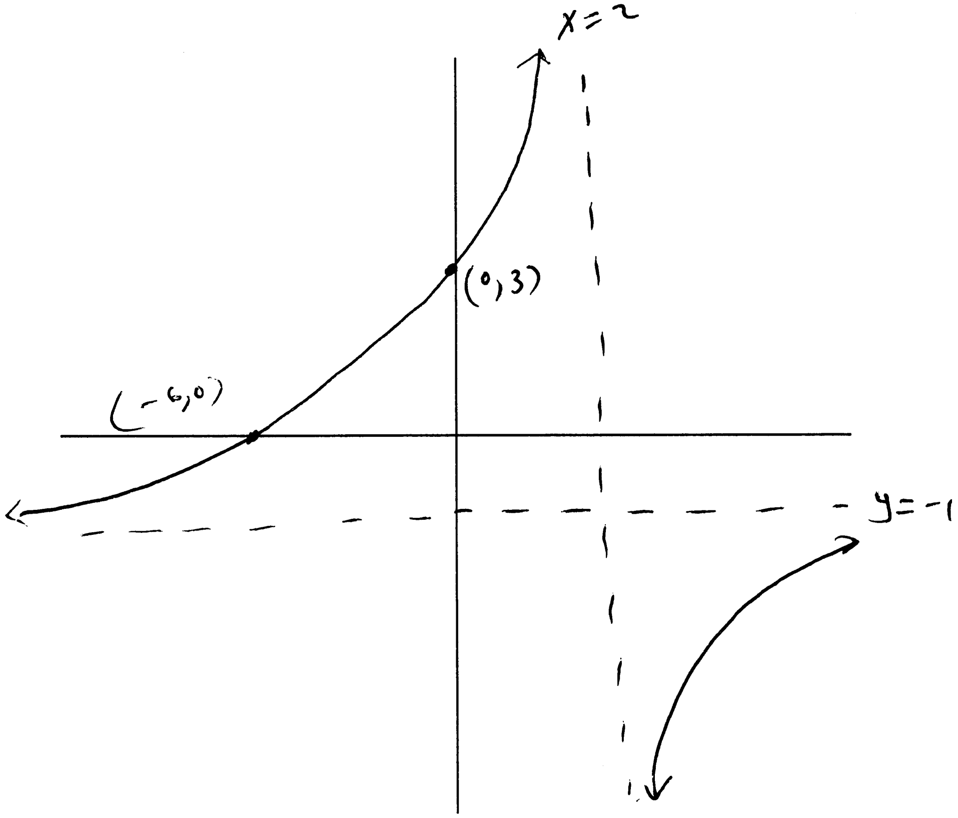
J) $f_{(x)} = -\frac{x+6}{x-2}$

VA	HA
$x-2=0$	$y= -\frac{1}{1}$
$x=2$	$y=-1$

x int	y int
$x+6=0$	$f(0) = -\left(\frac{6}{-2}\right)$
$x=-6$	$f(0) = 3$
$(-6,0)$	$(0,3)$

Range: $(-\infty, -1)\cup(-1,\infty)$

Domain: $(-\infty, 2)\cup(2,\infty)$



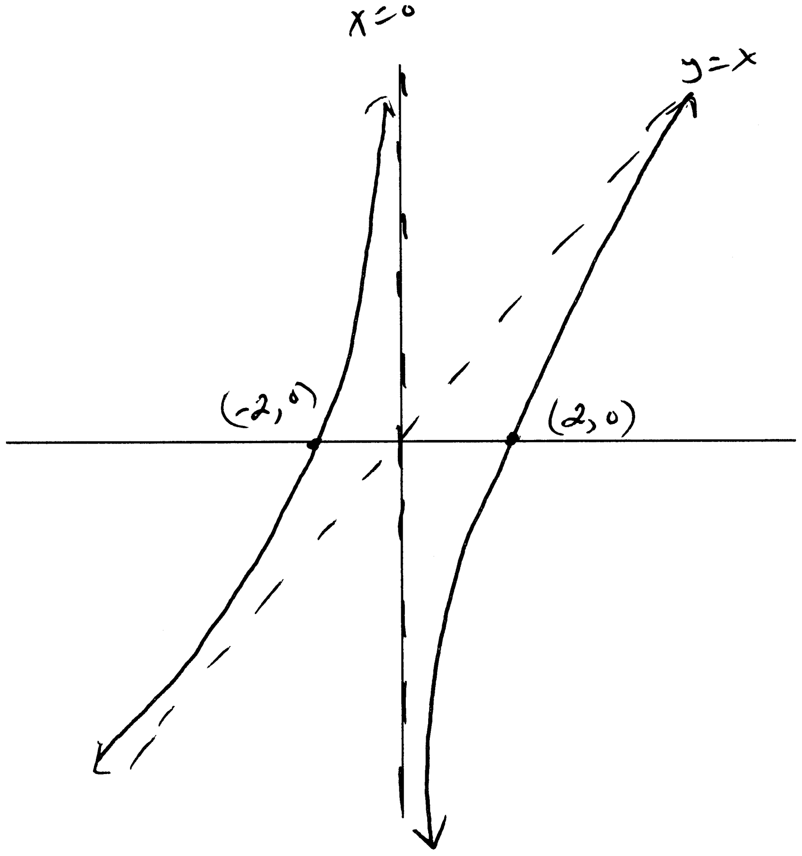
K) $f_{(x)} = \frac{x^2-4}{x}$

VA	OA
$x=0$	$x \overline{)\,x^2+0x-4}$ with quotient x
	$-x^2$
	$0 \quad -4$
	$y=x$

x int	y int
$x^2-4=0$	$f(0)= \frac{-4}{0}$
$(x+2)(x-2)=0$	undefined
$x=-2 \quad x=2$	none
$(-2,0) \quad (2,0)$	

Range: $(-\infty,\infty)$

Domain: $(-\infty, 0)\cup(0,\infty)$



L) $f_{(x)} = \frac{1}{x+1} + 1$

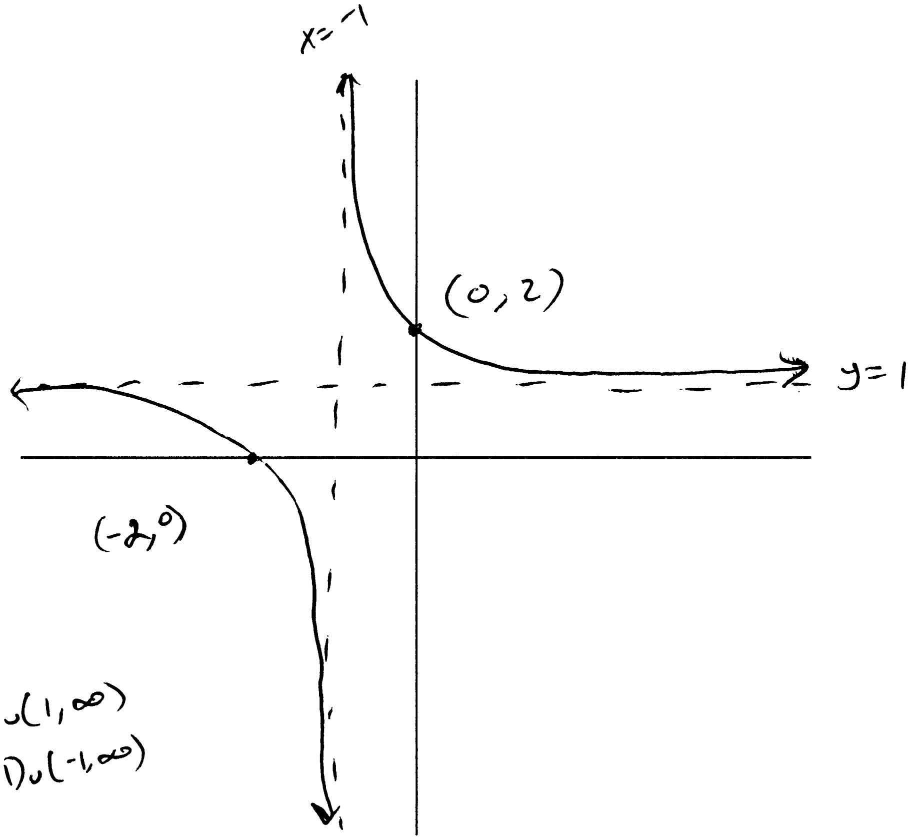
$\frac{1}{x+1} + \frac{x+1}{x+1}$

$f(x) = \frac{x+2}{x+1}$

VA	HA
$x+1=0$	$y=1$
$x=-1$	

x int	y int
$x+2=0$	$f(0)=\frac{2}{1}$
$x=-2$	$(0,2)$
$(-2,0)$	

Range: $(-\infty,1)\cup(1,\infty)$
Domain: $(-\infty,-1)\cup(-1,\infty)$



M) $f_{(x)} = \frac{1}{x+2} + 2\left(\frac{x+2}{x+2}\right)$

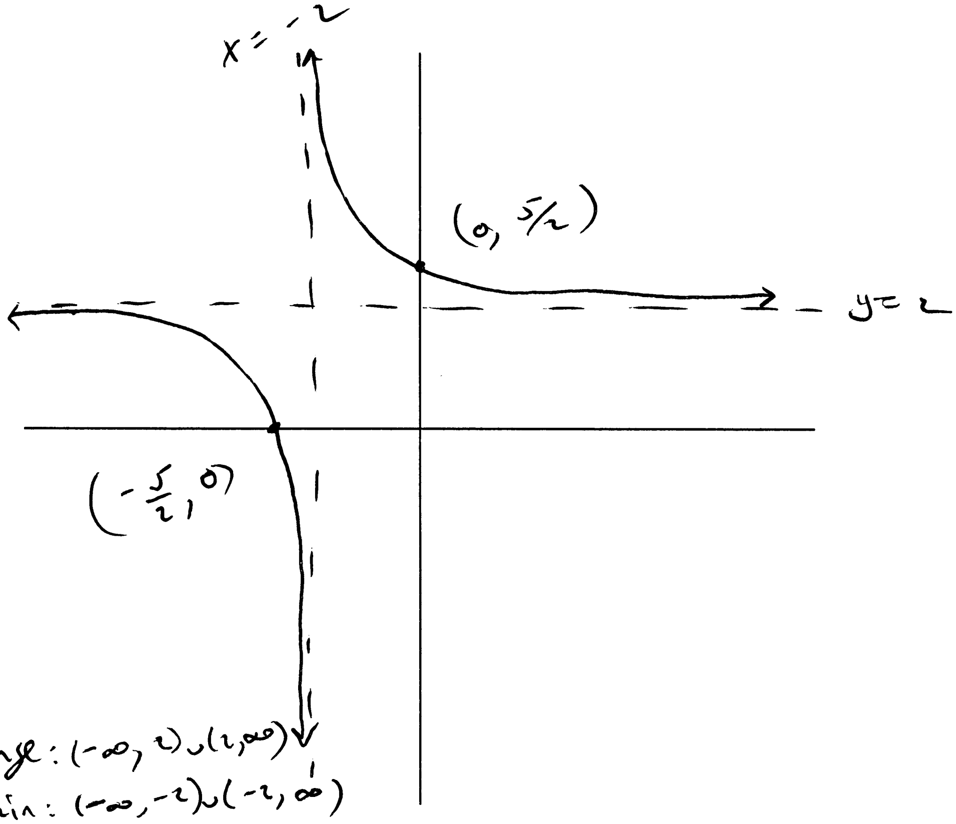
$\frac{1}{x+2} + \frac{2x+4}{x+2}$

$f(x) = \frac{2x+5}{x+2}$

VA	HA
$x+2=0$	$y=\frac{2}{1}$
$x=-2$	$y=2$

x int	y int
$2x+5=0$	$f(0)=\frac{5}{2}$
$x=-5/2$	$(0,5/2)$
$(-5/2,0)$	

Range: $(-\infty,2)\cup(2,\infty)$
Domain: $(-\infty,-2)\cup(-2,\infty)$



N) $f_{(x)} = \frac{1}{(x-2)^2}$

VA

$(x-2)^2=0$

$x-2=0$

$x=2$

HA

$y=0$

x int

$1 \neq 0$

none

y int

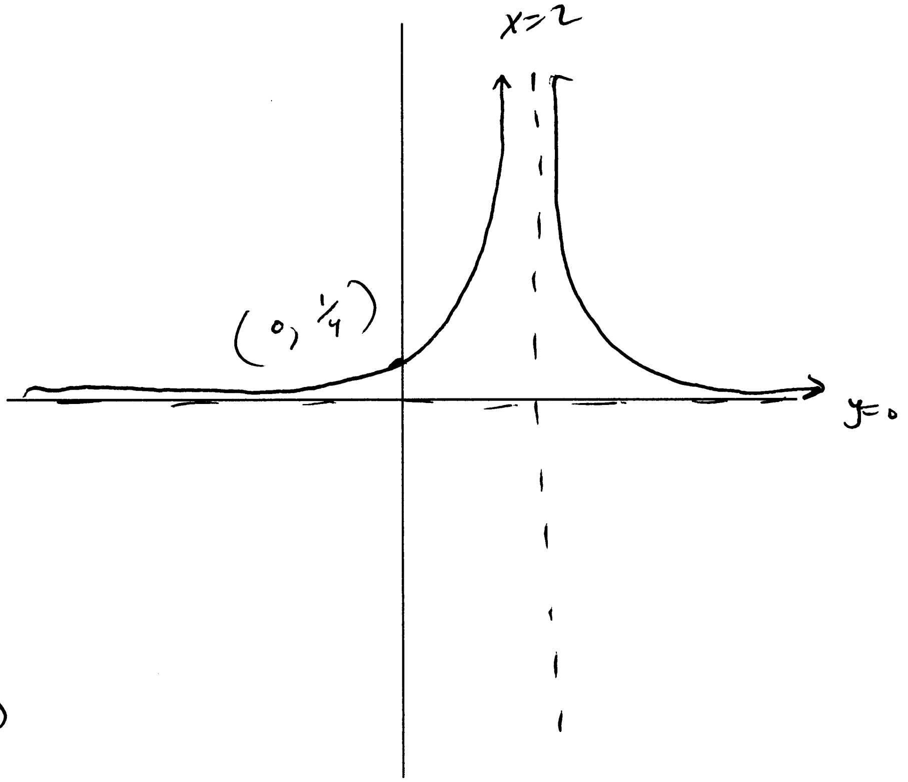
$f(0)=\frac{1}{(-2)^2}$

$f(0)=\frac{1}{4}$

$(0,\frac{1}{4})$

Range: $(0,\infty)$

Domain: $(-\infty,2)\cup(2,\infty)$



O) $f_{(x)} = \frac{1}{\sqrt{x+2}}$

VA

$x+2 \neq 0$

$x=-2$

HA

$y=0$

x int

$1 \neq 0$

none

y int

$f(0)=\frac{1}{\sqrt{2}}$

$f(0)=\frac{\sqrt{2}}{2}$

$(0,\frac{\sqrt{2}}{2})$

Range: $(0,\infty)$

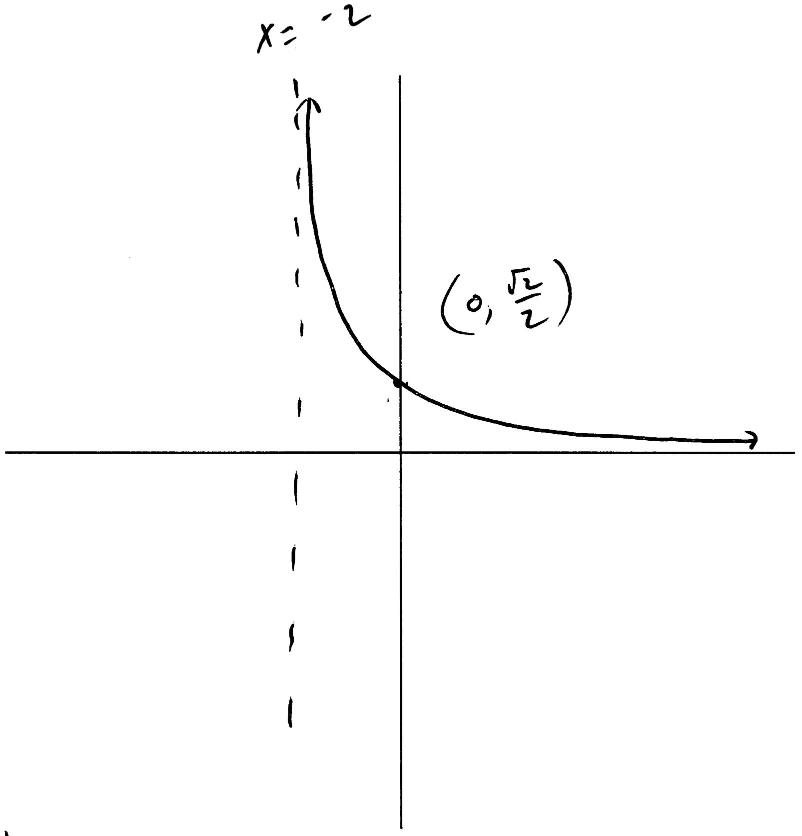
Domain: $(-2,\infty)$

↖

$\sqrt{x+2}$ $(x+2)$ must be greater than zero

$x+2>0$

$x>-2$



Checking Progress

You have now completed the "Rational Functions" section of the workbook. The following is a checklist so that you may check your progress. Check off each of the objectives you have accomplished.

The student should be able to…

- ☐ ***Find the vertical asymptotes of a rational function.***
- ☐ ***Determine if a rational function has a horizontal or oblique asymptote.***
- ☐ ***Find the horizontal asymptote of a rational function if it exists.***
- ☐ ***Find the oblique asymptote of a rational function if it exists.***
- ☐ ***Find the domain of a rational function.***
- ☐ ***Find the x intercepts of a rational function.***
- ☐ ***Find the y intercept of a rational function.***
- ☐ ***Graph a rational function.***

INTRODUCTION TO TRIGONOMETRY

Objectives

The following is a list of objectives for this section of the workbook.

By the time the student is finished with this section of the workbook, he/she should be able to…

- ***Give a graphical representation of any angle.***
- ***Find positive and negative coterminal angles.***
- ***Convert an angle measured in degrees to radians.***
- ***Convert an angle measured in radians to degrees.***
- ***Evaluate the basic trigonometric functions.***
- ***Use reference angles to evaluate the basic trigonometric functions.***
- ***Construct a unit circle.***
- ***Use the unit circle to evaluate basic trigonometric functions.***
- ***Use the unit circle to solve trigonometric equations.***

Math Standards Addressed

The following state standards are addressed in this section of the workbook.

Trigonometry

1.0 Students understand the notion of angle and how to measure it, in both degrees and radians. They can convert between degrees and radians.

2.0 Students know the definition of sine and cosine as *y*- and *x*- coordinates of points on the unit circle and are familiar with the graphs of the sine and cosine functions.

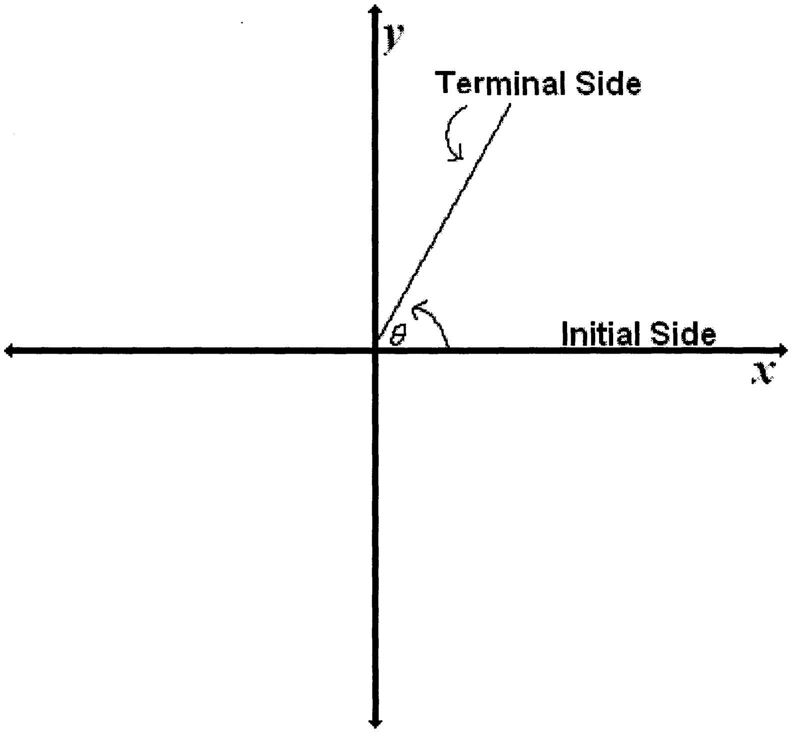
3.0 Students know the identity $\cos^2(x) + \sin^2(x) = 1$:

> 3.1 Students prove that this identity is equivalent to the Pythagorean theorem (i.e., students can prove this identity by using the Pythagorean theorem and, conversely, they can prove the Pythagorean theorem as a consequence of this identity).
>
> 3.2 Students prove other trigonometric identities and simplify others by using the identity $\cos^2(x) + \sin^2(x) = 1$. For example, students use this identity to prove that $\sec^2(x) = \tan^2(x) + 1$.

Angles

In trigonometry, we study angles and triangles. Before discussing angles, however, there are a few vocabulary terms that will be necessary. Each angle has an initial side and a terminal side. It will help to think of an angle in the following manner.

Begin by picturing a standard Cartesian Plane with two rays resting on the positive side of the x axis. As one of the sides moves in a counterclockwise direction, the other stays put. As the ray moves, an angle is being created at their vertex. The line segment that remains on the positive side of the x axis is called the Initial Side of the angle. The line segment that is moving is known as the Terminal Side of the angle.



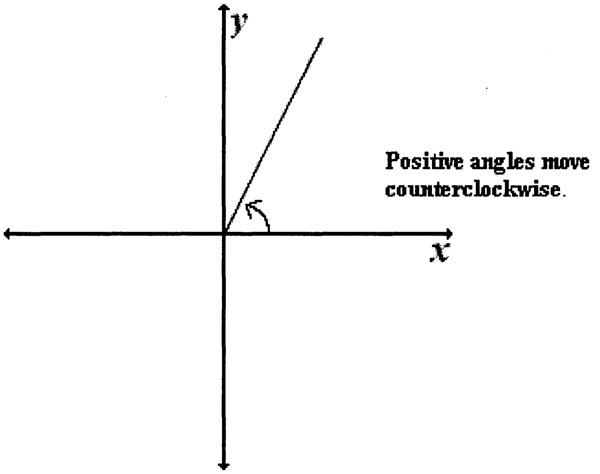
Notice the symbol used in the picture above. That symbol (θ) is the Greek letter theta. In trigonometry, Greek letters are often used to represent angles.

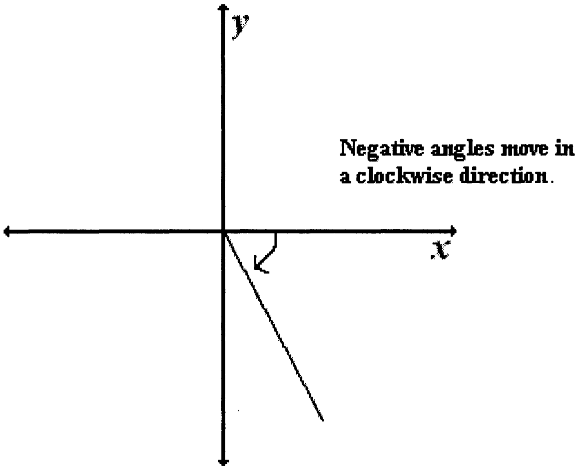
There are also some basic geometric terms that will be used in the study of trigonometry.

Recall that an Acute Angle is an angle that is less than 90 degrees, while an Obtuse Angle is an angle whose measure is between 90 and 180 degrees.

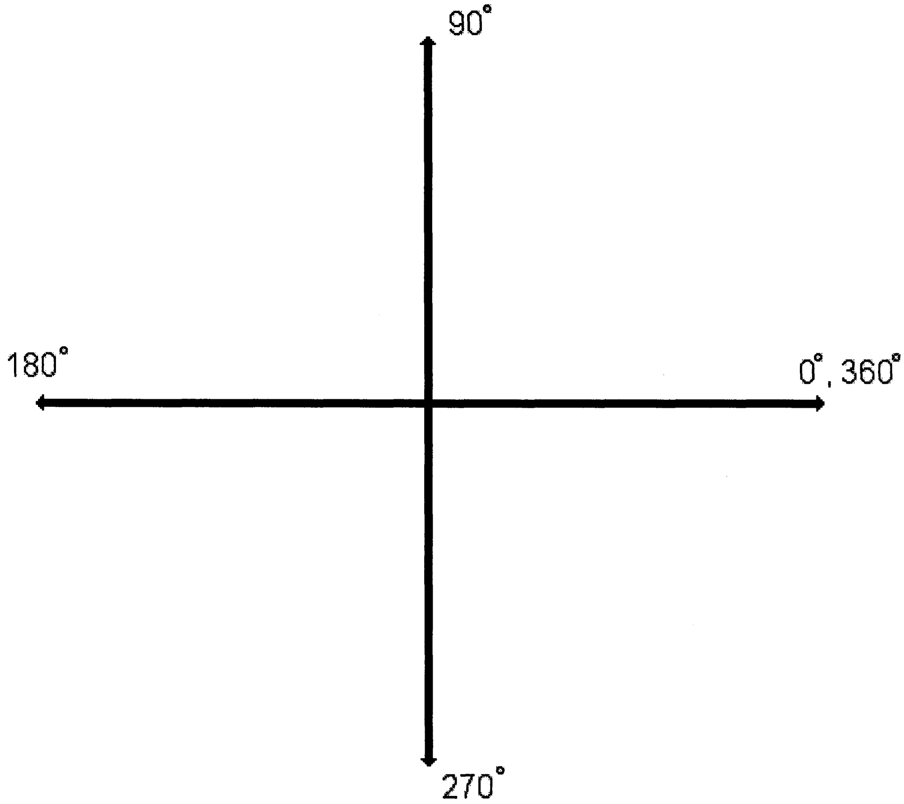
Supplementary Angles are two angles whose sum is 180 degree. ComplimentaryAngles are angles whose sum is 90 degrees.
In trigonometry, a plane is divided into four quadrants.

An angle whose initial side is on the positive side of the x axis is said to be in <u>Standard Position</u>. An angle is positive if the terminal side is moving in a counterclockwise direction. An angle is negative if the terminal side is moving in a clockwise direction.





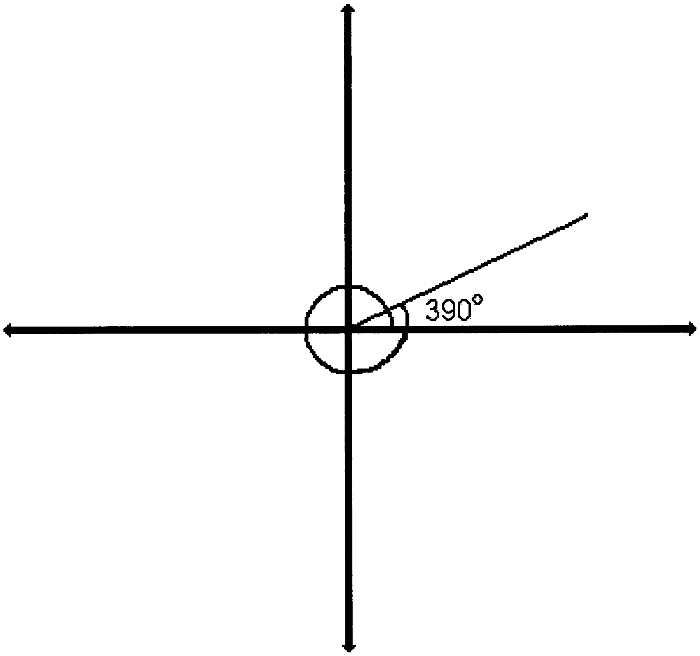
In trigonometry, a plane is divided into four quadrants.



According to the diagram above, the terminal side of a 20° angle would reside in quadrant I. However, an angle that measures 380° would also share the same terminal side. The only difference being, the terminal side of the 380° angle makes a complete revolution before finally coming to a stop.

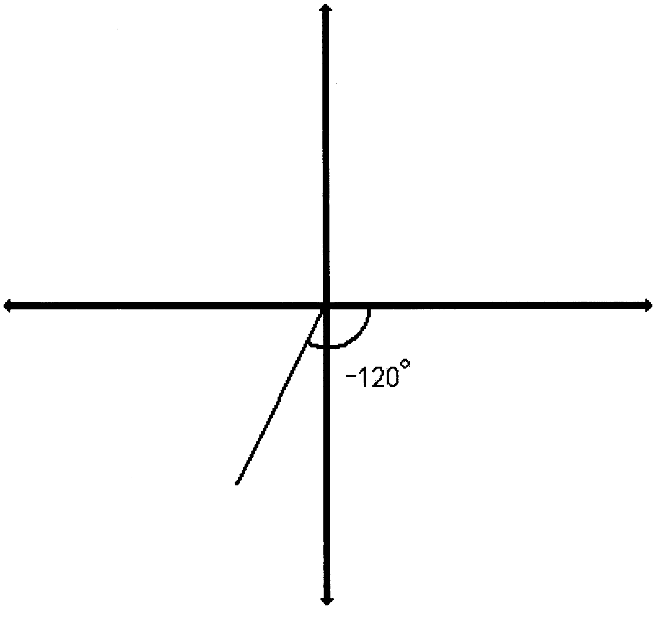
Here are a couple of examples of how to give a graphical representation of an angle.

Give a graphical representation of an angle that measures 390°.



In the above example, a 390° angle moves in a counterclockwise direction, and makes one complete revolution where the terminal side ends up in quadrant I.

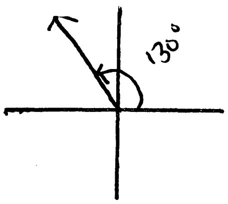
Give a graphical representation of an angle that measures -120°.



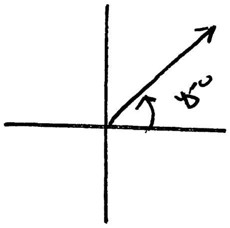
In the above example, a -120° moves in a clockwise direction, and the terminal side resides in quadrant III.

Give a graphical representation of each of the following angles.

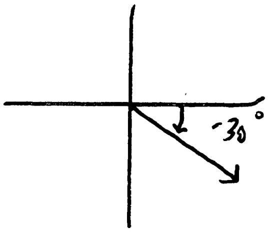
A) 130°



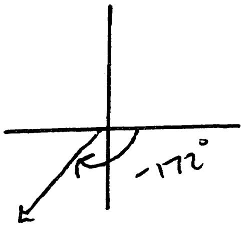
B) 45°



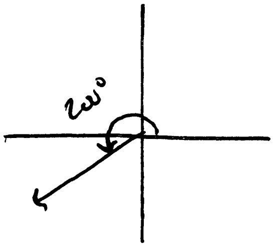
C) –30°



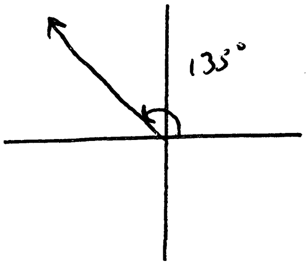
D) –172°



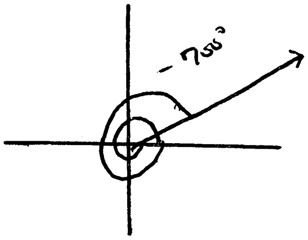
E) 200°



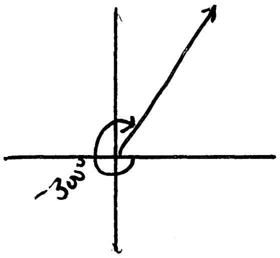
F) 135°



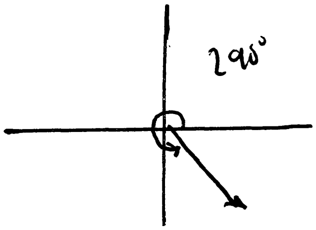
G) –700°



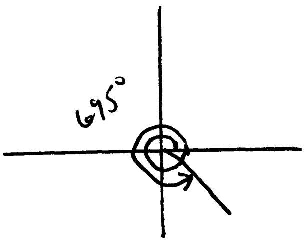
H) –300°



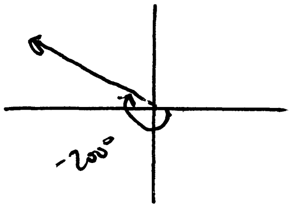
I) 290°



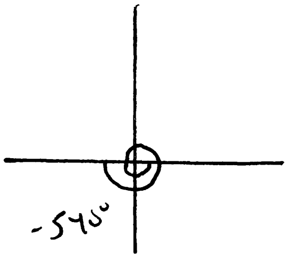
J) 695°



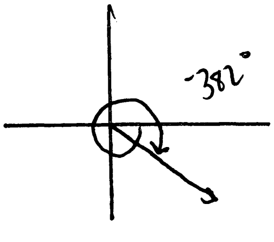
K) –200°



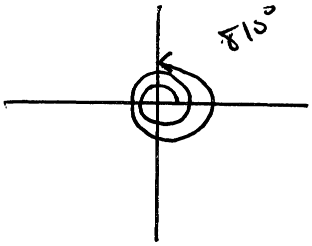
L) –540°



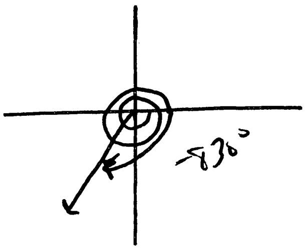
M) –382°



N) 810°



O) –830°



Determine the quadrant in which the terminal side of each of the following angles resides.

A) 172°

II

B) −315°

I

C) 718°

IV

D) 415°

I

E) −63°

IV

F) 135°

II

G) −700°

I

H) 1020°

IV

I) −284°

I

J) 615°

III

K) −200°

II

L) −540°

none

M) −450°

none

N) −700°

I

O) 840°

II

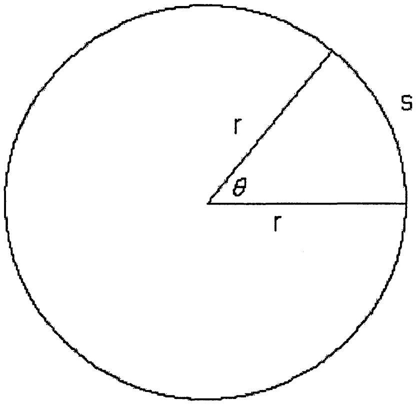
Coterminal Angles

***Coterminal Angles** are angles who share the same initial side and terminal sides. Finding coterminal angles is as simple as adding or subtracting $360°$ or 2π to each angle, depending on whether the given angle is in degrees or **radians**. There are an infinite number of coterminal angles that can be found. Following this procedure, all coterminal angles can be found. This is the basis for solving trigonometric equations which will be done in the future.*

Radians are often used in trigonometry to represent angle measures. Radian measures are very common in calculus, so it is important to have an understanding of what a radian is.

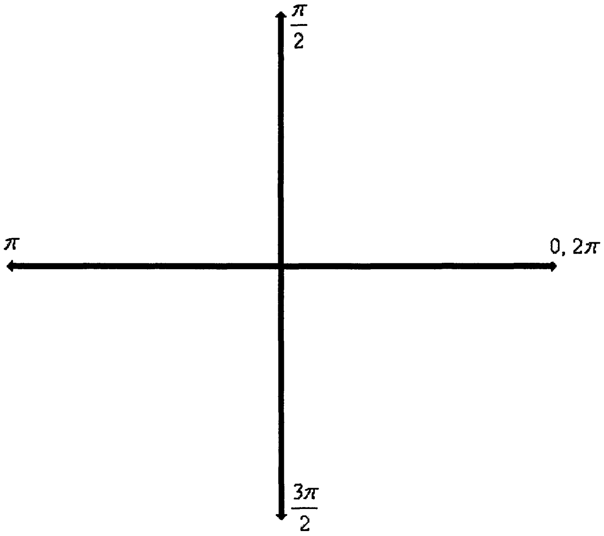
Definition of a Radian

A radian is the measure of a central angle θ that intercepts an arc s equal in length to the radius r of the circle. There are 2π, or approximately 6.28318, radians in a complete circle. Thus, one radian is about 57.296 angular degrees.



In other words, if we were to take the length of the radius of a circle, and lay in on the edge of a circle, that length would be one radian.

The number π is often used when describing radian measure. The approximate value of π is 3.14159... A plane, in trigonometry, can not only be divided into quadrants using degree measures, but radian as well. Observe the following moving in a counterclockwise direction.



When studying trigonometry, angles are usually measured in radians.

In relation to degrees, 180° is π radians. This means 2π radians is 360°. Since the approximate value of π is 3.14159..., it follows that 360° is approximately 6.28318...radians. When evaluating angles in trigonometry or calculus, always be aware of whether the question is given in terms of degrees or radians. If no degree symbol is given, the problem is in radians.

Examples of finding coterminal angles

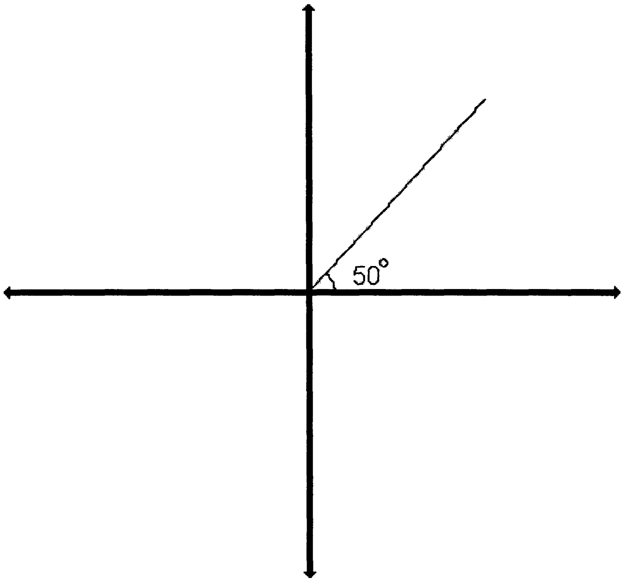
Find one positive angle that is coterminal to 50°.

Since the terminal side of a 50° angle resides in quadrant I, the terminal side of its coterminal angle must share that side. This means the new angle would make one complete revolution before having its terminal side come to rest at the same place.

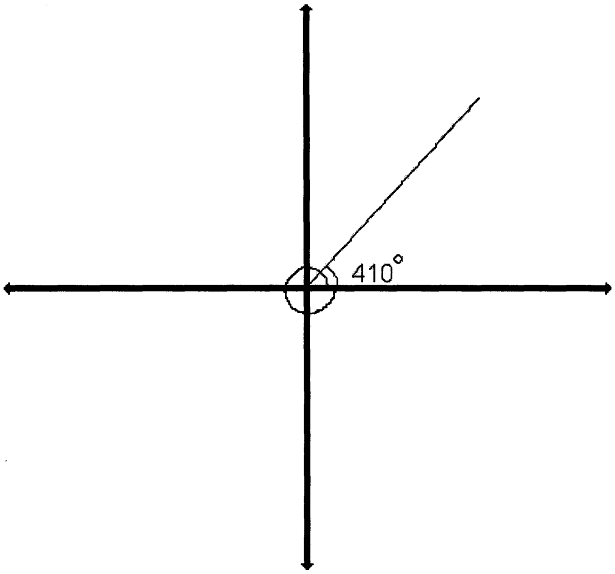
Therefore, to find the coterminal angle to a 50° angle, just add 360°.

$$50° + 360° = 410°$$

Below is the graphical representation of a 50° angle. Since its coterminal angle must share the same terminal side, it is reasonable to create a new angle that makes one complete revolution and ends up in the same place.



To find the coterminal angle of a 50° angle, add 360°. It would follow that 50°+360°=410°. A 410° angle is illustrated below. From the graphical representation of the angle, we can conclude that these two angle do indeed share the same terminal side, meaning they are coterminal.



Find one positive angle that is coterminal to 110°.

110° + 360° = 470°

Find two positive angles that are coterminal to -30°.

-30°+ 360° = 330°
330°+ 360° = 690°

In this case, the two positive coterminal angles to -30° are 330° and 690°.

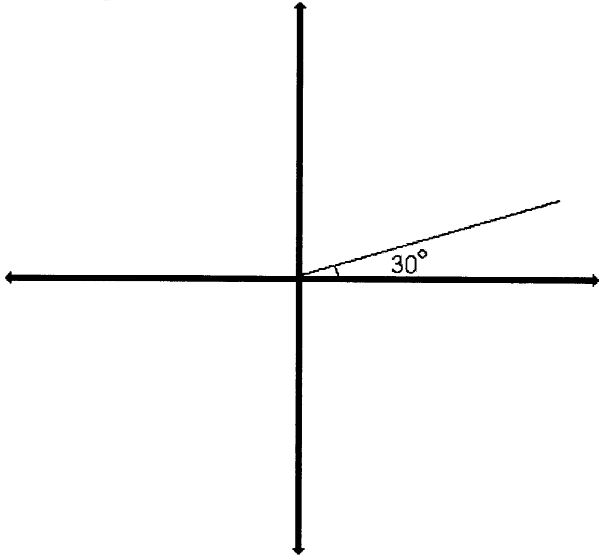
If more than one positive coterminal angle needs to be found, simply add another 360°. This would essentially make the new angle complete two full revolutions before its terminal side comes to rest.

Find one negative angle that is coterminal to 30°.

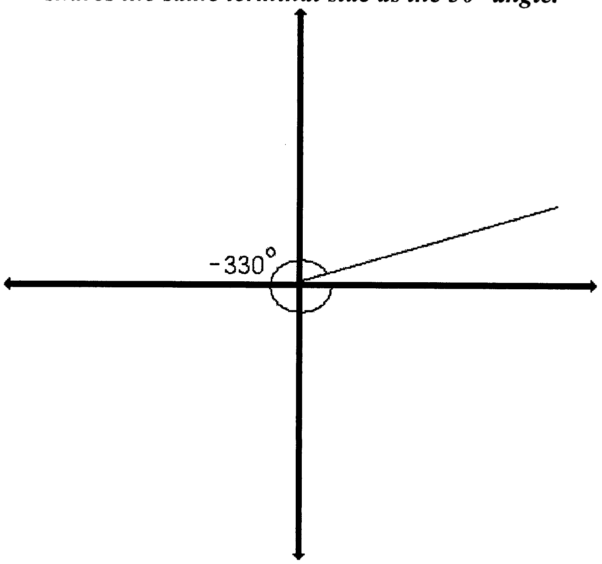
A negative angle moves in a clockwise direction. In this case, to find the negative coterminal angle, subtract 360° from 30°.

30° - 360° = -330°

Below is a 30° angle in standard position. This angle opens in a counterclockwise direction.



Here is a -330° angle. As the angle opens clockwise, it shares the same terminal side as the 30° angle.



Find one negative angle that is coterminal to 150°.

$$150° - 360° = -210°$$

Find one negative angle that is coterminal to 415°.

$$415° - 360° = 55°$$

Although 55° is a coterminal angle to 415°, this is not a solution to the problem. The problem specifically asked for a negative angle, so the process needs to take place one more time.

$$55° - 360° = -305°$$

These were all examples of finding coterminal angles. If the initial angle is given in the form or radians, add or subtract 2π instead of 360°.

Find a positive and negative angle that is coterminal to an angle that is $\frac{\pi}{6}$ radians.

$$\frac{\pi}{6}+2\pi$$

$$\frac{\pi}{6}+\frac{12\pi}{6}$$

$$\frac{13\pi}{6}$$

Adding 2π to the original angle yields the positive coterminal angle.

$$\frac{\pi}{6}-2\pi$$

$$\frac{\pi}{6}-\frac{12\pi}{6}$$

$$-\frac{11\pi}{6}$$

By subtracting 2π from the original angle, the negative coterminal angle has been found.

Find two positive angles that are coterminal to an angle that is $\frac{11\pi}{2}$ radians.

$$\frac{11\pi}{2}-2\pi$$

$$\frac{11\pi}{2}-\frac{4\pi}{2}$$

$$\frac{7\pi}{2}$$

$$\frac{7\pi}{2}-2\pi$$

$$\frac{7\pi}{2}-\frac{4\pi}{2}$$

$$\frac{3\pi}{2}$$

Since $\frac{11\pi}{2}$ is more than one complete revolution, 2π was subtracted from the initial angle yielding a coterminal angle of $\frac{7\pi}{2}$. This is still at least one full revolution, so 2π was subtracted yet again. This process resulted in the two positive coterminal angles of $\frac{7\pi}{2}$ and $\frac{3\pi}{2}$.

Find one positive and one negative coterminal angle of each of the following. **There is no need to graph the angles.**

A) 30°

$30° + 360°$ $30° - 360°$

$\boxed{390°, -330°}$

B) -40°

$-40° + 360°, -40° - 360°$

$\boxed{320°, -400°}$

C) 150°

$150° + 360°, 150° - 360°$

$\boxed{510°, -210°}$

D) 220°

$220° + 360°, 220° - 360°$

$\boxed{580°, -140°}$

E) -330°

$-330° + 360°, -330° - 360°$

$\boxed{30°, -690°}$

F) $\frac{\pi}{3}$

$\frac{\pi}{3} + 2\pi, \frac{\pi}{3} - 2\pi$

$\frac{\pi}{3} + \frac{6\pi}{3}, \frac{\pi}{3} - \frac{6\pi}{3}$

$\boxed{\frac{7\pi}{3}, -\frac{5\pi}{3}}$

G) $\frac{5\pi}{2}$

larger than 2π

$\frac{5\pi}{2} - 2\pi, \frac{5\pi}{2} - 4\pi$ ✓

$\frac{5\pi}{2} - \frac{4\pi}{2}$ $\frac{\pi}{2} - 2\pi$

$\frac{\pi}{2} - \frac{4\pi}{2}$

$\boxed{\frac{\pi}{2}, -\frac{3\pi}{2}}$

H) $-\frac{2\pi}{3}$

$-\frac{2\pi}{3} + 2\pi, -\frac{2\pi}{3} - 2\pi$

$-\frac{2\pi}{3} + \frac{6\pi}{3}, -\frac{2\pi}{3} - \frac{6\pi}{3}$

$\boxed{\frac{4\pi}{3}, -\frac{8\pi}{3}}$

I) $-\frac{5\pi}{6}$

$-\frac{5\pi}{6} + 2\pi, -\frac{5\pi}{6} - 2\pi$

$-\frac{5\pi}{6} + \frac{12\pi}{6}, -\frac{5\pi}{6} - \frac{12\pi}{6}$

$\boxed{\frac{7\pi}{6}, -\frac{17\pi}{6}}$

J) $\frac{5\pi}{3}$

$\frac{5\pi}{3} + 2\pi, \frac{5\pi}{3} - 2\pi$

$\frac{5\pi}{3} + \frac{6\pi}{3}, \frac{5\pi}{3} - \frac{6\pi}{3}$

$\boxed{\frac{11\pi}{3}, -\frac{\pi}{3}}$

K) $-\frac{4\pi}{3}$

$-\frac{4\pi}{3} + 2\pi, -\frac{4\pi}{3} - 2\pi$

$-\frac{4\pi}{3} + \frac{6\pi}{3}, -\frac{4\pi}{3} - \frac{6\pi}{3}$

$\boxed{\frac{2\pi}{3}, -\frac{10\pi}{3}}$

L) 300°

$300° + 360°, 300° - 360°$

$\boxed{660°, -60°}$

M) 700°

$700° - 360°, 700° - 360°$ ✓

$340° - 360°$

$\boxed{340°, -20°}$

N) $-\frac{17\pi}{6}$

smaller than -2π

$-\frac{17\pi}{6} + 2\pi, -\frac{5\pi}{6} + 2\pi$

$-\frac{17\pi}{6} + \frac{12\pi}{6}$ $-\frac{5\pi}{6} + \frac{12\pi}{6}$

$\boxed{-\frac{5\pi}{6}, \frac{7\pi}{6}}$

O) $\frac{7\pi}{3}$ larger than 2π

$\frac{7\pi}{3} - 2\pi, \frac{\pi}{3} - 2\pi$

$\frac{7\pi}{3} - \frac{6\pi}{3}, \frac{\pi}{3} - \frac{6\pi}{3}$

$\boxed{\frac{\pi}{3}, -\frac{5\pi}{3}}$

P) –410°

$-410° + 360°, -50° + 360°$

$\boxed{-50°, 310°}$

Q) 1000°

$1000° - 360°, 640° - 360°$ ✓

$280° - 360°$

$\boxed{640°, -80°}$

R) $\frac{31\pi}{6}$

larger than 2π

$\frac{31\pi}{6} - 2\pi, \frac{19\pi}{6} - \frac{12\pi}{6}$ ✓

$\frac{31\pi}{6} - \frac{12\pi}{6}, \frac{7\pi}{6} - \frac{12\pi}{6}$

$\boxed{\frac{19\pi}{6}, -\frac{5\pi}{6}}$

S) $-\frac{15\pi}{4}$

smaller than -2π

$-\frac{15\pi}{4} + 2\pi, -\frac{7\pi}{4} + 2\pi$

$-\frac{15\pi}{4} + \frac{8\pi}{4}, -\frac{7\pi}{4} + \frac{8\pi}{4}$

$\boxed{-\frac{7\pi}{4}, \frac{\pi}{4}}$

T) $\frac{5\pi}{6}$

$\frac{5\pi}{6} + 2\pi, \frac{5\pi}{6} - 2\pi$

$\frac{5\pi}{6} + \frac{12\pi}{6}, \frac{5\pi}{6} - \frac{12\pi}{6}$

$\boxed{\frac{17\pi}{6}, -\frac{7\pi}{6}}$

Conversions

Since both degrees and radian measures will be dealt with in trigonometry, it will sometimes be necessary to convert degrees to radians, or radians to degrees.

The following formulas are used for such conversions.

Degrees to Radians	***Radians to Degrees***
Degrees × $\dfrac{\pi}{180°}$	Radians × $\dfrac{180°}{\pi}$

Example

Convert 80° to radians.

$$80° \cdot \frac{\pi}{180°}$$

$$\frac{80\pi}{180}$$

$$\frac{4\pi}{9}$$

Once the final product is reduced, it is evident that 80° is equal to $\frac{4\pi}{9}$ ***radians.***

Convert $-\frac{3\pi}{4}$ ***radians to degrees.***

$$-\frac{3\pi}{4} \cdot \frac{180°}{\pi}$$

$$-\frac{540°}{4}$$

$$-135°$$

Notice the negative sign is kept in the problem. Once the final product is reduced, it can be concluded that $-\frac{3\pi}{4}$ ***radians is equal to -135°***

Convert 3 radians to degrees.

$$3 \cdot \frac{180°}{\pi} = \frac{540°}{\pi} \approx 171.887°$$

In this example, there is no degree symbol next to the 3. This means we are looking at 3 radians. Once 3 is multiplied by 180°, divide the result by π. ***This will result in a decimal estimate of the measure of the angle.***

Why are these conversion formulas necessary?

There are certain formulas used in trigonometry such as the, arc-length formula, where the angle used in the calculations must be in radians. These conversion formulas will allow this to be done. Also, it is sometimes difficult to tell in which quadrant the terminal side of an angle lies when it is written in radians. Converting from radians to degrees will make this process easier.

Convert each of the following to Radians.

A) 120°

$$120° \times \frac{\pi}{180°}$$
$$\frac{12\pi}{18}$$
$$\boxed{\frac{2\pi}{3}}$$

B) 210°

$$210° \cdot \frac{\pi}{180°}$$
$$\frac{21\pi}{18}$$
$$\boxed{\frac{7\pi}{6}}$$

C) -60°

$$-60° \cdot \frac{\pi}{180°}$$
$$\frac{-6\pi}{18}$$
$$\boxed{-\frac{\pi}{3}}$$

D) 420°

$$420° \cdot \frac{\pi}{180°}$$
$$\frac{42\pi}{18}$$
$$\boxed{\frac{7\pi}{3}}$$

E) −110°

$$-110° \cdot \frac{\pi}{180°}$$
$$\boxed{\frac{-11\pi}{18}}$$

F) 330°

$$330° \cdot \frac{\pi}{180°}$$
$$\frac{33\pi}{18}$$
$$\boxed{\frac{11\pi}{6}}$$

G) −45°

$$-45° \cdot \frac{\pi}{180°}$$
$$-\frac{45\pi}{180}$$
$$\boxed{-\frac{\pi}{4}}$$

H) 150°

$$150° \cdot \frac{\pi}{180°}$$
$$\frac{15\pi}{18}$$
$$\boxed{\frac{5\pi}{6}}$$

I) 300°

$$300° \cdot \frac{\pi}{180°}$$
$$\frac{30\pi}{18}$$
$$\boxed{\frac{5\pi}{3}}$$

J) −135°

$$-135° \cdot \frac{\pi}{180}$$
$$-\frac{27\pi}{36}$$
$$\boxed{-\frac{3\pi}{4}}$$

K) 450°

$$450° \cdot \frac{\pi}{180°}$$
$$\frac{45\pi}{18}$$
$$\boxed{\frac{5\pi}{2}}$$

L) −210°

$$-210° \cdot \frac{\pi}{180°}$$
$$\frac{21\pi}{18}$$
$$\boxed{\frac{7\pi}{6}}$$

M) 720°

$$720° \cdot \frac{\pi}{180}$$
$$\frac{72\pi}{18}$$
$$\boxed{4\pi}$$

N) 315°

$$315° \cdot \frac{\pi}{180°}$$
$$\frac{63\pi}{36}$$
$$\boxed{\frac{7\pi}{4}}$$

O) −30°

$$-30° \cdot \frac{\pi}{180°}$$
$$\frac{-3\pi}{18}$$
$$\boxed{-\frac{\pi}{6}}$$

P) 60°

$$60° \cdot \frac{\pi}{180°}$$
$$\frac{6\pi}{18}$$
$$\boxed{\frac{\pi}{3}}$$

Q) −15°

$$-15° \cdot \frac{\pi}{180°}$$
$$-\frac{15\pi}{180}$$
$$\boxed{-\frac{\pi}{12}}$$

R) 45°

$$45° \cdot \frac{\pi}{180°}$$
$$\frac{45\pi}{180}$$
$$\boxed{\frac{\pi}{4}}$$

S) 225°

$$225° \cdot \frac{\pi}{180°}$$
$$\frac{225\pi}{180}$$
$$\frac{45\pi}{36}$$
$$\boxed{\frac{5\pi}{4}}$$

T) 360°

$$360° \cdot \frac{\pi}{180°}$$
$$\frac{36\pi}{18}$$
$$\boxed{2\pi}$$

Convert each of the following from Radians to Degrees.

A) $\frac{\pi}{6}$

$\frac{\pi}{6} \cdot \frac{180^\circ}{\pi}$

$\boxed{30^\circ}$

B) $\frac{5\pi}{3}$

$\frac{5\cancel{\pi}}{\cancel{3}} \cdot \frac{\overset{60^\circ}{\cancel{180^\circ}}}{\cancel{\pi}}$

$\boxed{300^\circ}$

C) $-\frac{\pi}{2}$

$\frac{-\pi}{2} \cdot \frac{180^\circ}{\pi}$

$\boxed{-90^\circ}$

D) $\frac{3\pi}{4}$

$\frac{3\pi}{4} \cdot \frac{180^\circ}{\pi}$

$\frac{3 \cdot \overset{45}{\cancel{180}}}{\cancel{4}}$ $\boxed{135^\circ}$

E) $-\frac{\pi}{4}$

$\frac{-\pi}{4} \cdot \frac{180^\circ}{\pi}$

$\frac{-180^\circ}{4}$

$\boxed{-45^\circ}$

F) $-\frac{5\pi}{6}$

$-\frac{5\pi}{6} \cdot \frac{180^\circ}{\pi}$

$-5 \cdot 30^\circ$

$\boxed{-150^\circ}$

G) $\frac{7\pi}{6}$

$\frac{7\pi}{6} \cdot \frac{180^\circ}{\pi}$

$7 \cdot 30^\circ$

$\boxed{210^\circ}$

H) $-\frac{\pi}{6}$

$-\frac{\pi}{6} , \frac{180^\circ}{\pi}$

$\boxed{-30^\circ}$

I) 2.3

$2.3 \cdot \frac{180^\circ}{\pi}$

$\boxed{\approx 131.78^\circ}$

J) $\frac{11\pi}{6}$

$\frac{11\pi}{6} \cdot \frac{180^\circ}{\pi}$

$11 \cdot 30^\circ$

$\boxed{330^\circ}$

K) −1.28

$-1.28 \cdot \frac{180^\circ}{\pi}$

$\boxed{\approx -73.34^\circ}$

L) $-\frac{2\pi}{3}$

$-\frac{2\pi}{3} \cdot \frac{180^\circ}{\pi}$

$-2 \cdot 60^\circ$

$\boxed{-120^\circ}$

M) $\frac{7\pi}{4}$

$\frac{7\pi}{4} \cdot \frac{180^\circ}{\pi}$

$7 \cdot 45^\circ$

$\boxed{315^\circ}$

N) $-\frac{4\pi}{3}$

$\frac{-4\pi}{3} \cdot \frac{180^\circ}{\pi}$

$-4 \cdot 60^\circ$

$\boxed{-240^\circ}$

O) $\frac{5\pi}{4}$

$\frac{5\pi}{4} \cdot \frac{180^\circ}{\pi}$

$5 \cdot 45^\circ$

$\boxed{225^\circ}$

P) π

$\pi \cdot \frac{180^\circ}{\pi}$

$\boxed{180^\circ}$

Q) $\frac{13\pi}{6}$

$\frac{13\pi}{6} \cdot \frac{180^\circ}{\pi}$

$13 \cdot 30^\circ$

$\boxed{390^\circ}$

R) 4.3

$4.3 \cdot \frac{180^\circ}{\pi}$

$\approx 246.37^\circ$

S) 2π

$2\pi \cdot \frac{180^\circ}{\pi}$

$2 \cdot 180^\circ$

$\boxed{360^\circ}$

T) $-\frac{11\pi}{3}$

$\frac{-11\pi}{3} \cdot \frac{180^\circ}{\pi}$

$-11 \cdot 60^\circ$

$\boxed{-660^\circ}$

U) $-\frac{9\pi}{4}$

$-\frac{9\pi}{4} \cdot \frac{180^\circ}{\pi}$

$-9 \cdot 45^\circ$

$\boxed{-405^\circ}$

V) $\frac{31\pi}{6}$

$\frac{31\pi}{6} \cdot \frac{180^\circ}{\pi}$

$31 \cdot 30^\circ$

$\boxed{930^\circ}$

W) $\frac{11\pi}{2}$

$\frac{11\pi}{2} \cdot \frac{180^\circ}{\pi}$

$11 \cdot 90$

$\boxed{490^\circ}$

X) $\frac{23\pi}{6}$

$\frac{23\pi}{6} \cdot \frac{180^\circ}{\pi}$

$23 \cdot 30^\circ$

690°

Determine the quadrant in which the terminal side of following angles resides.

A) $\frac{\pi}{6}$

$\frac{\pi}{6} = 30^\circ$

$\boxed{\text{I}}$

B) $\frac{5\pi}{3}$

$\frac{5\pi}{3} = 300^\circ$

$\boxed{\text{IV}}$

C) $-\frac{\pi}{2}$

$\frac{-\pi}{2} = -90^\circ$

$\boxed{\text{none}}$

D) $\frac{3\pi}{4}$

$\frac{3\pi}{4} = 135^\circ$

$\boxed{\text{II}}$

E) $-\frac{\pi}{4}$

$-\frac{\pi}{4} = -45^\circ = 315^\circ$

$\boxed{\text{IV}}$

F) $-\frac{5\pi}{6}$

$-\frac{5\pi}{6} = -150^\circ = 210^\circ$

$\boxed{\text{III}}$

G) $\frac{7\pi}{6}$

$\frac{7\pi}{6} = 210^\circ$

$\boxed{\text{III}}$

H) $-\frac{\pi}{6}$

$\frac{-\pi}{6} = -30^\circ = 330^\circ$

$\boxed{\text{IV}}$

I) 2.3

$2.3 \approx 131.78^\circ$

$\boxed{\text{II}}$

J) $\frac{11\pi}{6}$

$\frac{11\pi}{6} = 330^\circ$

$\boxed{\text{IV}}$

K) -1.28

$-1.28 \approx -73.34^\circ$

$\approx 286.66^\circ$

$\boxed{\text{IV}}$

L) $-\frac{2\pi}{3}$

$\frac{-2\pi}{3} = -120^\circ = 240^\circ$

$\boxed{\text{III}}$

M) $\frac{7\pi}{4}$

$\frac{7\pi}{4} = 315^\circ$

$\boxed{\text{IV}}$

N) $-\frac{4\pi}{3}$

$-\frac{4\pi}{3} = -240^\circ = 120^\circ$

$\boxed{\text{II}}$

O) $\frac{5\pi}{4}$

$\frac{5\pi}{4} = 225^\circ$

$\boxed{\text{III}}$

P) π

$\pi = 180^\circ$

$\boxed{\text{none}}$

Q) $\frac{13\pi}{6}$

$\frac{13\pi}{6} = \frac{12\pi}{6} + \frac{\pi}{6}$

$\downarrow$

$2\pi + \frac{\pi}{6}$

$\boxed{\text{I}}$

R) 4.3

$4.3 \approx 246.37^\circ$

$\boxed{\text{III}}$

S) 2π

$2\pi = 360^\circ$

$\boxed{\text{none}}$

T) $-\frac{11\pi}{3}$

$-\frac{11\pi}{3} = -\frac{6\pi}{3} - \frac{5\pi}{3}$

$-2\pi \quad -300^\circ$

$+360 = 60^\circ$

$\boxed{\text{I}}$

U) $-\frac{9\pi}{4}$

$-\frac{9\pi}{4} = -\frac{8\pi}{4} - \frac{\pi}{4}$

$-2\pi \quad -\frac{\pi}{4}$

-45°

$\boxed{\text{IV}}$

V) $\frac{31\pi}{6}$

$\frac{31\pi}{6} = \frac{24\pi}{6} + \frac{7\pi}{6}$

$\swarrow$

$4\pi + \frac{7\pi}{6}$

$\downarrow$

210°

$\boxed{\text{III}}$

W) $\frac{11\pi}{2}$

$\frac{11\pi}{2} = \frac{8\pi}{2} + \frac{3\pi}{2}$

$4\pi + \frac{3\pi}{2}$

$\swarrow$

270°

$\boxed{\text{none}}$

X) $\frac{23\pi}{6}$

$\frac{23\pi}{6} = \frac{12\pi}{6} + \frac{11\pi}{6}$

$2\pi + \frac{11\pi}{6}$

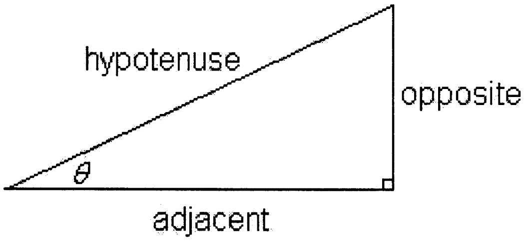
330°

$\boxed{\text{IV}}$

Basic Trigonometric Functions

There are six basic trigonometric functions used in trigonometry. Below are the names of the six functions and their three letter abbreviation.

Sine (sin)	**Cosecant (csc)**
Cosine(cos)	**Secant (sec)**
Tangent (tan)	**Cotangent (cot)**



These six trigonometric functions are used to evaluate acute angles in a right triangle. The ratio of the lengths of two sides of a right triangle will be used to evaluate a given angle θ. We will go back to something introduced in geometry for this.

Soh-Cah-Toa

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

The second set of trigonometric functions, are reciprocal functions.

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

Therefore, it follows that...

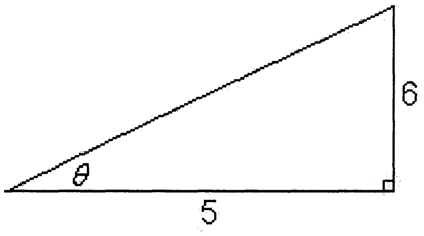
$$\csc\theta = \frac{\text{hyp}}{\text{opp}} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}}$$

Exercises that require finding the exact value of the six trigonometric functions follow on the next few pages. Sometimes, only two of the three sides of a triangle will be given requiring the student to find the third.

When given the lengths of two sides of a right triangle, how can the length of the third side be found?

Example

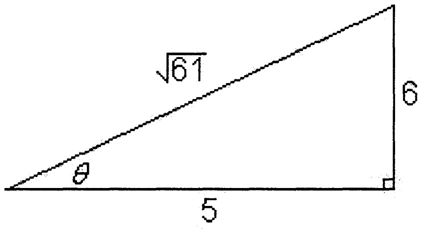
Find the exact values of the six trigonometric functions of θ.



In this example, the length of the side adjacent to the angle θ measures 5 units, while the opposite side measures 6 units. In order to find the missing side, the Pythagorean Theorem must be used.

$$a^2+b^2=c^2$$
$$(5)^2+(6)^2=c^2$$
$$25+36=c^2$$
$$61=c^2$$
$$\sqrt{61}=c$$

Using the Pythagorean Theorem, the length of the hypotenuse is $\sqrt{61}$ units.



Once the value of the hypotenuse is found, we can find the exact value of the six trigonometric functions of the angle θ.

$$\sin\theta=\frac{6}{\sqrt{61}}=\frac{6\sqrt{61}}{61} \qquad \csc\theta=\frac{\sqrt{61}}{6}$$
$$\cos\theta=\frac{5}{\sqrt{61}}=\frac{5\sqrt{61}}{61} \qquad \sec\theta=\frac{\sqrt{61}}{5}$$
$$\tan\theta=\frac{6}{5} \qquad \cot\theta=\frac{5}{6}$$

Evaluate the first three functions using Soh-Cah-Toa.

$$\sin\theta=\frac{\text{opp}}{\text{hyp}} \qquad \cos\theta=\frac{\text{adj}}{\text{hyp}} \qquad \tan\theta=\frac{\text{opp}}{\text{adj}}$$

Evaluate second set of functions by finding the reciprocals of the first three.

Do not forget to rationalize any denominators if needed.

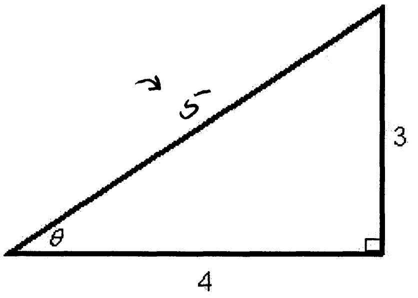
$$\sin\theta=\frac{6\sqrt{61}}{61} \qquad \csc\theta=\frac{\sqrt{61}}{6}$$
$$\cos\theta=\frac{5\sqrt{61}}{61} \qquad \sec\theta=\frac{\sqrt{61}}{5}$$
$$\tan\theta=\frac{6}{5} \qquad \cot\theta=\frac{5}{6}$$

Here are the exact values of the six trigonometric functions of the angle θ. Radicals are left in the solutions because we need the <u>exact</u> values, not estimates.

Find the exact values of the six trigonometric functions of θ.

Soh- Cah- Toa

A)



$\sin\theta = 3/5$ $\csc\theta = 5/3$

$\cos\theta = 4/5$ $\sec\theta = 5/4$

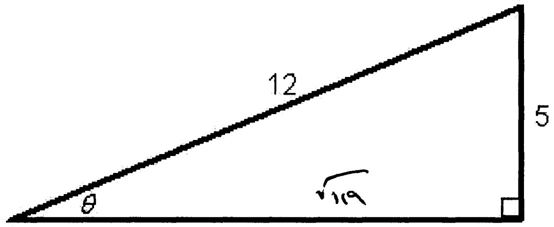
$\tan\theta = 3/4$ $\cot\theta = 4/3$

$x^2+25=144$

$x^2=119$

$x=\sqrt{119}$

B)

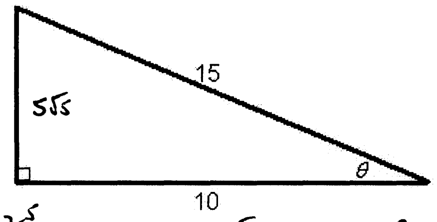


$\sin\theta = 5/12$ $\csc\theta = 12/5$

$\cos\theta = \frac{\sqrt{119}}{12}$ $\sec\theta = \frac{12}{\sqrt{119}} = \frac{12\sqrt{119}}{119}$

$\tan\theta = \frac{5}{\sqrt{119}} = \frac{5\sqrt{119}}{119}$ $\cot\theta = \frac{\sqrt{119}}{5}$

C)



$x^2+100=225$

$x^2=125$

$x=5\sqrt{5}$

$\sin\theta = \frac{5\sqrt{5}}{15} = \frac{\sqrt{5}}{3}$ $\csc\theta = \frac{3}{\sqrt{5}} = \frac{3\sqrt{5}}{5}$

$\cos\theta = \frac{10}{15} = \frac{2}{3}$ $\sec\theta = 3/2$

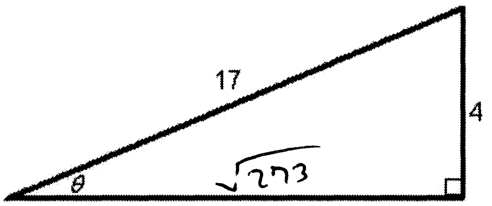
$\tan\theta = \frac{5\sqrt{5}}{10} = \frac{\sqrt{5}}{2}$ $\cot\theta = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5}$

$x^2+16=289$

$x^2=273$

$x=\sqrt{273}$

D)



$\sin\theta = 4/17$ $\csc\theta = 17/4$

$\cos\theta = \frac{\sqrt{273}}{17}$ $\sec\theta = \frac{17}{\sqrt{273}} = \frac{17\sqrt{273}}{273}$

$\tan\theta = \frac{4}{\sqrt{273}} = \frac{4\sqrt{273}}{273}$ $\cot\theta = \frac{\sqrt{273}}{4}$

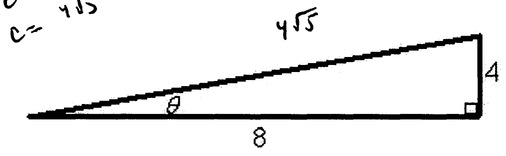
$4^2+8^2=c^2$

$16+64=c^2$

$c^2=80$

$c=4\sqrt{5}$

E)



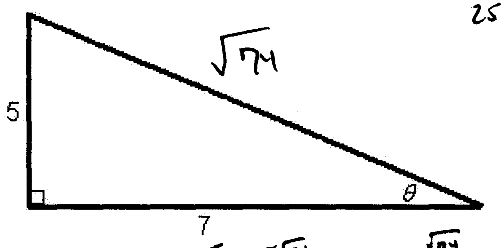
$5^2+7^2=c^2$

$25+49=c^2$

$c^2=74$

$c=\sqrt{74}$

F)



$\frac{4}{4\sqrt{5}} = \frac{1}{\sqrt{5}}$

$\frac{8}{4\sqrt{5}} = \frac{2}{\sqrt{5}}$

$\sin\theta = \frac{4}{4\sqrt{5}} = \frac{\sqrt{5}}{5}$ $\csc\theta = \sqrt{5}$

$\cos\theta = \frac{8}{4\sqrt{5}} = \frac{2\sqrt{5}}{5}$ $\sec\theta = \sqrt{5}/2$

$\tan\theta = \frac{4}{8} = \frac{1}{2}$ $\cot\theta = 2$

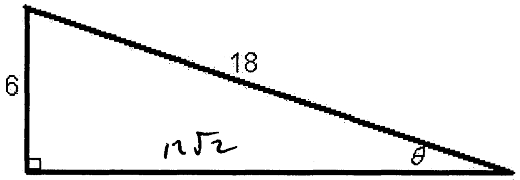
$\sin\theta = \frac{5}{\sqrt{74}} = \frac{5\sqrt{74}}{74}$ $\csc\theta = \frac{\sqrt{74}}{5}$

$\cos\theta = \frac{7}{\sqrt{74}} = \frac{7\sqrt{74}}{74}$ $\sec\theta = \frac{\sqrt{74}}{7}$

$\tan\theta = \frac{5}{7}$ $\cot\theta = \frac{7}{5}$

Continued

G)



$x^2+36=324$

$x^2=288$

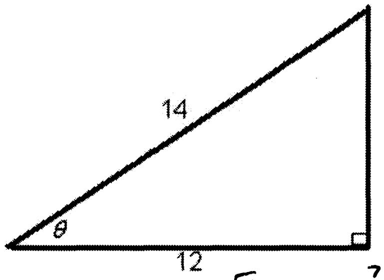
$x=12\sqrt{2}$

$\sin\theta = \frac{6}{18} = \frac{1}{3}$ $\quad \csc\theta = 3$

$\cos\theta = \frac{12\sqrt{2}}{18} = \frac{2\sqrt{2}}{3}$ $\quad \sec\theta = \frac{3}{2\sqrt{2}} = \frac{3\sqrt{2}}{4}$

$\tan\theta = \frac{6}{12\sqrt{2}} = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}$ $\quad \cot\theta = \frac{1}{2\sqrt{2}} = 2\sqrt{2}$

H)



$x^2+144=196$

$x^2=52$

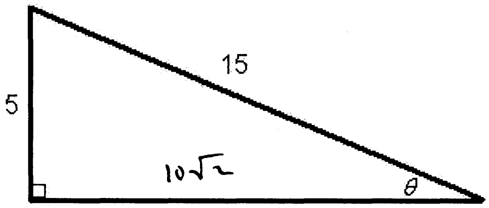
$x=2\sqrt{13}$

$\sin\theta = \frac{2\sqrt{13}}{14} = \frac{\sqrt{13}}{7}$ $\quad \csc\theta = \frac{7}{\sqrt{13}} = \frac{7\sqrt{13}}{13}$

$\cos\theta = \frac{12}{14} = \frac{6}{7}$ $\quad \sec\theta = 7/6$

$\tan\theta = \frac{2\sqrt{13}}{12} = \frac{\sqrt{13}}{6}$ $\quad \cot\theta = \frac{6}{\sqrt{13}} = \frac{6\sqrt{13}}{13}$

I)



$x^2+25=225$

$x^2=200$

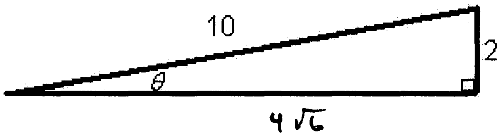
$x=10\sqrt{2}$

$\sin\theta = \frac{5}{15} = \frac{1}{3}$ $\quad \csc\theta = 3$

$\cos\theta = \frac{10\sqrt{2}}{15} = \frac{2\sqrt{2}}{3}$ $\quad \sec\theta = \frac{3}{2\sqrt{2}} = \frac{3\sqrt{2}}{4}$

$\tan\theta = \frac{5}{10\sqrt{2}} = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}$ $\quad \cot\theta = 2\sqrt{2}$

J)



$x^2+4=100$

$x^2=96$

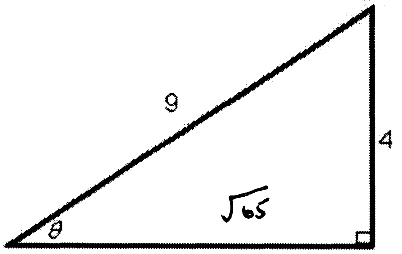
$x=4\sqrt{6}$

$\sin\theta = \frac{2}{10} = \frac{1}{5}$ $\quad \csc\theta = 5$

$\cos\theta = \frac{4\sqrt{6}}{10} = \frac{2\sqrt{6}}{5}$ $\quad \sec\theta = \frac{5}{2\sqrt{6}} = \frac{5\sqrt{6}}{12}$

$\tan\theta = \frac{2}{4\sqrt{6}} = \frac{1}{2\sqrt{6}} = \frac{\sqrt{6}}{12}$ $\quad \cot\theta = 2\sqrt{6}$

K)



$x^2+16=81$

$x^2=65$

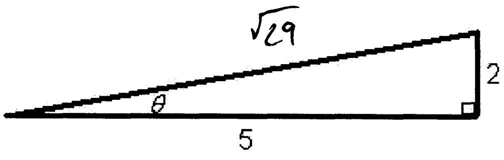
$x=\sqrt{65}$

$\sin\theta = \frac{4}{9}$ $\quad \csc\theta = \frac{9}{4}$

$\cos\theta = \frac{\sqrt{65}}{9}$ $\quad \sec\theta = \frac{9}{\sqrt{65}} = \frac{9\sqrt{65}}{65}$

$\tan\theta = \frac{4}{\sqrt{65}} = \frac{4\sqrt{65}}{65}$ $\quad \cot\theta = \frac{\sqrt{65}}{4}$

L)



$4+25=c^2$

$c^2=29$

$c=\sqrt{29}$

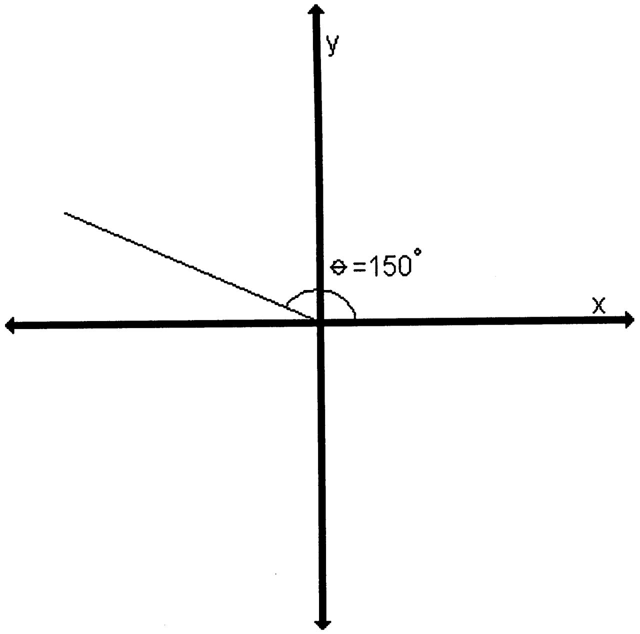
$\sin\theta = \frac{2}{\sqrt{29}} = \frac{2\sqrt{29}}{29}$ $\quad \csc\theta = \frac{\sqrt{29}}{2}$

$\cos\theta = \frac{5}{\sqrt{29}} = \frac{5\sqrt{29}}{29}$ $\quad \sec\theta = \frac{\sqrt{29}}{5}$

$\tan\theta = \frac{2}{5}$ $\quad \cot\theta = \frac{5}{2}$

Reference Angles

The angles we will evaluate in trigonometry will always rest between the terminal side of that angle, and the x axis. Below is a graphical representation of a 150° angle.

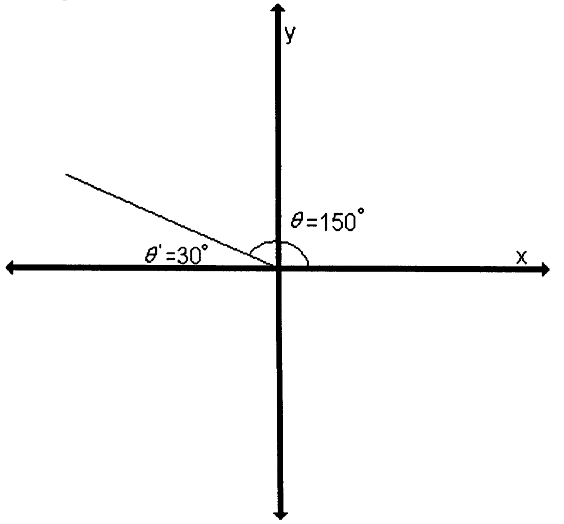


In the illustration above, the terminal side of a 150° angle, in standard position, resides in quadrant II. In order to evaluate the six trigonometric functions of an angle θ, we are required to use a reference angle.

A reference angle is the acute angle θ' (read as theta prime) formed by the terminal side of the angle θ, and the x axis.

REFERENCE ANGLES ARE ALWAYS DRAWN IN RELATION TO THE X AXIS.

Therefore, to evaluate the six trigonometric functions of a 150° angle in standard position, a 30° angle will be used.



Since the horizontal in quadrant II represents 180°, evaluate 180° - 150° to find θ', which is this case is 30°. This is not the same method that will be used for every angle.

If for example, we need to find a reference angle of a 200° angle, evaluate 200° - 180° = 20°.

Here are some guidelines for finding reference angles. The method used to find a reference angle depends on the quadrant in which the terminal side of the angle resides.

- **If the terminal side of an angle θ rests in quadrant I,** $\theta' = \theta$**.**
- **If the terminal side of an angle θ rests in quadrant II,** $\theta' = 180° - \theta$ **or** $\theta' = \pi - \theta$**.**
- **If the terminal side of an angle θ rests in quadrant III,** $\theta' = \theta - 180°$ **or** $\theta' = \theta - \pi$**.**
- **If the terminal side of an angle θ rests in quadrant IV,** $\theta' = 360° - \theta$ **or** $\theta' = 2\pi - \theta$**.**

Example 1

The angle $\theta = -150°$. Find the reference angle θ'.

Begin by finding the positive coterminal angle to a -150° angle.

$$-150° + 360° = 210°$$

The terminal side of a 210° angle resides in quadrant III. Therefore, to find the reference angle use $\theta' = \theta - 180°$.

$$210° - 180° = 30°$$

$$\theta' = 30°$$

Example 2

The angle $\theta = 2.5$. Find the reference angle θ'.

In this case, there is no degree symbol. This means the measure of angle θ is 2.5 radians.

First use the formula to convert radians to degrees.

$$2.5 \cdot \frac{180°}{\pi} \approx 143.239°$$

Since the question was asked in terms of radians, the answer must be given in the same way. Converting the angle measure to degrees allows us to get a clear picture of where the terminal side of the angle will lie. In this case, the terminal side of an angle that measures 2.5 radians lies in quadrant II.

To find the reference angle, evaluate $\theta' = \pi - \theta$.

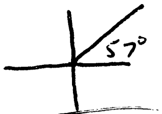
$$\theta' = \pi - 2.5$$

$$\theta' \approx 0.642 \quad \textit{radians}$$

As illustrated in example 1 on the previous page, it is sometimes necessary to find a coterminal angle first. If θ is negative, first find the coterminal angle, then use that to find the reference angle. If the measure of the original angle is given in degrees, its reference angle must also be in degrees. If the measure of the original angle is given in radians, then the reference angle found must also be in radians. Exact solutions should be found whenever possible. In example 2 on the previous page, it was impossible to give an exact solution, because the measure of angle θ did not include π. Therefore, a decimal approximation had to be made.

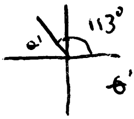
For each of the following, find the reference angle θ'.

A) $\theta = 57°$



$\boxed{\theta' = 57°}$

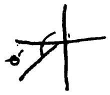
B) $\theta = 113°$



$\theta' = 180° - 113°$

$\boxed{\theta' = 67°}$

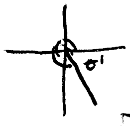
C) $\theta = \frac{7\pi}{6}$ π



$\frac{7\pi}{6} - \frac{6\pi}{6} = \frac{\pi}{6}$

$\boxed{\theta' = \frac{\pi}{6}}$

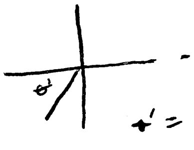
D) $\theta = \frac{5\pi}{3}$ 2π



$\frac{6\pi}{3} - \frac{5\pi}{3} = \frac{\pi}{3}$

$\boxed{\theta' = \frac{\pi}{3}}$

E) $\theta = -\frac{2\pi}{3}$ 2π



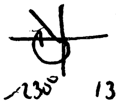
$-\frac{2\pi}{3} + \frac{6\pi}{3} = \frac{4\pi}{3}$

$\theta' = \frac{4\pi}{3} - \frac{3\pi}{3} = \frac{\pi}{3}$

π

$\boxed{\theta' = \frac{\pi}{3}}$

F) $\theta = -230°$



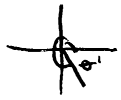
$-230° + 360°$

130°

$\theta' = 180° - 130°$

$\boxed{\theta' = 50°}$

G) $\theta = 300°$



$360° - 300° = 60°$

$\boxed{\theta' = 60°}$

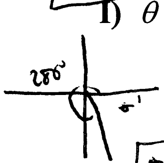
H) $\theta = 2.3$

$2.3 \approx 131.78°$

$\theta' = \pi - 2.3$

$\boxed{\theta' \approx .8416 \text{ rad}}$

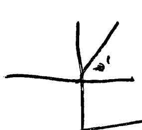
I) $\theta = 280°$



$\theta' = 360° - 280°$

$\boxed{\theta' = 80°}$

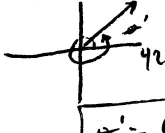
J) $\theta = 1.2$



$1.2 \approx 68.75°$

$\boxed{\theta' = 1.2 \text{ rad}}$

K) $\theta = 420°$

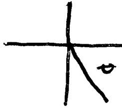


$420° - 360° = 60°$

$\boxed{\theta' = 60°}$

L) $\theta = -60°$

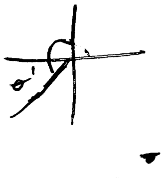
$-60° + 360° = 300°$



$\theta' = 360° - 300° = 60°$

$\boxed{\theta' = 60°}$

M) $\theta = -2$



$-2 \approx -114.59°$

$+360$

$\approx 245.41°$

$-2 + 2\pi \approx 4.283 \text{ rad}$

$\theta' = 4.283 - \pi$

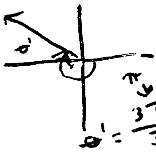
$\boxed{\theta' \approx 1.141 \text{ rad}}$

N) $\theta = 100°$

$180° - 100°$

$\boxed{\theta' = 80°}$

O) $\theta = -\frac{4\pi}{3}$ 2π



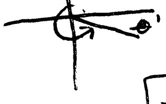
$-\frac{4\pi}{3} + \frac{6\pi}{3} = \frac{2\pi}{3}$

π

$\theta' = \frac{3\pi}{3} - \frac{2\pi}{3} = \frac{\pi}{3}$

$\boxed{\theta' = \frac{\pi}{3}}$

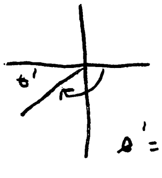
P) $\theta = \frac{11\pi}{6}$ 2π



$\frac{12\pi}{6} - \frac{11\pi}{6} = \frac{\pi}{6}$

$\boxed{\theta' = \frac{\pi}{6}}$

Q) $\theta = -135°$

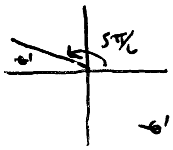


$-135° + 360° = 225°$

$\theta' = 225° - 180°$

$\boxed{\theta' = 45°}$

R) $\theta = \frac{17\pi}{6}$



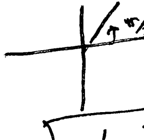
$\frac{12\pi}{6} + \frac{5\pi}{6}$

π

$\theta' = \frac{6\pi}{6} - \frac{5\pi}{6}$

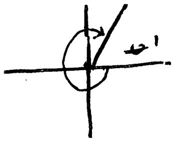
$\boxed{\theta' = \frac{\pi}{6}}$

S) $\theta = \frac{\pi}{3}$



$\boxed{\theta' = \frac{\pi}{3}}$

T) $\theta = -\frac{5\pi}{3}$ 2π

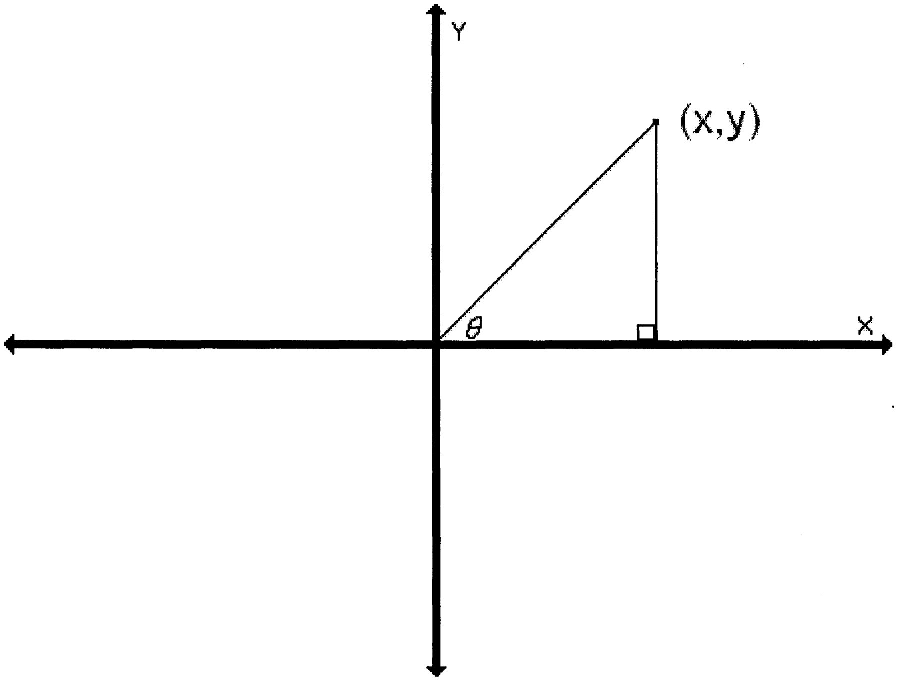


$-\frac{5\pi}{3} + \frac{6\pi}{3}$

$\frac{\pi}{3}$

$\boxed{\theta' = \frac{\pi}{3}}$

Trigonometric Functions of any Angle



When evaluating any angle θ, in standard position, whose terminal side is given by the coordinates (x,y), a reference angle is always used. Notice how a right triangle has been created. This will allow us to evaluate the six trigonometric functions of any angle.

Notice the side opposite the angle θ has a length of the y value of the given coordinates. The adjacent side has a length of the x value of the coordinates. The length of the hypotenuse is given by $\sqrt{x^2+y^2}$.

Lets say, for the sake of argument, the length of the hypotenuse is 1 unit. This would mean the following would be true.

$$\sin\theta = y \qquad \csc\theta = \frac{1}{y}$$

$$\cos\theta = x \qquad \sec\theta = \frac{1}{x}$$

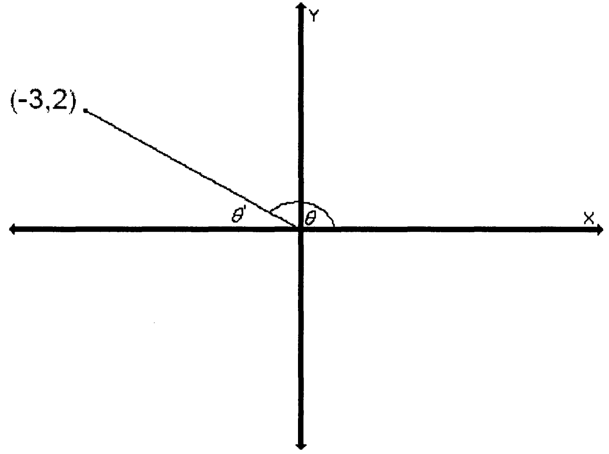
$$\tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{x}{y}$$

You must think of the sine function as giving you the y value, whereas the cosine function yields the x value. This is how we will determine whether the sine, cosine, tangent, cosecant, secant or cotangent of a given angle is a positive or negative value.

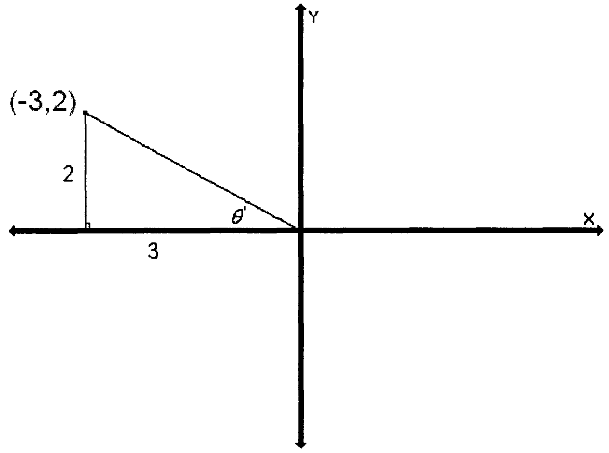
If the angle to be evaluated is in quadrant IV, for instance, the sine of the angle θ will be negative. The cosine of θ, in this instance, will be positive, while the tangent of the angle θ will be negative.

Example

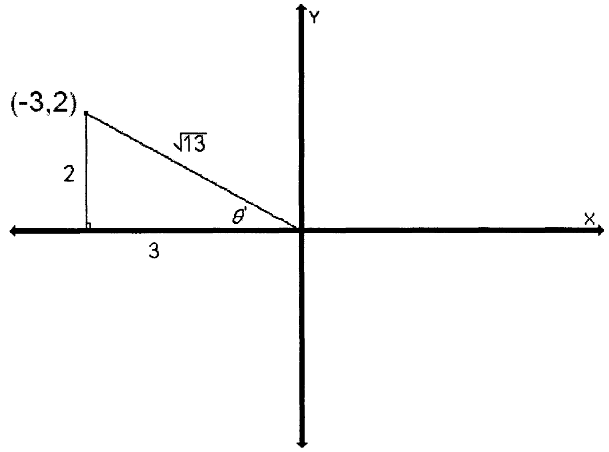
Evaluate the six trigonometric functions of an angle θ, in standard position, whose terminal side has an endpoint of (-3,2).



The angle with terminal side is first drawn. Remember, in order to evaluate the six trigonometric functions for θ, use the reference angle θ'.



From the endpoint of the terminal side of the angle, a line is drawn to the x axis. This is the reason reference angles are always drawn in relation to the x axis. It will always create a right triangle with which to work. Now all that is needed to solve the problem, is to find the length of the hypotenuse then the values of the six trigonometric functions can be found.



Using the Pythagorean Theorem, the length of the hypotenuse may be found.

$$2^2 + 3^2 = c^2$$

$$4 + 9 = c^2$$

$$13 = c^2$$

$$\sqrt{13} = c$$

$$\sin\theta = \frac{2}{\sqrt{13}} = \frac{2\sqrt{13}}{13} \qquad \csc\theta = \frac{\sqrt{13}}{2}$$

$$\cos\theta = -\frac{3}{\sqrt{13}} = -\frac{3\sqrt{13}}{13} \qquad \sec\theta = -\frac{\sqrt{13}}{3}$$

$$\tan\theta = -\frac{2}{3} \qquad \cot\theta = -\frac{3}{2}$$

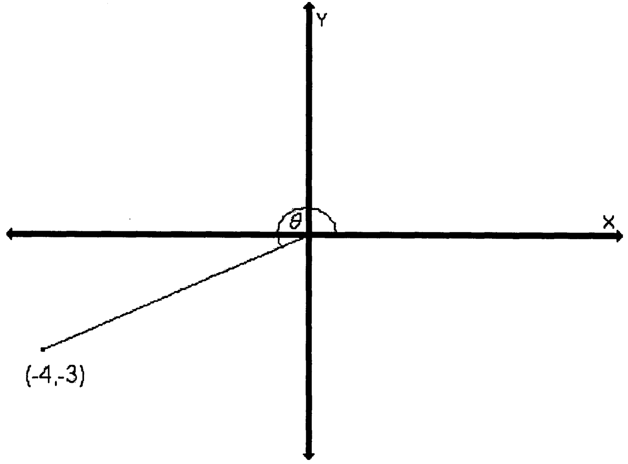
The first three functions are evaluated using Soh-Cah-Toa.

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} \qquad \cos\theta = \frac{\text{adj}}{\text{hyp}} \qquad \tan\theta = \frac{\text{opp}}{\text{adj}}$$

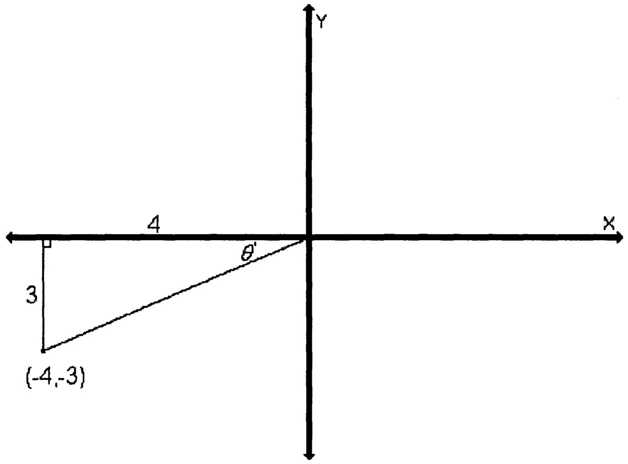
To find the second set of functions take the reciprocals of the first three. Rationalize any denominators if needed. Note the terminal side to this angle is in quadrant II. This means cosine, tangent, secant and cotangent are all negative values.

Example

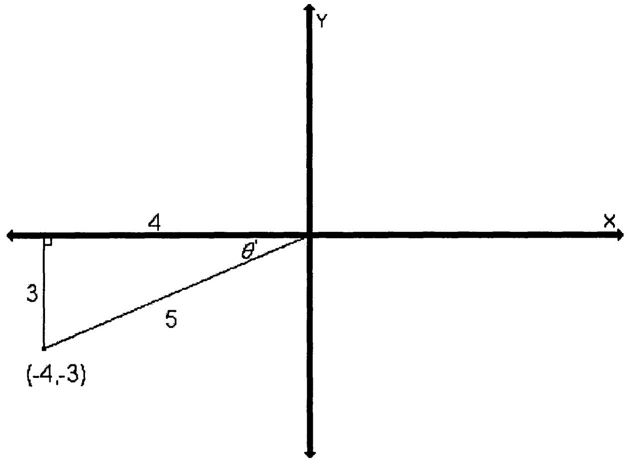
Evaluate the six trigonometric functions of the angle θ, in standard position, whose terminal side has an endpoint of (-4,-3).



Begin by drawing the angle θ in standard position whose terminal side has the endpoint of (-4,-3).



A right triangle is formed by drawing a line segment to the x axis. Now use the reference angle that is drawn in relation to the x axis to evaluate the six trigonometric functions.



Since this is obviously a 3-4-5 right triangle, there is no need to use the Pythagorean Theorem in this case.

$\sin\theta = -\frac{3}{5}$ $\quad\csc\theta = -\frac{5}{3}$

$\cos\theta = -\frac{4}{5}$ $\quad\sec\theta = -\frac{5}{4}$

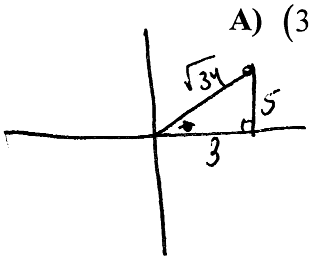
$\tan\theta = \frac{3}{4}$ $\quad\cot\theta = \frac{4}{3}$

Since this angle resides in quadrant III, sine, cosine, cosecant and secant are negative values. Tangent is $\frac{y}{x}$ and cotangent is $\frac{x}{y}$. This means both tangent and cotangent will be positive values.

Evaluate the six trigonometric functions of the angle θ, in standard position, that has a terminal side with the following endpoints. ***(Remember, reference angles are always drawn in relation to the x axis.)***

A) $(3,5)$

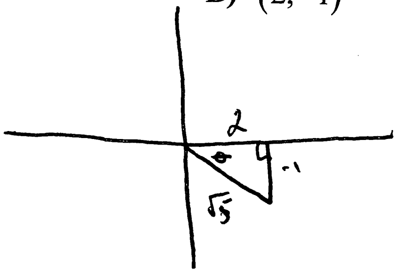
$9+25=c^2$
$c^2=34$
$c=\sqrt{34}$



$\sin\theta=\frac{5}{\sqrt{34}}=\frac{5\sqrt{34}}{34}$ $\quad\csc\theta=\frac{\sqrt{34}}{5}$
$\cos\theta=\frac{3}{\sqrt{34}}=\frac{3\sqrt{34}}{34}$ $\quad\sec\theta=\frac{\sqrt{34}}{3}$
$\tan\theta=\frac{5}{3}$ $\quad\cot\theta=\frac{3}{5}$

B) $(2,-1)$

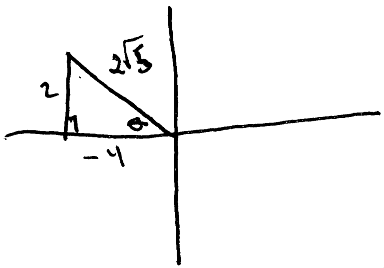
$1+4=c^2$
$c^2=5$
$c=\sqrt{5}$



$\sin\theta=\frac{-1}{\sqrt{5}}=\frac{-\sqrt{5}}{5}$ $\quad\csc\theta=-\sqrt{5}$
$\cos\theta=\frac{2}{\sqrt{5}}=\frac{2\sqrt{5}}{5}$ $\quad\sec\theta=\sqrt{5}/2$
$\tan\theta=-\frac{1}{2}$ $\quad\cot\theta=-2$

C) $(-4,2)$

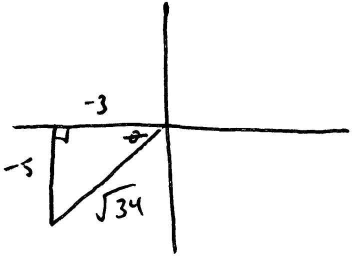
$4+16=c^2$
$c^2=20$
$c=2\sqrt{5}$



$\sin\theta=\frac{2}{2\sqrt{5}}=\frac{1}{\sqrt{5}}=\frac{\sqrt{5}}{5}$ $\quad\csc\theta=\sqrt{5}$
$\cos\theta=\frac{-4}{2\sqrt{5}}=\frac{-2}{\sqrt{5}}=\frac{-2\sqrt{5}}{5}$ $\quad\sec\theta=-\frac{\sqrt{5}}{2}$
$\tan\theta=\frac{2}{-4}=-\frac{1}{2}$ $\quad\cot\theta=-2$

D) $(-3,-5)$

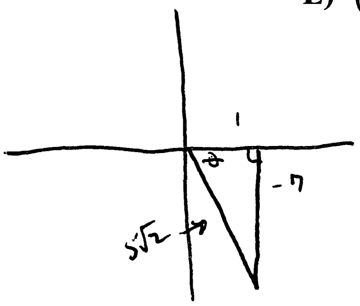
$9+25=c^2$
$c^2=34$
$c=\sqrt{34}$



$\sin\theta=-\frac{5}{\sqrt{34}}=\frac{-5\sqrt{34}}{34}$ $\quad\csc\theta=-\frac{\sqrt{34}}{5}$
$\cos\theta=-\frac{3}{\sqrt{34}}=\frac{-3\sqrt{34}}{34}$ $\quad\sec\theta=-\frac{\sqrt{34}}{3}$
$\tan\theta=\frac{-5}{-3}=\frac{5}{3}$ $\quad\cot\theta=\frac{3}{5}$

E) $(1,-7)$

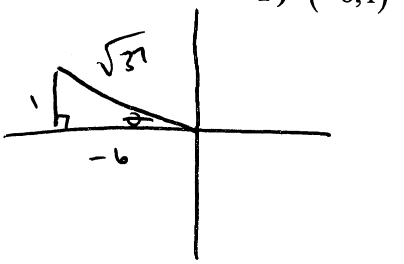
$1+49=c^2$
$c^2=50$
$c=5\sqrt{2}$



$\sin\theta=\frac{-7}{5\sqrt{2}}=\frac{-7\sqrt{2}}{10}$ $\quad\csc\theta=-\frac{5\sqrt{2}}{7}$
$\cos\theta=\frac{1}{5\sqrt{2}}=\frac{\sqrt{2}}{10}$ $\quad\sec\theta=5\sqrt{2}$
$\tan\theta=-7$ $\quad\cot\theta=-\frac{1}{7}$

F) $(-6,1)$

$1+36=c^2$
$c^2=37$
$c=\sqrt{37}$



$\sin\theta=\frac{1}{\sqrt{37}}=\frac{\sqrt{37}}{37}$ $\quad\csc\theta=\sqrt{37}$
$\cos\theta=-\frac{6}{\sqrt{37}}=\frac{-6\sqrt{37}}{37}$ $\quad\sec\theta=-\frac{\sqrt{37}}{6}$
$\tan\theta=-\frac{1}{6}$ $\quad\cot\theta=-6$

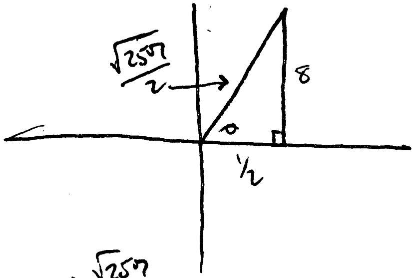
G) $\left(\frac{1}{2},8\right)$

$64+\frac{1}{4}=c^2$

$\frac{256}{4}+\frac{1}{4}=c^2$

$c^2=\frac{257}{4}$

$c=\frac{\sqrt{257}}{2}$



$\sin\theta = 8 \div \frac{\sqrt{257}}{2}$

$8\cdot\frac{2}{\sqrt{257}}=\frac{16}{\sqrt{259}}$

$\sin\theta = \frac{16\sqrt{257}}{257}$ $\quad$ $\csc\theta = \frac{\sqrt{257}}{16}$

$\cos\theta = \frac{\sqrt{257}}{257}$ $\quad$ $\sec\theta = \sqrt{257}$

$\tan\theta = \frac{8}{\frac{1}{2}} = 16$ $\quad$ $\cot\theta = \frac{1}{16}$

$\cos\theta = \frac{1}{2} \div \frac{\sqrt{257}}{2}$

$\frac{1}{2}\cdot\frac{2}{\sqrt{257}}=\frac{1}{\sqrt{257}}$

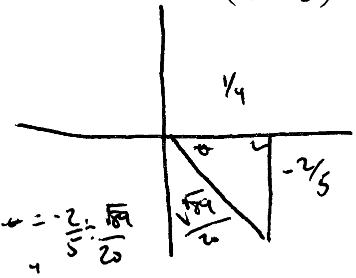
H) $\left(\frac{1}{4},-\frac{2}{5}\right)$

$\frac{1}{16}+\frac{4}{25}=c^2$

$\frac{25}{400}+\frac{64}{400}=c^2$

$c^2=\frac{89}{400}$

$c=\frac{\sqrt{89}}{20}$



$\sin\theta = -\frac{2}{5} \div \frac{\sqrt{89}}{20}$

$-\frac{2}{\cancel{5}}\cdot\frac{\overset{4}{\cancel{20}}}{\sqrt{89}}$

$\frac{-8}{\sqrt{89}}$

$\sin\theta = \frac{-8\sqrt{89}}{89}$ $\quad$ $\csc\theta = -\frac{\sqrt{89}}{8}$

$\cos\theta = \frac{5\sqrt{89}}{89}$ $\quad$ $\sec\theta = \frac{\sqrt{89}}{5}$

$\tan\theta = -\frac{8}{5}$ $\quad$ $\cot\theta = -\frac{5}{8}$

$\cos\theta = \frac{1}{4} \div \frac{\sqrt{89}}{20}$

$\frac{1}{\cancel{4}}\cdot\frac{\overset{5}{\cancel{20}}}{\sqrt{89}}$

$\frac{5}{\sqrt{89}}$

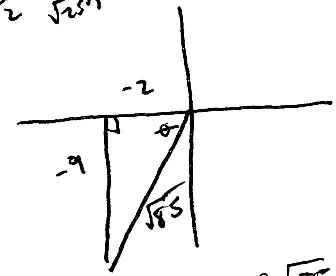
$\tan\theta = -\frac{2}{5}\cdot\frac{4}{1}$

I) $(-2,-9)$

$4+81=c^2$

$c^2=85$

$c=\sqrt{85}$



$\sin\theta = \frac{-9}{\sqrt{85}}$

$\sin\theta = \frac{-9\sqrt{85}}{85}$ $\quad$ $\csc\theta = -\frac{\sqrt{85}}{9}$

$\cos\theta = -\frac{2\sqrt{85}}{85}$ $\quad$ $\sec\theta = -\frac{\sqrt{85}}{2}$

$\tan\theta = \frac{9}{2}$ $\quad$ $\cot\theta = \frac{2}{9}$

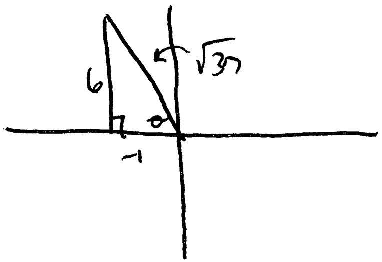
$\cos\theta = \frac{-2}{\sqrt{85}}$

J) $(-1,6)$

$1+36=c^2$

$c^2=37$

$c=\sqrt{37}$



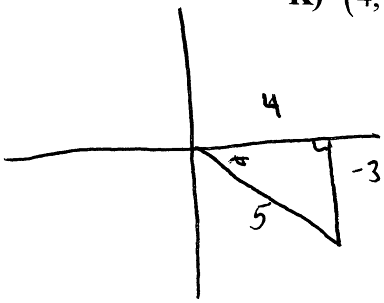
$\sin\theta = \frac{6}{\sqrt{37}} = \frac{6\sqrt{37}}{37}$ $\quad$ $\csc\theta = \frac{\sqrt{37}}{6}$

$\cos\theta = \frac{-1}{\sqrt{37}} = \frac{-\sqrt{37}}{37}$ $\quad$ $\sec\theta = -\sqrt{37}$

$\tan\theta = -6$ $\quad$ $\cot\theta = -\frac{1}{6}$

K) $(4,-3)$

3-4-5 triangle



$\sin\theta = -\frac{3}{5}$ $\quad$ $\csc\theta = -\frac{5}{3}$

$\cos\theta = \frac{4}{5}$ $\quad$ $\sec\theta = \frac{5}{4}$

$\tan\theta = -\frac{3}{4}$ $\quad$ $\cot\theta = -\frac{4}{3}$

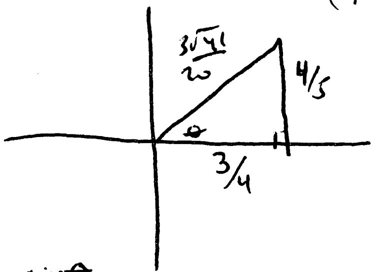
L) $\left(\frac{3}{4},\frac{4}{5}\right)$

$\frac{9}{16}+\frac{16}{25}=c^2$

$\frac{225}{400}+\frac{144}{400}=c^2$

$c^2=\frac{369}{400}$

$c=\frac{3\sqrt{41}}{20}$



$\sin\theta$

$\frac{4}{\cancel{5}}\cdot\frac{\overset{4}{\cancel{20}}}{3\sqrt{41}}$

$\frac{16}{3\sqrt{41}}\cdot\frac{\sqrt{41}}{\sqrt{41}}$

$\sin\theta = \frac{16\sqrt{41}}{123}$ $\quad$ $\csc\theta = \frac{3\sqrt{41}}{16}$

$\cos\theta = \frac{5\sqrt{41}}{41}$ $\quad$ $\sec\theta = \frac{\sqrt{41}}{5}$

$\tan\theta = \frac{16}{15}$ $\quad$ $\cot\theta = \frac{15}{16}$

$\cos\theta = \frac{3}{\cancel{4}}\cdot\frac{\overset{5}{\cancel{20}}}{3\sqrt{41}} = \frac{15}{3\sqrt{41}} = \frac{5}{\sqrt{41}}$

$\tan\theta = \frac{4}{5}\cdot\frac{4}{3} = \frac{16}{15}$

Once again, think of the sine of an angle θ as yielding the y value, while the cosine yields the x value when the hypotenuse is 1. Since the tangent of an angle is y over x, as $\sin\theta = y$ and $\cos\theta = x$; it is a trigonometric identity that $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$.

If the hypotenuse is any other length, the following is true.

$$\sin\theta = \frac{y}{h} \qquad \csc\theta = \frac{h}{y}$$

$$\cos\theta = \frac{x}{h} \qquad \sec\theta = \frac{h}{x}$$

$$\tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{x}{y}$$

These are the actual equations used for evaluating the six trigonometric functions. The reason we think of sine being the y value, cosine being the x value, and tangent being sine divided by cosine is to determine whether the value of a trigonometric function is positive or negative. This of course all depends on where the terminal side of the angle lies.

The following questions will require evaluating the six trigonometric functions of an angle θ given different types of information. Understand that these are the same types of questions encountered on the previous pages, just asked in a different manner.

Evaluate the six trigonometric functions of an angle θ, *in standard position, where* $\sin\theta = \frac{2}{3}$ *and* $\cos\theta < 0$.

This question will be done shortly. Since the sine of an angle can be thought of as the y value there are two quadrants in which sine is positive. It is therefore necessary to have one more piece of information to answer the question. There is some vital information that is needed to answer this type of question.

The following guidelines will help determine in which quadrant an angle lies.

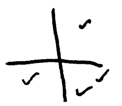
- ***Sine is positive in quadrants I and II.***
- ***Sine is negative in quadrants III and IV.***
- ***Cosine is positive in quadrants I and IV.***
- ***Cosine is negative in quadrants II and III.***
- ***Tangent is positive in quadrants I and III.***
- ***Tangent is negative in quadrants II and IV.***

For this particular problem, $\sin\theta > 0$ and $\tan\theta < 0$, this means the angle θ must reside in quadrant II. This information tells us where to construct our triangle.

Before we answer the sample question, here is some practice using this information.

Given the following information, determine the quadrant in which the angle θ resides.

A) $\cos\theta > 0$ and $\sin\theta < 0$

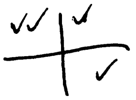


IV

B) $\tan\theta > 0$ and $\cos\theta < 0$

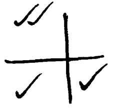
III

C) $\sin\theta > 0$ and $\tan\theta < 0$



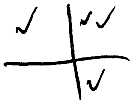
II

D) $\tan\theta < 0$ and $\cos\theta<0$



II

E) $\sec\theta > 0$ and $\csc\theta > 0$

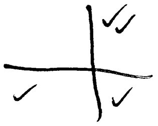


$\cos\theta > 0$

$\sin\theta > 0$

I

F) $\cot\theta > 0$ and $\sec\theta > 0$

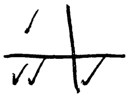


$\tan\theta > 0$

$\cos\theta > 0$

I

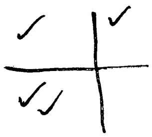
G) $\csc\theta < 0$ and $\cos\theta < 0$



$\sin\theta < 0$

III

H) $\cot\theta > 0$ and $\sec\theta < 0$



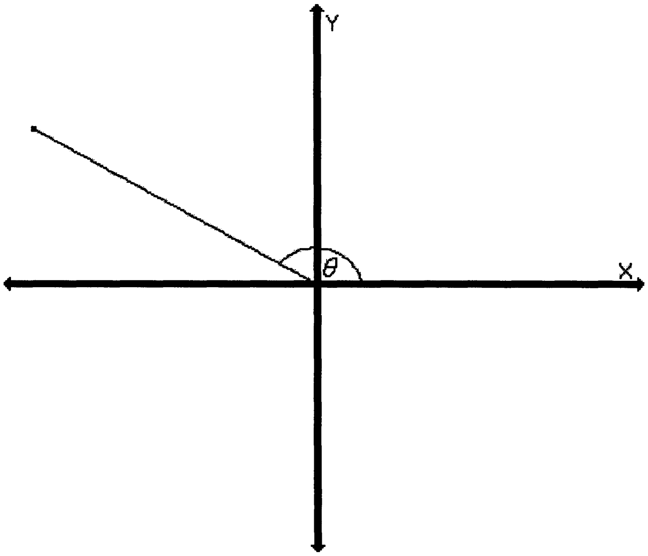
$\tan\theta > 0$

$\cos\theta < 0$

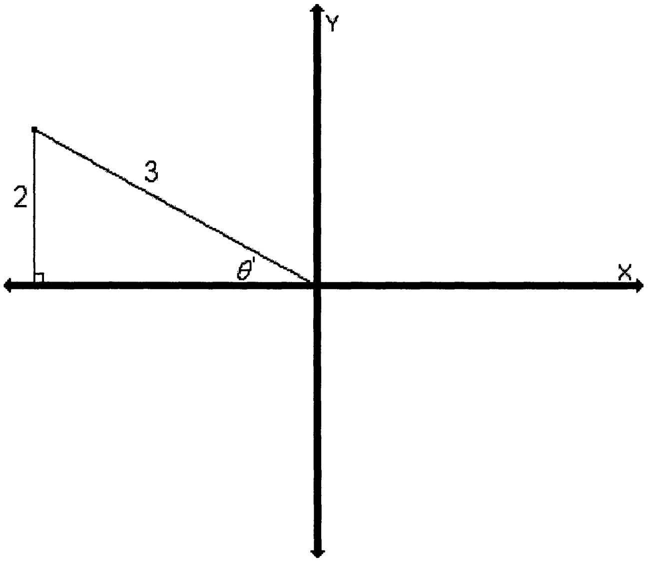
III

Example

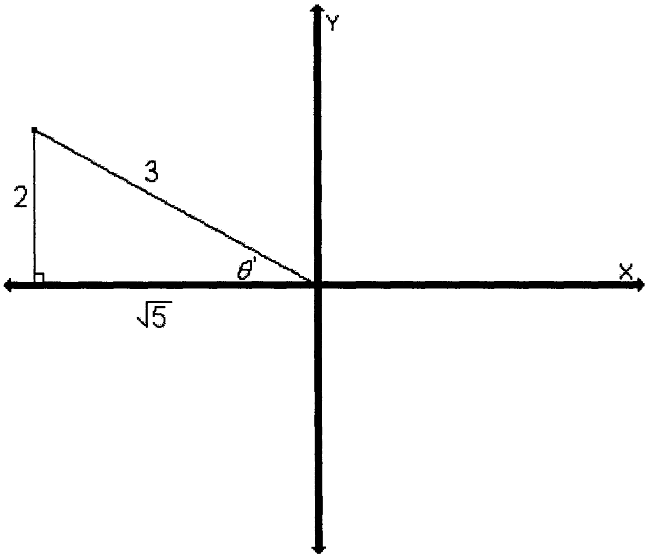
Evaluate the six trigonometric functions of an angle θ, *in standard position, where* $\sin\theta = \frac{2}{3}$ ***and*** $\cos\theta < 0$.



According to the information given, the sine of the angle is positive and cosine is negative. This means the terminal side of the angle to be evaluated must be in quadrant II.



From here, we will use the reference angle drawn in relation to the x axis. A right triangle is then constructed. Since the sine of an angle is opposite over hypotenuse, the 2 and the 3 can be placed on the appropriate sides of the triangle.



Using the Pythagorean Theorem, the adjacent side is found to be $\sqrt{5}$ units.

$$x^2 + 2^2 = 3^2$$

$$x^2 + 4 = 9$$

$$x^2 = 5$$

$$x = \sqrt{5}$$

$$\sin\theta = \frac{2}{3} \qquad \csc\theta = \frac{3}{2}$$

$$\cos\theta = -\frac{\sqrt{5}}{3} \qquad \sec\theta = -\frac{3}{\sqrt{5}} = -\frac{3\sqrt{5}}{5}$$

$$\tan\theta = -\frac{2}{\sqrt{5}} = -\frac{2\sqrt{5}}{5} \qquad \cot\theta = -\frac{\sqrt{5}}{2}$$

At the beginning of the problem, the value of sine was given. Therefore, we can fill in the values of sine and cosecant right away. From that point, the other values can be found using: Soh-Cah-Toa.

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} \qquad \cos\theta = \frac{\text{adj}}{\text{hyp}} \qquad \tan\theta = \frac{\text{opp}}{\text{adj}}$$

$$\sin\theta = \frac{2}{3} \qquad \csc\theta = \frac{3}{2}$$

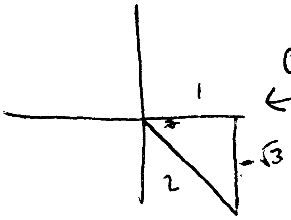
$$\cos\theta = -\frac{\sqrt{5}}{3} \qquad \sec\theta = -\frac{3\sqrt{5}}{5}$$

$$\tan\theta = -\frac{2\sqrt{5}}{5} \qquad \cot\theta = -\frac{\sqrt{5}}{2}$$

Here are the values of the six trigonometric functions of the angle θ.

Find the exact value of the six trigonometric functions of an angle θ, in standard position, given the following information.

A) $\cos\theta = \frac{1}{2}$, $\sin\theta < 0$



$\cos = \frac{adj}{hyp}$

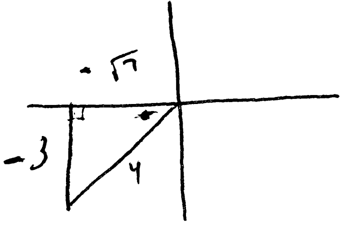
$1 + b^2 = 4$
$b^2 = 3$
$b = \sqrt{3}$

angle in quad IV → $-\sqrt{3}$

$\sin\theta = \frac{-\sqrt{3}}{2}$
$\cos\theta = \frac{1}{2}$
$\tan\theta = -\sqrt{3}$

$\csc\theta = \frac{-2}{\sqrt{3}} = \frac{-2\sqrt{3}}{3}$
$\sec\theta = 2$
$\cot\theta = \frac{-1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$

B) $\sin\theta = -\frac{3}{4}$, $\tan\theta > 0$



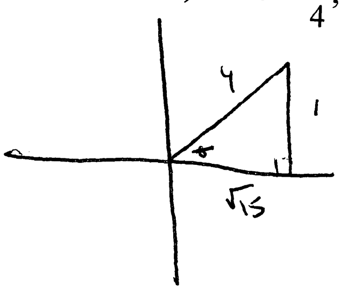
$\sin\theta = \frac{opp}{hyp}$

$9 + b^2 = 16$
$b^2 = 7$
$b = \sqrt{7}$

$\sin\theta = -3/4$
$\cos\theta = -\frac{\sqrt{7}}{4}$
$\tan\theta = \frac{-3}{-\sqrt{7}} = \frac{3\sqrt{7}}{7}$

$\csc\theta = -4/3$
$\sec\theta = -\frac{4}{\sqrt{7}} = \frac{-4\sqrt{7}}{7}$
$\cot\theta = \frac{\sqrt{7}}{3}$

C) $\sin\theta = \frac{1}{4}$, $\cos\theta > 0$



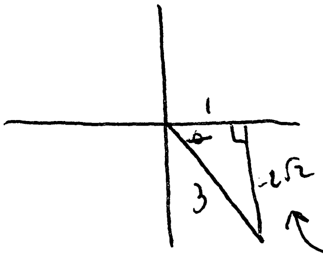
$1 + b^2 = 16$
$b^2 = 15$
$b = \sqrt{15}$

$\tan\theta = \frac{1}{\sqrt{15}}$

$\sin\theta = 1/4$
$\cos\theta = \frac{\sqrt{15}}{4}$
$\tan\theta = \frac{\sqrt{15}}{15}$

$\csc\theta = 4$
$\sec\theta = \frac{4}{\sqrt{15}} = \frac{4\sqrt{15}}{15}$
$\cot\theta = \sqrt{15}$

D) $\sec\theta = 3$, $\csc\theta < 0$



Angle in Quad. IV

$\cos\theta = \frac{1}{3}$

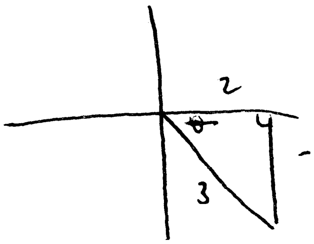
$a^2 + 1 = 9$
$a^2 = 8$
$a = 2\sqrt{2}$

$\sin\theta = \frac{-2\sqrt{2}}{3}$
$\cos\theta = \frac{1}{3}$
$\tan\theta = -2\sqrt{2}$

sin
$\csc\theta = \frac{-3}{2\sqrt{2}} = \frac{-3\sqrt{2}}{4}$
$\sec\theta = 3$
$\cot\theta = \frac{1}{-2\sqrt{2}} = \frac{-\sqrt{2}}{4}$

E) $\tan\theta = -\frac{\sqrt{5}}{2}$, $\sin\theta < 0$

$(2)^2 + (\sqrt{5})^2 = c^2$
$4 + 5 = c^2$
$c^2 = 9$
$c = 3$

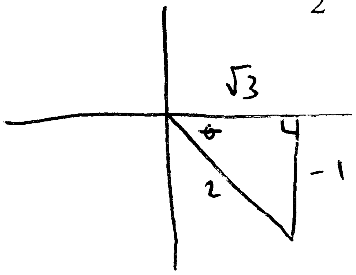


angle in quad IV → value (-)

$\sin\theta = \frac{-\sqrt{5}}{3}$
$\cos\theta = 2/3$
$\tan\theta = -\frac{\sqrt{5}}{2}$

$\csc\theta = \frac{-3}{\sqrt{5}} = \frac{-3\sqrt{5}}{5}$
$\sec\theta = 3/2$
$\cot\theta = -\frac{2}{\sqrt{5}} = \frac{-2\sqrt{5}}{5}$

F) $\cos\theta = \frac{\sqrt{3}}{2}$, $\sin\theta < 0$



$a^2 + 3 = 4$
$a^2 = 1$
$a = -1$

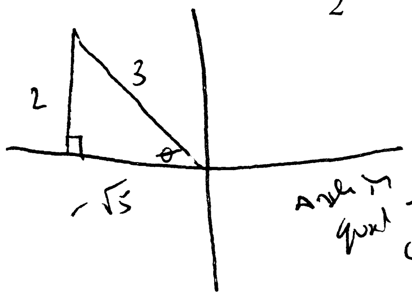
Angle in quad IV so

$\sin\theta = -1/2$
$\cos\theta = \frac{\sqrt{3}}{2}$
$\tan\theta = \frac{-1}{\sqrt{3}} = \frac{-\sqrt{3}}{3}$

$\csc\theta = -2$
$\sec\theta = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$
$\cot\theta = -\sqrt{3}$

Continued

G) $\csc\theta=\frac{3}{2},\ \cos\theta<0$



$a^2+4=9$

$a^2=5$

$a=\sqrt5$

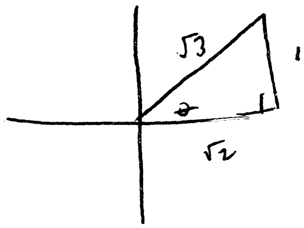
Angle in Quad II (−) x value

$\sin\theta=$ $2/3$ $\csc\theta=$ $3/2$

$\cos\theta=$ $-\sqrt5/3$ $\sec\theta=$ $\frac{-3}{\sqrt5}=\frac{-3\sqrt5}{5}$

$\tan\theta=$ $\frac{2}{-\sqrt5}=\frac{-2\sqrt5}{5}$ $\cot\theta=$ $-\frac{\sqrt5}{2}$

H) $\cot\theta=\sqrt2,\ \cos\theta>0$



$1+2=c^2$

$c^2=3$

$c=\sqrt3$

$\cos\theta=\frac{\sqrt2}{\sqrt3}\cdot\frac{\sqrt3}{\sqrt3}=\frac{\sqrt6}{3}$

$\sec\theta=\frac{\sqrt3}{\sqrt2}\cdot\frac{\sqrt2}{\sqrt2}$

$\sin\theta=$ $\frac{1}{\sqrt3}=\frac{\sqrt3}{3}$ $\csc\theta=$ $\sqrt3$

$\cos\theta=$ $\frac{\sqrt6}{3}$ $\sec\theta=$ $\sqrt6/2$

$\tan\theta=$ $\frac{1}{\sqrt2}=\frac{\sqrt2}{2}$ $\cot\theta=$ $\sqrt2$

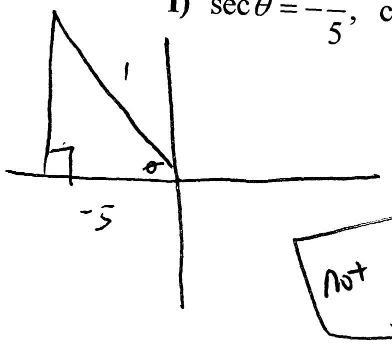
I) $\sec\theta=-\frac{1}{5},\ \cot\theta<0$

$a^2+(-5)^2=(1)^2$

$a^2+25=1$

$a^2=-24$

Not possible
No solution



$\sin\theta=$ $\csc\theta=$

$\cos\theta=$ $\sec\theta=$

$\tan\theta=$ $\cot\theta=$

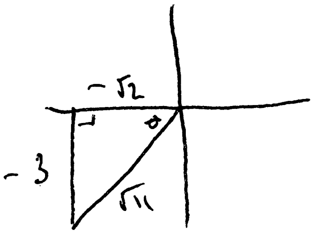
J) $\cot\theta=\frac{\sqrt2}{3},\ \sin\theta<0$

$(3)^2+(\sqrt2)^2=c^2$

$9+2=c^2$

$c^2=11$

$c=\sqrt{11}$

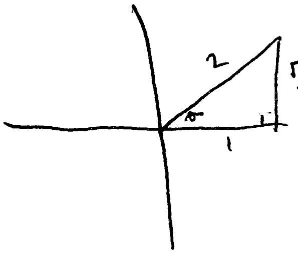


$\sin\theta=$ $\frac{-3}{\sqrt{11}}=\frac{-3\sqrt{11}}{11}$ $\csc\theta=$ $-\frac{\sqrt{11}}{3}$

$\cos\theta=$ $-\frac{\sqrt2}{\sqrt{11}}=\frac{-\sqrt{22}}{11}$ $\sec\theta=$ $-\frac{\sqrt{11}}{\sqrt2}=\frac{-\sqrt{22}}{2}$

$\tan\theta=$ $\frac{3}{\sqrt2}=\frac{3\sqrt2}{2}$ $\cot\theta=$ $\frac{\sqrt2}{3}$

K) $\csc\theta=\frac{2\sqrt3}{3},\ \cos\theta>0$



$a^2+3=4$

$a^2=1$

$a=1$

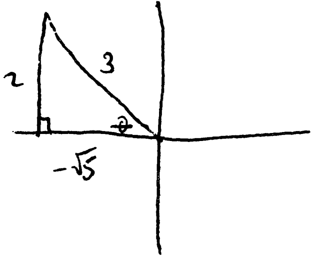
$\frac{3}{2\sqrt3}\cdot\frac{\sqrt3}{\sqrt3}=\frac{\sqrt3}{2}$

$\sin\theta=$ $\frac{\sqrt3}{2}$ $\csc\theta=$ $\frac{2\sqrt3}{3}$

$\cos\theta=$ $1/2$ $\sec\theta=$ 2

$\tan\theta=$ $\sqrt3$ $\cot\theta=$ $\frac{1}{\sqrt3}=\frac{\sqrt3}{3}$

L) $\sec\theta=-\frac{3\sqrt5}{5},\ \tan\theta<0$



$a^2+5=9$

$a^2=4$

$a=2$

$-\frac{\cancel{5}}{3\cancel{\sqrt5}}\cdot\frac{\sqrt5}{\sqrt5}=\frac{-\sqrt5}{3}$

$\sin\theta=$ $2/3$ $\csc\theta=$ $3/2$

$\cos\theta=$ $-\frac{\sqrt5}{3}$ $\sec\theta=$ $-\frac{3\sqrt5}{5}$

$\tan\theta=$ $\frac{2}{-\sqrt5}=\frac{-2\sqrt5}{5}$ $\cot\theta=$ $-\frac{\sqrt5}{2}$

Answer the following questions, keeping the following information in mind.

$$\sin\theta = \frac{y}{h} \qquad \csc\theta = \frac{h}{y}$$

$$\cos\theta = \frac{x}{h} \qquad \sec\theta = \frac{h}{x}$$

$$\tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{x}{y}$$

Describe all angles that satisfy the questions below in the interval $0° \le \theta \le 360°$.

What angle would cause the value of $\tan\theta$ to be undefined?

90° or 270° for each, the terminal side is ~~on~~ the y axis ⇒ x value is zero

What angle would cause the value of $\tan\theta$ equal zero?

terminal side on x axis, ⇒ all y values zero

0°, 180°, 360°

What angle would cause $\cot\theta$ to be undefined?

0°, 180°, 360° since $\cot\theta = \frac{x}{y}$ y values are zero here

What angle would cause the value of $\cot\theta$ to be equal to zero?

90° or 270° $\cot\theta = \frac{x}{y}$ when x values are zero $\cot\theta = 0$

What angle would cause the value of $\sin\theta$ to be equal to zero?

0°, 180°, 360° $\sin\theta = \frac{y}{h} \rightarrow$ the y-values of the terminal sides of these angles are zero

What angle would cause the value of $\csc\theta$ to be undefined?

0°, 180°, 360° $\csc\theta = \frac{h}{y} \leftarrow$ when y value is zero the function is undefined.

What angle would cause the value of $\cos\theta$ to be equal to zero.

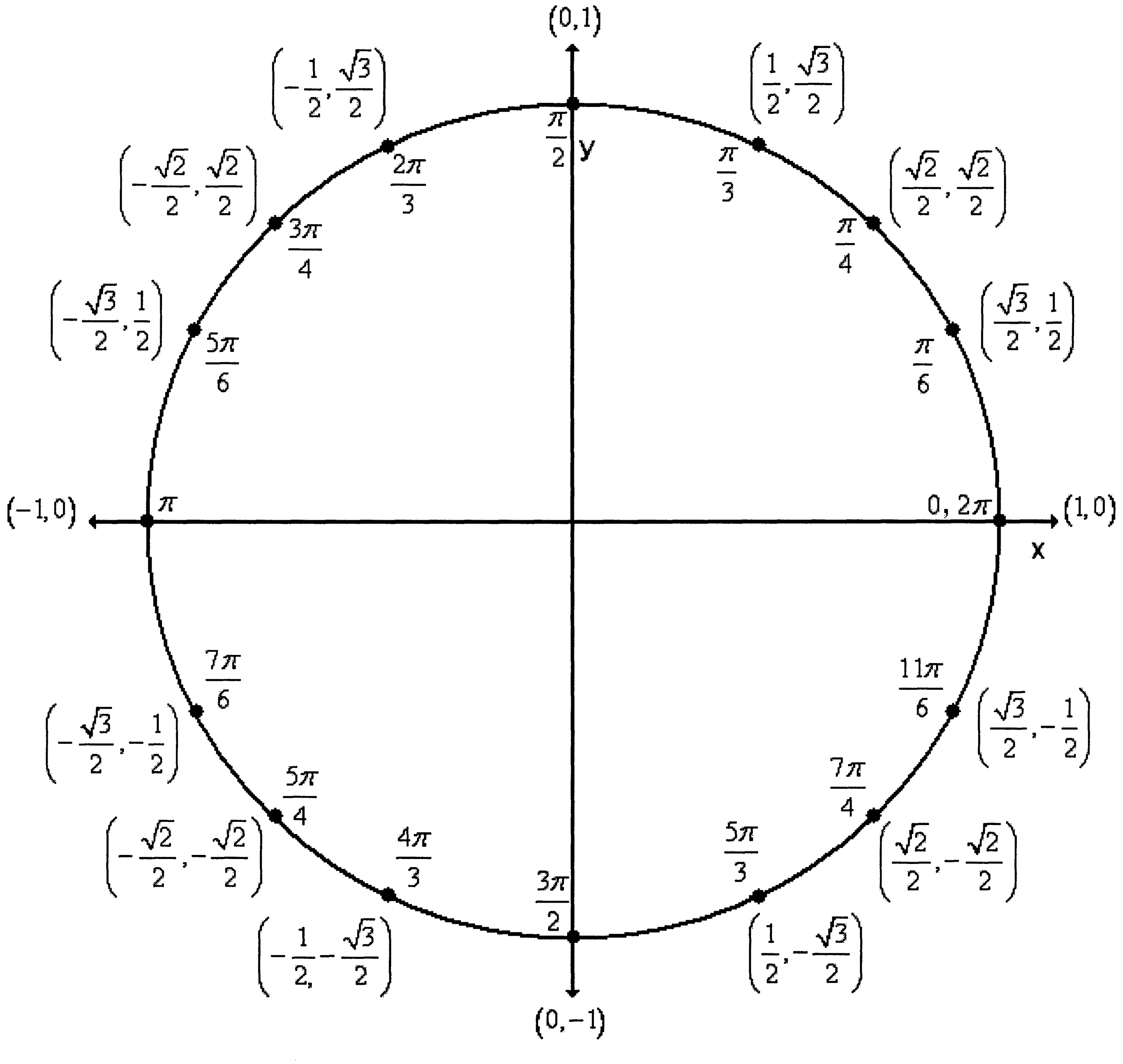
90°, 270° $\cos\theta = \frac{x}{h} \longleftarrow$ when x-values are zero the value of the function is zero,

What angle would cause the value of $\sec\theta$ to be undefined?

90°, 270° $\sec\theta = \frac{h}{x} \leftarrow$ when x value is zero the f value of the function is undefined

The Unit Circle

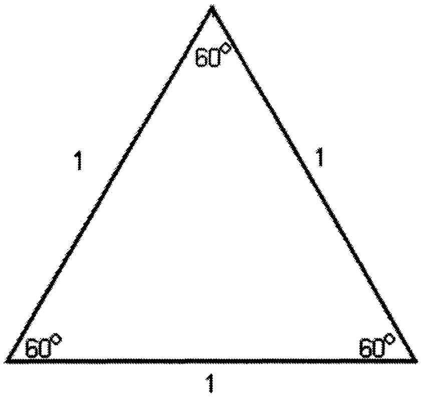
The unit circle is without a doubt the most critical topic a student must understand in trigonometry. The unit circle is the foundation on which trigonometry is based. If someone were to look at the unit circle and try to memorize it, they may find it difficult. In this section, we will discuss how to construct the unit circle, and exactly where those numbers on the unit circle come from.



This is called the unit circle, because the radius of the circle is exactly one unit. The numbers on the outside of the circle represent coordinates. These will be the x and y values with which various trigonometric functions can be evaluated. The numbers on the inside represent the radian measure of the angle. The construction of the unit circle entails the use of a conversion formula, and two different triangles. The two triangle used in the construction of a unit circle are a 30°-60°-90° right triangle, and a 45°-45°-90° right triangle. The lengths of the sides of the 30°-60°-90° triangle can be derived from a standard equilateral triangle.

The 30°-60°-90° triangle

To the right is a standard equilateral triangle. In this particular triangle, the length of each side is one unit.



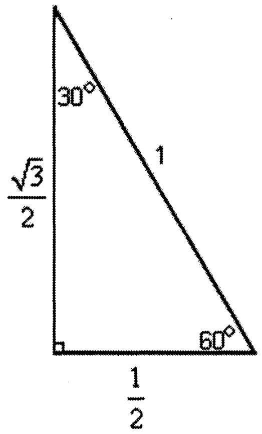
If the equilateral triangle is bisected, a 30°-60°-90° right triangle is formed. Since the triangle is bisected, the base is cut in half. The find the height of the triangle, use the Pythagorean Theorem.

$$x^2 + \left(\frac{1}{2}\right)^2 = 1^2$$

$$x^2 + \frac{1}{4} = 1$$

$$x^2 = \frac{3}{4}$$

$$x = \frac{\sqrt{3}}{2}$$



As a result, the side opposite the 60° angle has a length of $\frac{\sqrt{3}}{2}$ *units, while the side opposite the 30° angle has a length of* $\frac{1}{2}$ *units. The hypotenuse was never touched, so the length of the hypotenuse remains 1 unit.*

The 45°-45°-90° triangle

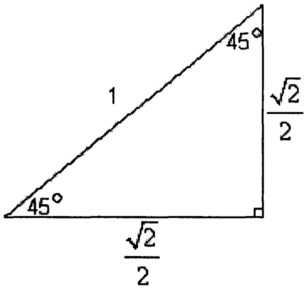
The lengths of the legs of a 45°-45°-90° triangle can be found using the Pythagorean Theorem. Since this is an isosceles triangle, the length of the two legs are equal to each other.

$$x^2 + x^2 = 1^2$$

$$2x^2 = 1$$

$$x^2 = \frac{1}{2}$$

$$x = \frac{\sqrt{2}}{2}$$



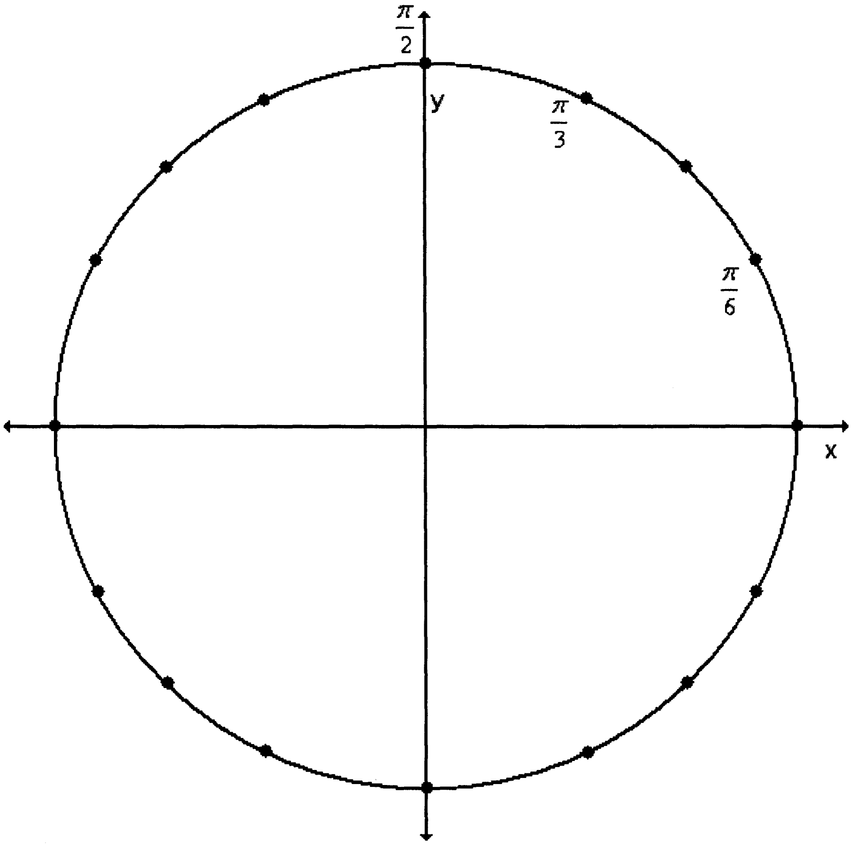
When dealing with a 45°-45°-90° triangle, the length of the sides opposite the 45° angles is $\frac{\sqrt{2}}{2}$

Building the Unit Circle

The first objective when building the unit circle is to use the conversion formula to find out the radian measures for a 30° angle, and a 45° angle. All of the angles used on the unit circle are multiples of the 30° angle and the 45° angle. Therefore, all that is needed it to add the required set. In other words, 120° is made up by 4 30° angles. A 30° angle is $\frac{\pi}{6}$ radians. Adding four of these together results in $\frac{4\pi}{6}$ radians which reduces to $\frac{2\pi}{3}$.

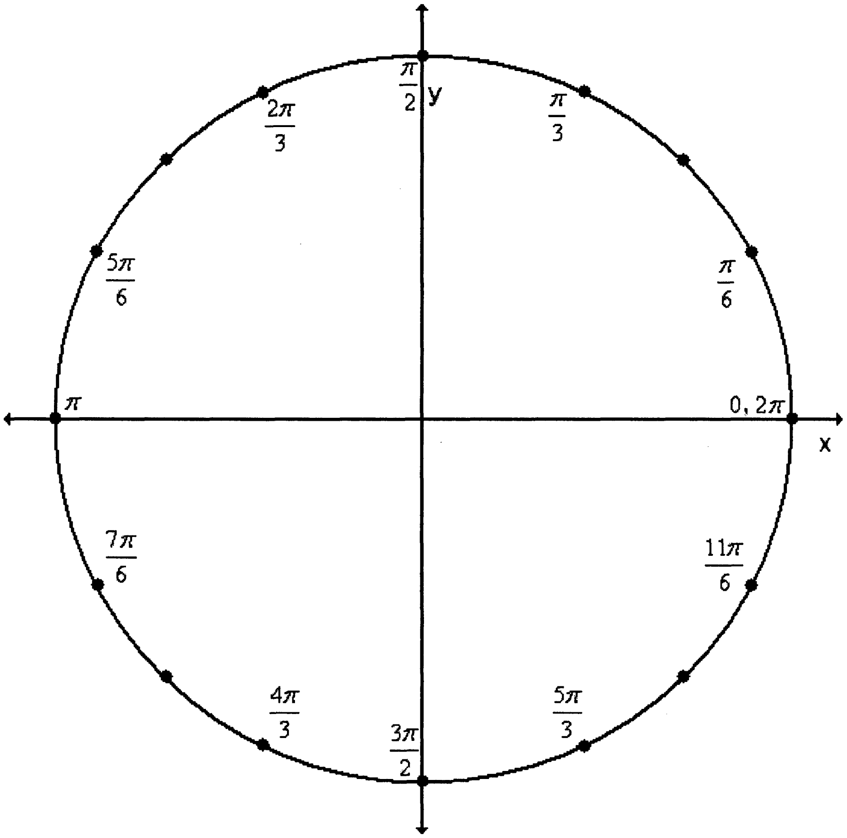
$30° \cdot \frac{\pi}{180°}$	$45° \cdot \frac{\pi}{180°}$
$\frac{30\pi}{180}$	$\frac{45\pi}{180}$
$\frac{\pi}{6}$	$\frac{\pi}{4}$

Using the conversion formula, a 30° angle is $\frac{\pi}{6}$ radians, and a 45° angle is $\frac{\pi}{4}$ radians.



Begin at the 30° angle. Place $\frac{\pi}{6}$ at that location and move around the circle in a counterclockwise direction adding by $\frac{\pi}{6}$ at every 30° increment. Make sure to reduce the totals when possible. For example, in the above diagram, to find the radian measure for 60°, add $\frac{\pi}{6}$ together twice. The results in $\frac{2\pi}{6}$ which is reduced to $\frac{\pi}{3}$.

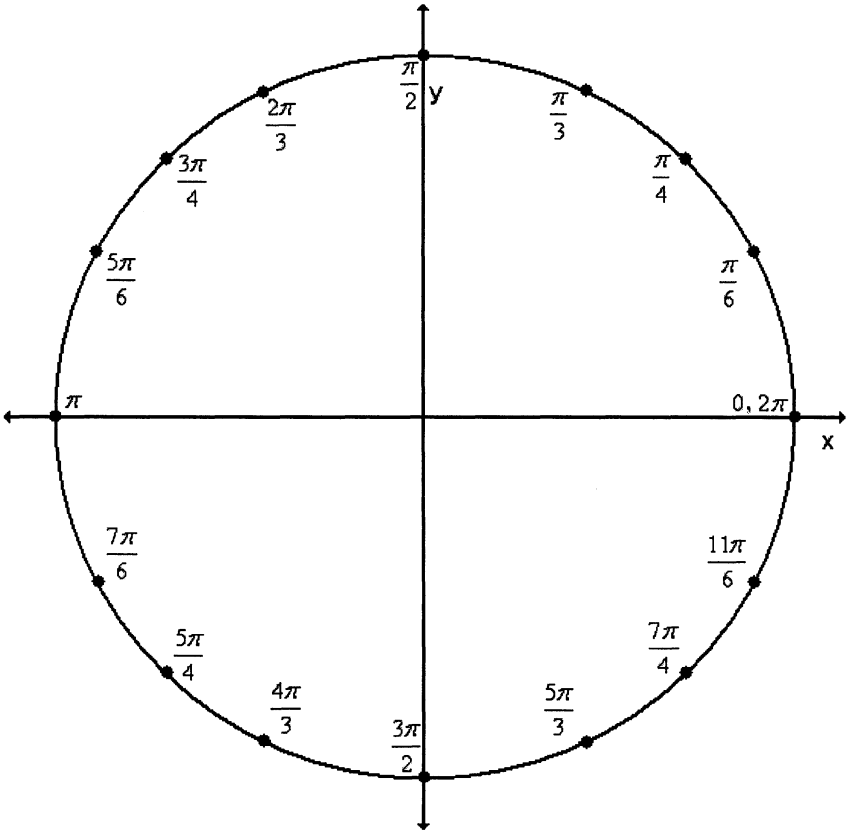
In the diagram on the right, all multiples of the 30° angles have been completed.



The next step is to continue moving counterclockwise, inputting all 45° angles.

A 45° angle is $\frac{\pi}{4}$ *radians.*

Remember to reduce any fractions if possible.

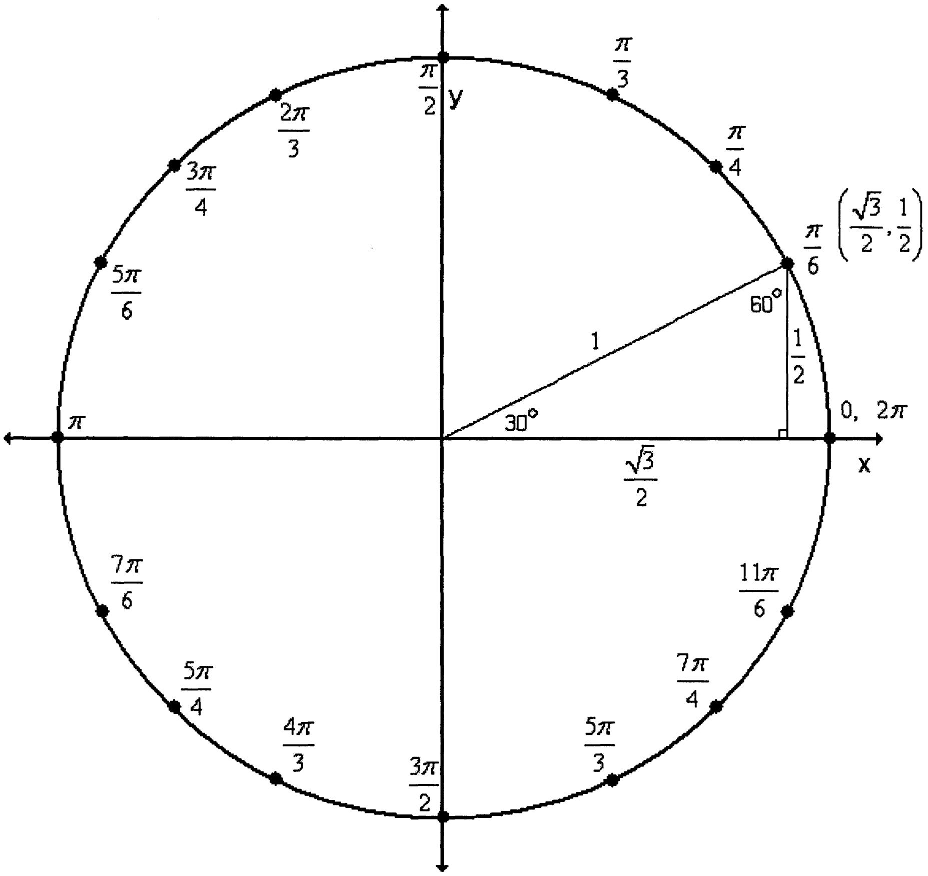


If a diagonal is drawn at the 30° angle, it will intercept the circle the first point. From there, a line is drawn to the x axis. This creates a 30°-60°-90° triangle. As demonstrated before:

The side opposite the 30° angle has a length of $\frac{1}{2}$.

The side opposite the 60° angle has a length of $\frac{\sqrt{3}}{2}$.

Since the length of the hypotenuse is 1 unit, we have found the x and y values at $\frac{\pi}{6}$. This yields to coordinates $\left(\frac{\sqrt{3}}{2},\frac{1}{2}\right)$.

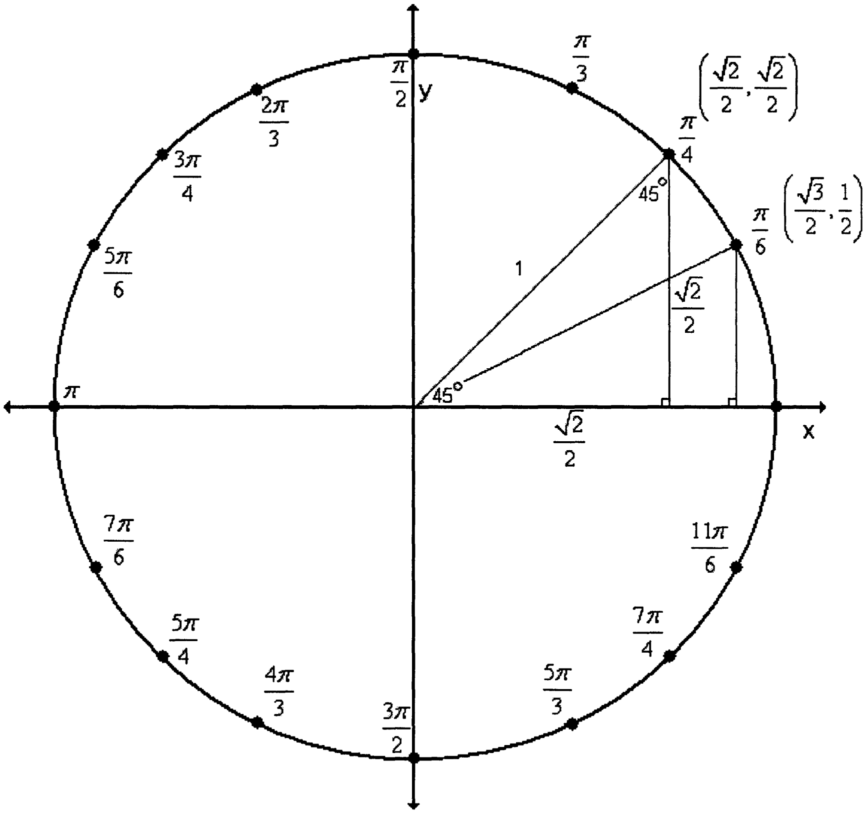


In this instance, a diagonal is drawn at the 45° angle to intercept the arc of the circle. The second line is then drawn to the x axis creating a 45°-45°-90° triangle.

The hypotenuse of the triangle has a length of 1 unit.

The length of the legs of the 45°-45°-90° triangle is $\frac{\sqrt{2}}{2}$ units.

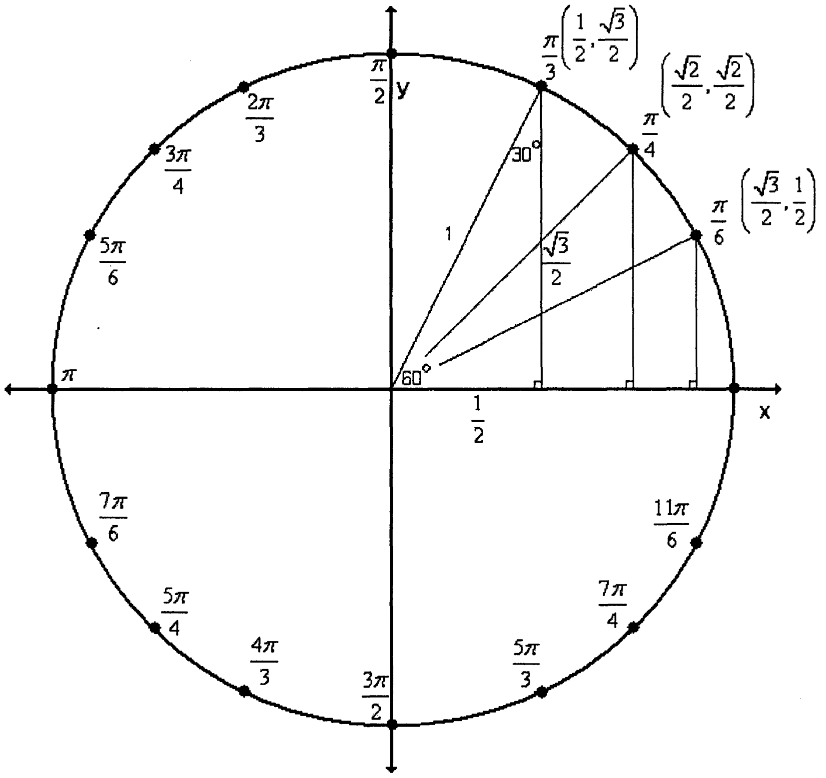
The x and y values here can be put together to find the coordinates of $\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)$.



The same process is done for $\frac{\pi}{3}$.

Once again, a triangle is formed from which the lengths of the sides of the triangle are determined.

The coordinates here are $\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$.

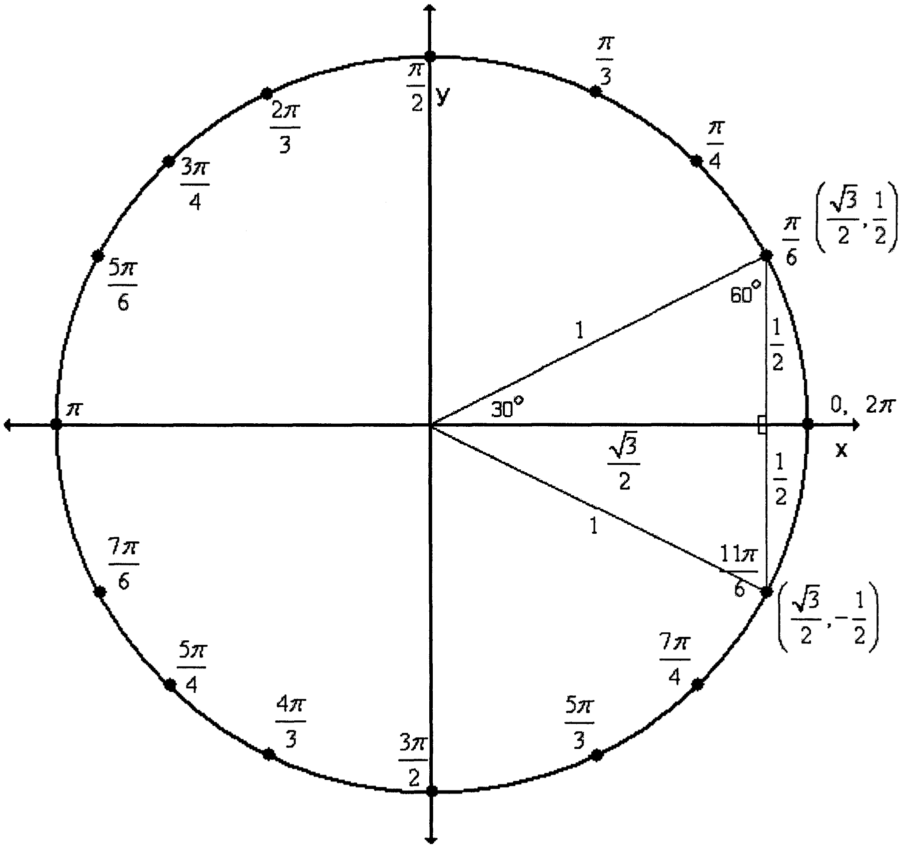


To fill in the remaining coordinates use reflections of the triangle. As illustrated here, the lengths of the sides of the triangle formed at $\frac{11\pi}{6}$ *are the same as those for* $\frac{\pi}{6}$.

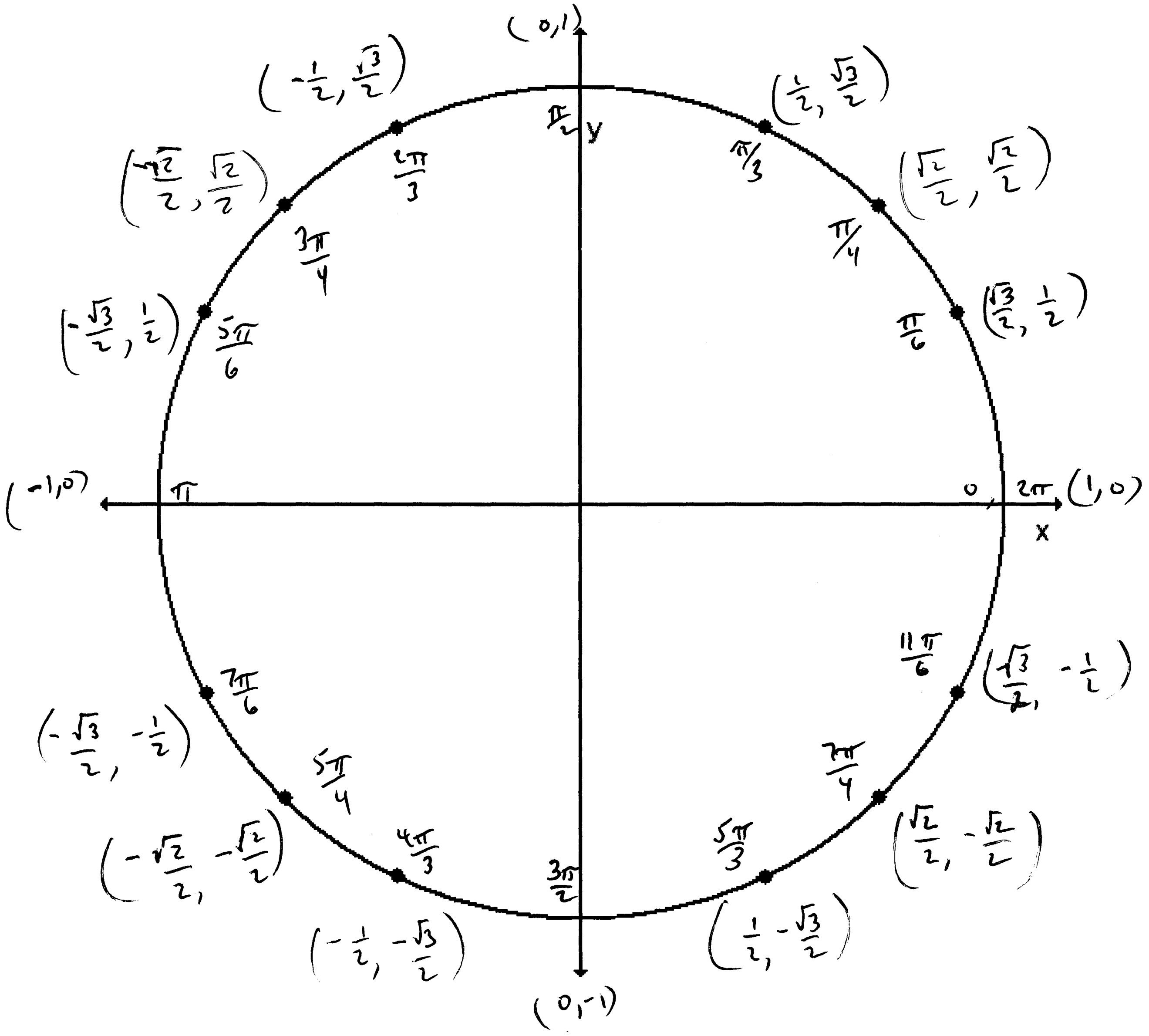
When labeling the coordinate here, however, the y value must be negative because the angle is in quadrant IV.

Once the coordinates are found in quadrant I, all others are reflections. Just take care with the sign being used.

Since the hypotenuse is always one, the coordinates on the axes are simple to find.



Complete the unit circle. Label all required radians and the coordinates for each.



Using the Unit Circle

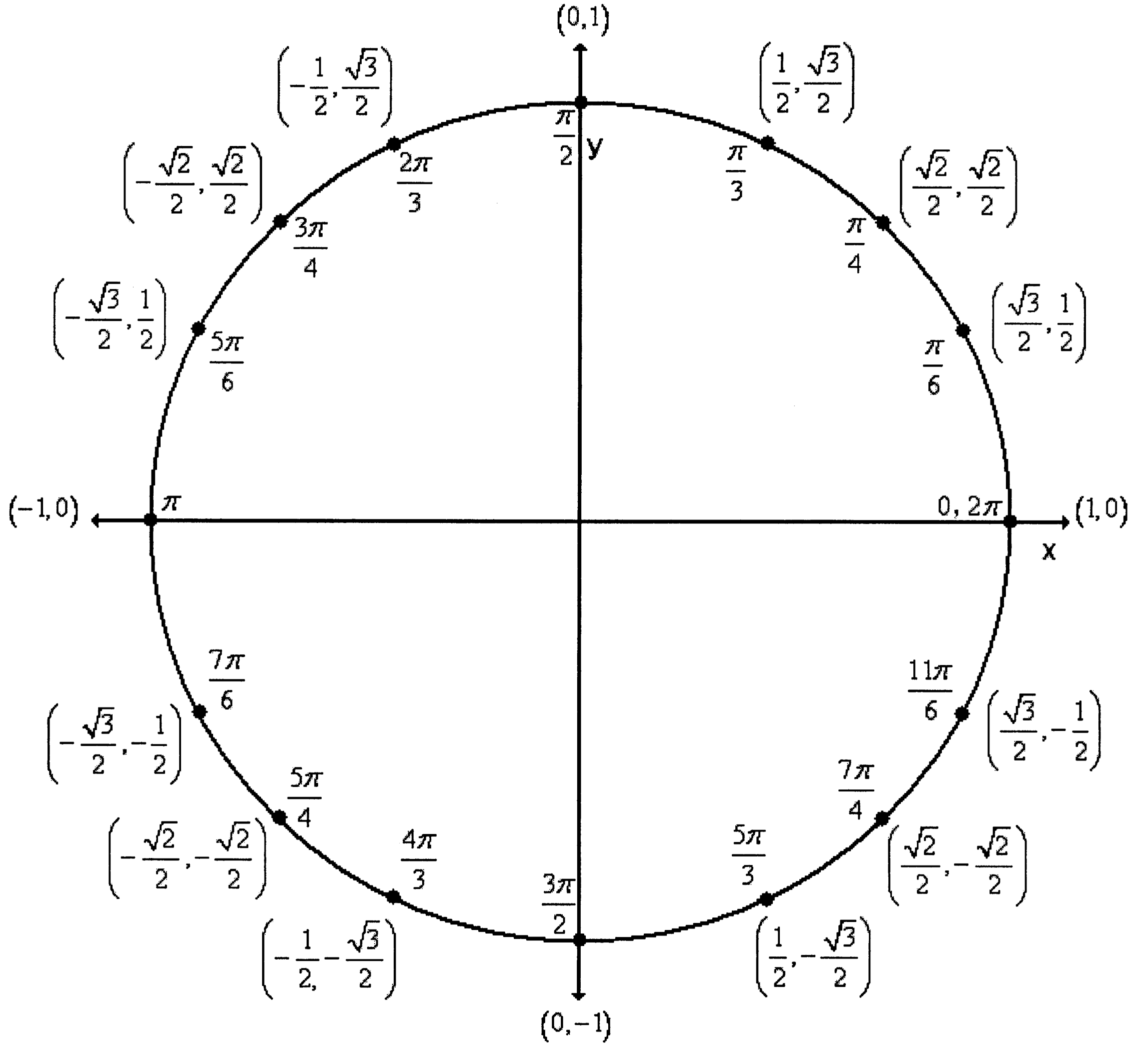
The hypotenuse of the unit circle has a length of one unit. Therefore, whenever any angle needs to be evaluated using any of the trigonometric functions, the following will be used.

$$\sin\theta = y \qquad \csc\theta = \frac{1}{y}$$

$$\cos\theta = x \qquad \sec\theta = \frac{1}{x}$$

$$\tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{x}{y}$$

Think of the sine of an angle being the y value of the coordinate, the cosine of an angle as being the x value of the coordinate, and the tangent of an angle being x over y. Then the reciprocals will be taken for the second set of functions.



When reading through the following examples, refer to the unit circle on the previous page.

Examples

Find the exact value of the six trigonometric functions for $\frac{4\pi}{3}$.

$$\sin\frac{4\pi}{3}=-\frac{\sqrt{3}}{2} \qquad \csc\frac{4\pi}{3}=-\frac{2}{\sqrt{3}}=-\frac{2\sqrt{3}}{3}$$

$$\cos\frac{4\pi}{3}=-\frac{1}{2} \qquad \sec\frac{4\pi}{3}=-2$$

$$\tan\frac{4\pi}{3}=\left(-\frac{\sqrt{3}}{2}\div-\frac{1}{2}\right)=\left(-\frac{\sqrt{3}}{2}\cdot-2\right)=\sqrt{3} \qquad \cot\frac{4\pi}{3}=\frac{1}{\sqrt{3}}=\frac{\sqrt{3}}{3}$$

$$\sin\frac{4\pi}{3}=-\frac{\sqrt{3}}{2} \qquad \csc\frac{4\pi}{3}=-\frac{2\sqrt{3}}{3}$$

$$\cos\frac{4\pi}{3}=-\frac{1}{2} \qquad \sec\frac{4\pi}{3}=-2$$

$$\tan\frac{4\pi}{3}=\sqrt{3} \qquad \cot\frac{4\pi}{3}=\frac{\sqrt{3}}{3}$$

Locate the coordinates at $\frac{4\pi}{3}$. The y value at $\frac{4\pi}{3}$ is $-\frac{\sqrt{3}}{2}$. The x value at $\frac{4\pi}{3}$ is $-\frac{1}{2}$.

Therefore, sin $\frac{4\pi}{3}=-\frac{\sqrt{3}}{2}$, and cos $\frac{4\pi}{3}=-\frac{1}{2}$.

Tangent is y over x, so the quotient of the two is found. The remaining three are evaluated using the reciprocal. All denominators <u>must</u> be rationalized. The exact value of the function means do not use decimal approximations..

Find the exact value of the six trigonometric functions for $-\frac{11\pi}{6}$.

$$\sin-\frac{11\pi}{6}=\frac{1}{2} \qquad \csc-\frac{11\pi}{6}=2$$

$$\cos-\frac{11\pi}{6}=\frac{\sqrt{3}}{2} \qquad \sec-\frac{11\pi}{6}=\frac{2}{\sqrt{3}}=\frac{2\sqrt{3}}{3}$$

$$\tan-\frac{11\pi}{6}=\left(\frac{1}{2}\div\frac{\sqrt{3}}{2}\right)=\left(\frac{1}{2}\cdot\frac{2}{\sqrt{3}}\right)=\frac{1}{\sqrt{3}}=\frac{\sqrt{3}}{3} \qquad \cot-\frac{11\pi}{6}=\sqrt{3}$$

$$\sin-\frac{11\pi}{6}=\frac{1}{2} \qquad \csc-\frac{11\pi}{6}=2$$

$$\cos-\frac{11\pi}{6}=\frac{\sqrt{3}}{2} \qquad \sec-\frac{11\pi}{6}=\frac{2\sqrt{3}}{3}$$

$$\tan-\frac{11\pi}{6}=\frac{\sqrt{3}}{3} \qquad \cot-\frac{11\pi}{6}=\sqrt{3}$$

In this case, $-\frac{11\pi}{6}$ is located in quadrant I. Moving in a clockwise direction, it is evident that $-\frac{11\pi}{6}$ is the same as $\frac{\pi}{6}$. This can also be found using coterminal angles. If we add 2π to $-\frac{11\pi}{6}$, the result is $\frac{\pi}{6}$. From this point, evaluate the six trigonometric functions.

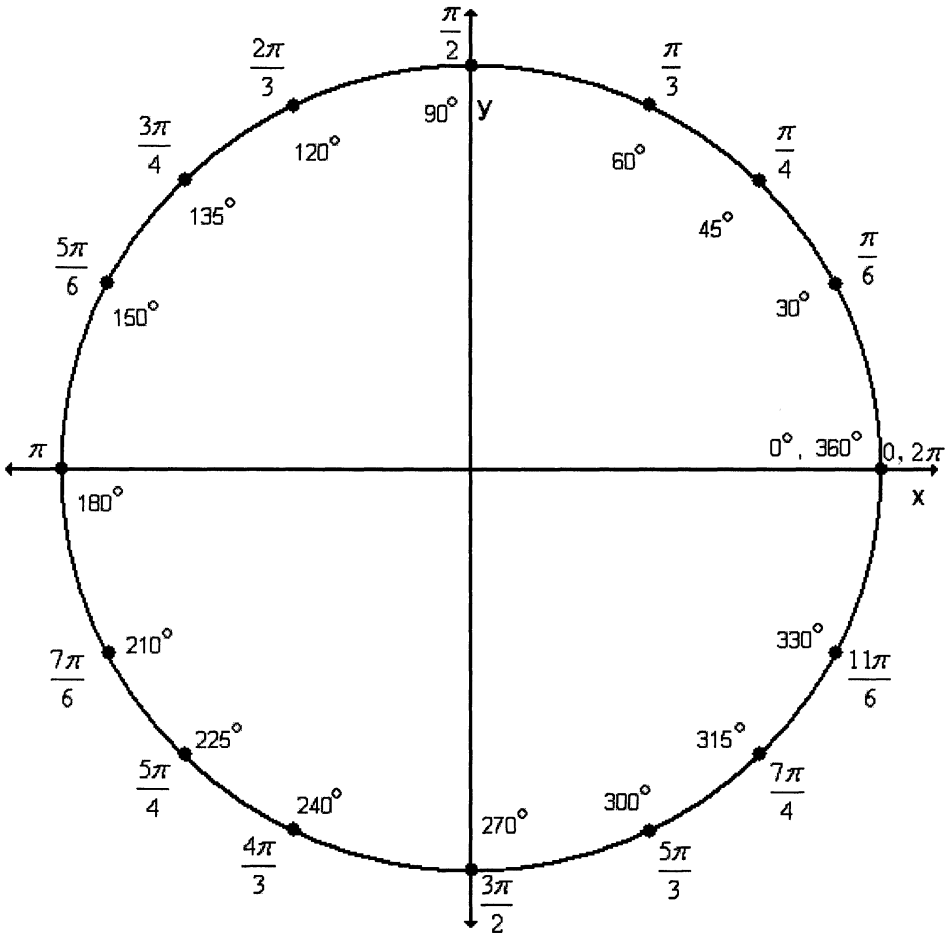
Find the exact value of the six trigonometric functions for $\frac{19\pi}{6}$

$$\sin\frac{19\pi}{6}=-\frac{1}{2} \qquad \csc\frac{19\pi}{6}=-2$$

$$\cos\frac{19\pi}{6}=-\frac{\sqrt{3}}{2} \qquad \sec\frac{19\pi}{6}=-\frac{2\sqrt{3}}{3}$$

$$\tan\frac{19\pi}{6}=\frac{\sqrt{3}}{3} \qquad \cot\frac{19\pi}{6}=\sqrt{3}$$

In this example, it is obvious that $\frac{19\pi}{6}$ is greater than 2π. This is called a periodic function. This means the angle makes at least one complete revolution before coming to rest. To find the angle that must be used, in this case, subtract 2π from $\frac{19\pi}{6}$. The result of this operation is $\frac{7\pi}{6}$. Therefore, in order to find the exact value of the six trigonometric functions of $\frac{19\pi}{6}$ use the angle $\frac{7\pi}{6}$.



Use the unit circle above to find the exact value of the six trigonometric functions for each of the following angles.

A) $\frac{3\pi}{4}$

$\sin\theta = \frac{\sqrt{2}}{2}$

$\csc\theta = \frac{1}{\sqrt{2}} = \frac{\cancel{2}\sqrt{2}}{\cancel{2}} = \sqrt{2}$

$\cos\theta = -\frac{\sqrt{2}}{2}$

$\sec\theta = -\sqrt{2}$

$\tan\theta = -1$

$\cot\theta = -1$

B) 300°

$\sin\theta = \frac{-\sqrt{3}}{2}$

$\csc\theta = -\frac{2}{\sqrt{3}} = \frac{-2\sqrt{3}}{3}$

$\cos\theta = 1/2$

$\sec\theta = 2$

$\tan\theta = -\sqrt{3}$

$\cot\theta = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$

$-\frac{\sqrt{3}}{2} \div \frac{1}{2} = \frac{-\sqrt{3}}{\cancel{2}} \cdot \frac{\cancel{2}}{1} = -\sqrt{3}$

C) $-\frac{5\pi}{6}$

$\sin\theta = -\frac{1}{2}$ $\csc\theta = -2$

$\cos\theta = -\frac{\sqrt{3}}{2}$ $\sec\theta = -\frac{2}{\sqrt{3}} = -\frac{2\sqrt{3}}{3}$

$\tan\theta = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$ $\cot\theta = \sqrt{3}$

$-\frac{1}{2} \div -\frac{\sqrt{3}}{2} = -\frac{1}{\cancel{2}} \cdot -\frac{\cancel{2}}{\sqrt{3}} = \frac{1}{\sqrt{3}}$

D) $\frac{2\pi}{3}$

$\sin\theta = \frac{\sqrt{3}}{2}$ $\csc\theta = \frac{2\sqrt{3}}{3}$

$\cos\theta = -\frac{1}{2}$ $\sec\theta = -2$

$\tan\theta = -\sqrt{3}$ $\cot\theta = -\frac{\sqrt{3}}{3}$

E) $\frac{13\pi}{3} = \overset{4\pi}{\frac{12\pi}{3}} + \frac{\pi}{3}$

$\sin\theta = \sqrt{3}/2$ $\csc\theta = \frac{2\sqrt{3}}{3}$

$\cos\theta = 1/2$ $\sec\theta = 2$

$\tan\theta = \sqrt{3}$ $\cot\theta = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

$\frac{\sqrt{3}}{2} \div \frac{1}{2} = \frac{\sqrt{3}}{\cancel{2}} \cdot \frac{\cancel{2}}{1} = \sqrt{3}$

F) $-240°$

$\sin\theta = \sqrt{3}/2$ $\csc\theta = 2\sqrt{3}/3$

$\cos\theta = -1/2$ $\sec\theta = -2$

$\tan\theta = -\sqrt{3}$ $\cot\theta = -\frac{\sqrt{3}}{3}$

G) $-\frac{7\pi}{2} = -\overset{-2\pi}{\frac{4\pi}{2}} - \frac{3\pi}{2}$

$\sin\theta = 1$ $\csc\theta = 1$

$\cos\theta = 0$ $\sec\theta =$ undef

$\tan\theta =$ undef. $\cot\theta = 0$

H) $135°$

$\sin\theta = \frac{\sqrt{2}}{2}$ $\csc\theta = \frac{2}{\sqrt{2}} = \frac{2\sqrt{2}}{2} = \sqrt{2}$

$\cos\theta = -\frac{\sqrt{2}}{2}$ $\sec\theta = -\sqrt{2}$

$\tan\theta = -1$ $\cot\theta = -1$

I) $\frac{13\pi}{6} = \overset{2\pi}{\frac{12\pi}{6}} + \frac{\pi}{6}$

$\sin\theta = 1/2$ $\csc\theta = 2$

$\cos\theta = \sqrt{3}/2$ $\sec\theta = \frac{2\sqrt{3}}{3}$

$\tan\theta = \frac{\sqrt{3}}{3}$ $\cot\theta = \sqrt{3}$

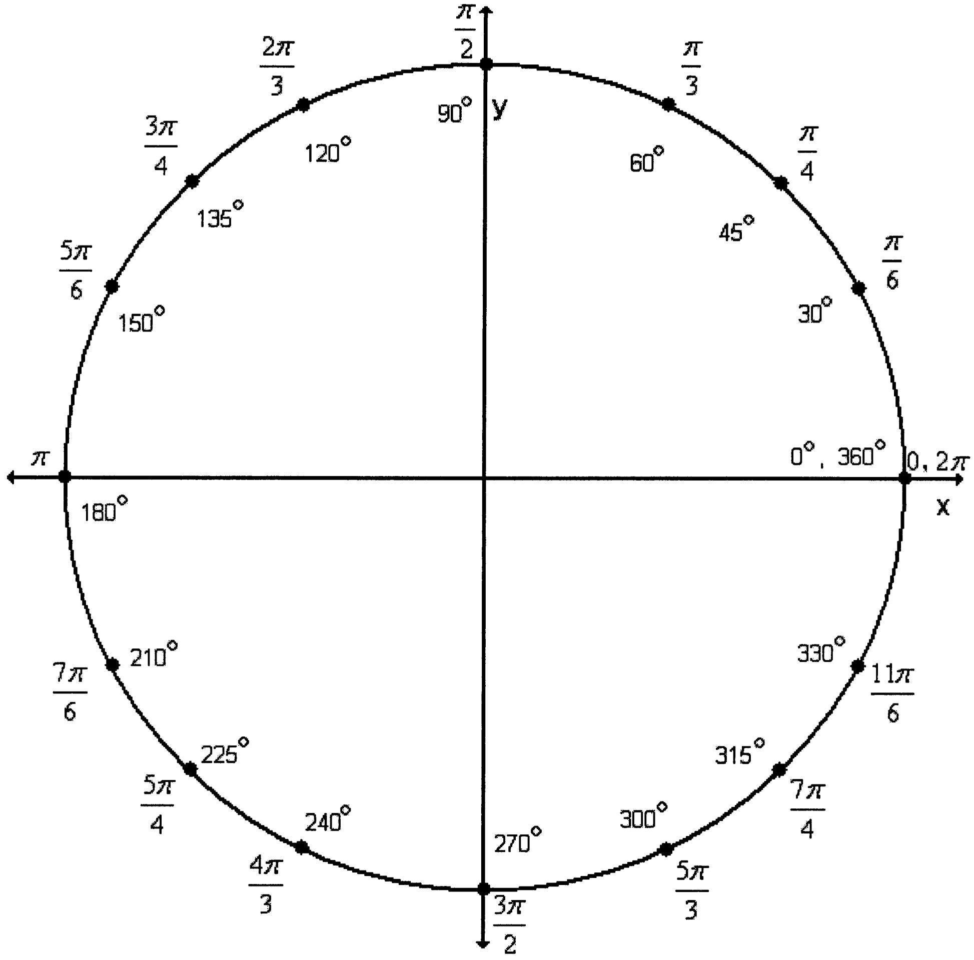
$\frac{1}{2} \div \frac{\sqrt{3}}{2} = \frac{1}{2} \cdot \frac{2}{\sqrt{3}} = \frac{1}{\sqrt{3}}$

J) $-\frac{2\pi}{3}$

$\sin\theta = -\sqrt{3}/2$ $\csc\theta = -\frac{2\sqrt{3}}{3}$

$\cos\theta = -1/2$ $\sec\theta = -2$

$\tan\theta = \sqrt{3}$ $\cot\theta = -\frac{\sqrt{3}}{3}$



Use the unit circle above to find the exact value of each of the following.

A) $\tan\frac{\pi}{4} =$ 1

$\frac{\sqrt{2}}{2} \div \frac{\sqrt{2}}{2}$

B) $\cos\frac{2\pi}{3} =$ $-\frac{1}{2}$

C) $\cos\pi =$ -1

D) $\sin\frac{11\pi}{6} =$ $-\frac{1}{2}$

E) $\tan\left(-\frac{2\pi}{3}\right) =$ $-\sqrt{3}$

$\frac{\sqrt{3}}{2} \div -\frac{1}{2} = \frac{\sqrt{3}}{\cancel{2}} \cdot -\frac{\cancel{2}}{1}$

F) $\csc\frac{\pi}{3} =$ $\frac{2\sqrt{3}}{3}$

$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$ $\qquad \frac{2}{\sqrt{3}}$

G) $\sec\frac{4\pi}{3} =$ -2

$\cos\frac{4\pi}{3} = -\frac{1}{2}$

H) $\cos\left(-\frac{11\pi}{6}\right) =$ $\frac{\sqrt{3}}{2}$

I) $\sin\frac{13\pi}{4} =$ $-\frac{\sqrt{2}}{2}$

$2\pi \searrow \frac{8\pi}{4} + \frac{5\pi}{4}$

$\sin\frac{5\pi}{4} = -\frac{\sqrt{2}}{2}$

J) $\csc\left(-\frac{5\pi}{6}\right) = \boxed{-2}$

$\sin -\frac{5\pi}{6} = -\frac{1}{2}$

K) $\tan\left(-\frac{\pi}{6}\right) = \boxed{-\frac{\sqrt{3}}{3}}$

$-\frac{1}{2} \div \frac{\sqrt{3}}{2}$

$-\frac{1}{2} \cdot \frac{2}{\sqrt{3}} = -\frac{1}{\sqrt{3}}$

L) $\cot\frac{2\pi}{3} = \boxed{-\frac{\sqrt{3}}{3}}$

$\tan\frac{2\pi}{3} = \frac{\sqrt{3}}{2} \div -\frac{1}{2}$

$\frac{\sqrt{3}}{2} \cdot -\frac{2}{1} = -\sqrt{3}$ $\quad \frac{1}{-\sqrt{3}}$

M) $\sec\left(-\frac{19\pi}{3}\right) = \boxed{2}$

$-\frac{18\pi}{3} - \frac{\pi}{3}$ $\quad \cos -\frac{\pi}{3} = \frac{1}{2}$

$\overset{6\pi}{\cancel{6\pi}} - \frac{\pi}{3}$

N) $\cot\frac{\pi}{4} = \boxed{1}$

O) $\cot\frac{11\pi}{6} = \boxed{-\sqrt{3}}$

$\tan\frac{11\pi}{6} = -\frac{1}{2} \div \frac{\sqrt{3}}{2}$

$-\frac{1}{2} \cdot \frac{2}{\sqrt{3}} = -\frac{1}{\sqrt{3}}$

P) $\cos\left(-\frac{9\pi}{2}\right) = \boxed{0}$

$-\frac{8\pi}{2} - \frac{\pi}{2}$

$-4\pi - \frac{\pi}{2}$

$\cos\left(-\frac{\pi}{2}\right)$

Q) $\sin\frac{21\pi}{4} = \boxed{-\frac{\sqrt{2}}{2}}$

$\frac{16\pi}{4} + \frac{5\pi}{4}$

$4\pi + \frac{5\pi}{4}$

$\sin\frac{5\pi}{4}$

R) $\cot\frac{7\pi}{4} = \boxed{-1}$

S) $\sin\left(-\frac{7\pi}{6}\right) = \boxed{\frac{1}{2}}$

T) $\cot\frac{26\pi}{3} = \boxed{-\frac{\sqrt{3}}{3}}$

$\frac{24\pi}{3} + \frac{2\pi}{3}$ $\quad \frac{\sqrt{3}}{2} \div -\frac{1}{2}$

$8\pi + \frac{2\pi}{3}$ $\quad \frac{\sqrt{3}}{2} \cdot -\frac{2}{1} = -\sqrt{3}$

$\tan\frac{2\pi}{3} \longrightarrow$

U) $\cos\frac{\pi}{3} = \boxed{\frac{1}{2}}$

V) $\sec\left(-\frac{11\pi}{6}\right) = \boxed{\frac{2\sqrt{3}}{3}}$

$\cos -\frac{11\pi}{6} = \frac{\sqrt{3}}{2}$

W) $\sin(-7\pi) = \boxed{0}$

$-6\pi - \pi$

$\sin -\pi$

X) $\cot 2\pi = \boxed{\text{undef.}}$

$\cot\theta = \frac{x}{y}$

$\cot 2\pi = \frac{1}{0}$

Y) $\cot\left(-\frac{2\pi}{3}\right) = \boxed{\frac{\sqrt{3}}{3}}$

$\tan -\frac{2\pi}{3} = -\frac{\sqrt{3}}{2} \div -\frac{1}{2}$

$-\frac{\sqrt{3}}{2} \cdot -\frac{2}{1} = \sqrt{3}$

Z) $\csc\frac{5\pi}{3} = \boxed{-\frac{2\sqrt{3}}{3}}$

$\sin\frac{5\pi}{3} = -\frac{\sqrt{3}}{2}$

a) $\sec\left(-\frac{3\pi}{2}\right) = \boxed{\text{undef.}}$

$\cos\left(-\frac{3\pi}{2}\right) = 0$

b) $\sec\left(-\frac{23\pi}{6}\right) = \boxed{\frac{2\sqrt{3}}{3}}$

$-\frac{12\pi}{6} - \frac{11\pi}{6}$

$-2\pi - \frac{11\pi}{6}$

$\cos -\frac{11\pi}{6} = \frac{\sqrt{3}}{2}$

c) $\tan\frac{3\pi}{4} = \boxed{-1}$

d) $\csc\frac{7\pi}{6} = \boxed{-2}$

$\sin\frac{7\pi}{6} = -\frac{1}{2}$

e) $\sin\frac{14\pi}{3} = \boxed{\frac{\sqrt{3}}{2}}$

$\frac{12\pi}{3} + \frac{2\pi}{3}$

$4\pi + \frac{2\pi}{3}$

$\sin\frac{2\pi}{3}$

f) $\cos\left(-\frac{17\pi}{6}\right) = \boxed{-\frac{\sqrt{3}}{2}}$

$-\frac{12\pi}{6} - \frac{5\pi}{6}$

-2π

$\cos\left(-\frac{5\pi}{6}\right)$

g) $\cot\frac{3\pi}{2} = \boxed{0}$

$\cot\theta = \frac{x}{y}$

$\cot\frac{3\pi}{2} = \frac{0}{-1}$

It is also possible to use the unit circle going backwards. The previous exercises require a student to evaluate the trigonometric function of an angle using the unit circle. The samples below require the student to work backwards. Given the value of the trigonometric function of an angle θ, refer to the unit circle, and find the angle θ that makes the statement true. Given a statement such as $\sin\theta = \frac{1}{2}$, we will work backwards to try to determine the angle θ that would make the statement true.

For Example

Find all values of θ in the interval $(0, 2\pi]$ that make the statement $\cos\theta = \frac{1}{2}$ true.

Referring to the unit circle, look for x coordinates of $\frac{1}{2}$. This happens in two places, $\frac{\pi}{3}$ and $\frac{5\pi}{3}$.

As a result, $\theta = \frac{\pi}{3}, \frac{5\pi}{3}$

Find all values of θ in the interval $(0, 2\pi]$ that make the statement $\tan\theta = -\sqrt{3}$ true.

The only coordinate that has $\sqrt{3}$ in it is $\frac{\sqrt{3}}{2}$. That rules out any and all of the 45° angles or multiples thereof.

It would therefore follow, that the $\sqrt{3}$ is a result of either $\left(\frac{\sqrt{3}}{2} \div \frac{1}{2}\right)$ or $\left(\frac{1}{2} \div \frac{\sqrt{3}}{2}\right)$.

Working backwards will reveal what the result is.

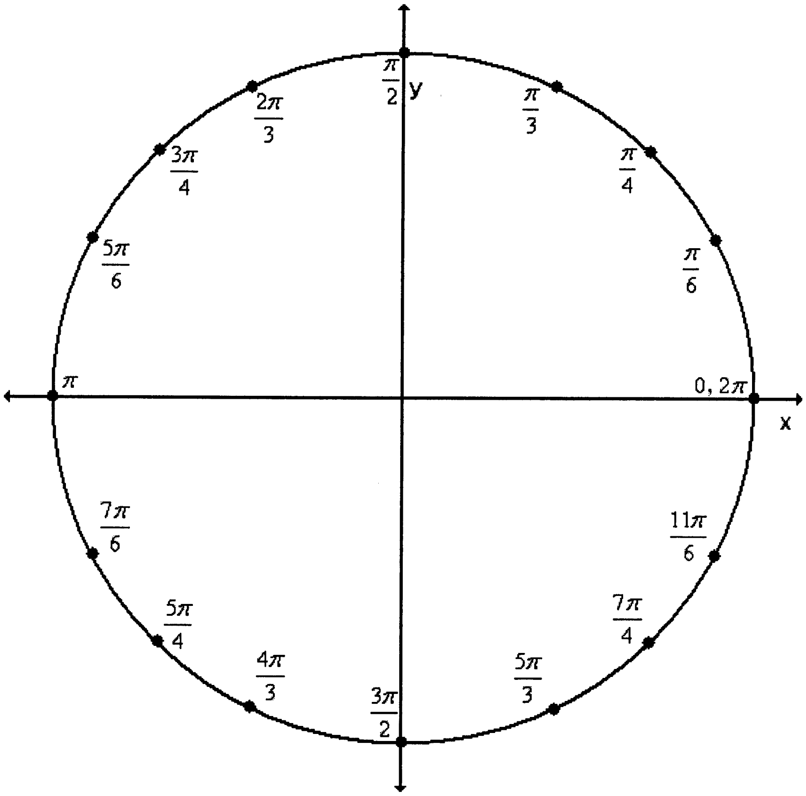
$$\left(\frac{\sqrt{3}}{2} \div \frac{1}{2}\right) = \left(\frac{\sqrt{3}}{2} \cdot \frac{2}{1}\right) = \sqrt{3} \qquad \left(\frac{1}{2} \div \frac{\sqrt{3}}{2}\right) = \left(\frac{1}{2} \div \frac{2}{\sqrt{3}}\right) = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

According to the work above a y value of $\frac{\sqrt{3}}{2}$ divided by an x value of $\frac{1}{2}$ would yield a result of $\sqrt{3}$.

This occurs at $\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}$ and $\frac{5\pi}{3}$.

Since tangent is y divided by x, and in this case, the tangent of θ is negative, it would stand to reason that one of the coordinates used will be a negative, while the other is a positive. As discussed earlier, the tangent of θ will be negative in quadrants II and IV.

The solution to this problem is: $\theta = \frac{2\pi}{3}, \frac{5\pi}{3}$



For each of the following, find all values of θ in the interval $(0, 2\pi]$ that make the following statements true.

A) $\sin\theta = \frac{1}{2}$ $\boxed{\frac{\pi}{6}, \frac{5\pi}{6}}$

y value $\frac{1}{2}$

B) $\cos\theta = -\frac{\sqrt{3}}{2}$ $\boxed{\frac{5\pi}{6}, \frac{7\pi}{6}}$

x value $-\frac{\sqrt{3}}{2}$

C) $\cos\theta = -1$ $\boxed{\pi}$

x value -1

D) $\sin\theta = -\frac{\sqrt{3}}{2}$ $\boxed{\frac{4\pi}{3}, \frac{5\pi}{3}}$

y value $-\frac{\sqrt{3}}{2}$

E) $\csc\theta = 2$ $\boxed{\frac{\pi}{6}, \frac{5\pi}{6}}$

$\sin\theta = \frac{1}{2}$

y value $\frac{1}{2}$

F) $\cot\theta = -1$ $\boxed{\frac{3\pi}{4}, \frac{7\pi}{4}}$

$\cot\theta = \frac{x}{y}$

$\tan\theta = -1$

tan is (-) in quad II and IV

G) $\tan\theta = \textit{undefined}$ $\boxed{\frac{\pi}{2}, \frac{3\pi}{2}}$

undef if $\frac{y}{x} = \frac{\#}{0}$

$\nearrow$ x values zero

H) $\csc\theta = -\frac{2\sqrt{3}}{3}$ $\boxed{\frac{4\pi}{3}, \frac{5\pi}{3}}$

$\sin\theta = -\frac{3}{2\sqrt{3}} = -\frac{\cancel{3}\sqrt{3}}{2\cancel{(3)}} = -\frac{\sqrt{3}}{2}$

y value $-\frac{\sqrt{3}}{2}$

I) $\tan\theta = -1$ $\boxed{\frac{3\pi}{4}, \frac{7\pi}{4}}$

tan is (-) in quad II and IV

J) $\tan\theta = \frac{\sqrt{3}}{3}$ $\boxed{\frac{\pi}{6}, \frac{7\pi}{6}}$

either

$y \div x$

$\frac{\sqrt{3}}{2} \div \frac{1}{2}$

$\frac{\sqrt{3}}{2} \cdot \frac{2}{1}$

or

$y \div x$

$\frac{1}{2} \div \frac{\sqrt{3}}{2}$

$\frac{1}{2} \cdot \frac{2}{\sqrt{3}}$

y value $\frac{1}{2}$

x value $\sqrt{3}/2$

tan is (+) in quad I and III

K) $\csc\theta = \textit{undefined}$ $\boxed{\pi, 2\pi}$

$\sin\theta = 0$

y value on x axis

0 is not included because of the domain $(0, 2\pi)$

$\uparrow$ exclusive

L) $\sin\theta = \pm\frac{1}{2}$ $\boxed{\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}}$

y value $\pm\frac{1}{2}$

Trigonometric Equations

Many of the skills used for solving algebraic equations will be used to solve trigonometric equations. Trigonometric equations are solved using inverse operations. The ultimate objective of solving trigonometric equations is to find the angle that makes the statement true.

Examples

Solve the following equations in the interval $(0, 2\pi]$.

$$2\sin\theta + 1 = 0$$

$$\begin{aligned} 2\sin\theta + 1 &= 0 \\ 2\sin\theta &= -1 \\ \sin\theta &= -\frac{1}{2} \end{aligned}$$

Begin solving a trigonometric equation by isolating the trigonometric function involved.

$$\theta = \frac{7\pi}{6}, \ \frac{11\pi}{6}$$

At this point, find all angles in the interval $(0, 2\pi]$ ***which make the equation true.***

Solve the following equations in the interval $(0, 2\pi]$.

$$\sin^2\theta - 1 = 0$$

When dealing with trigonometric equations, $\sin^2\theta$ ***is the same type of thing as*** x^2

In other words, $(\sin 30°)^2$ ***is written as***

$\sin^2 30°$.

$$\begin{aligned} \sin^2\theta - 1 &= 0 \\ \sin^2\theta &= 1 \\ \sqrt{\sin^2\theta} &= \pm\sqrt{1} \\ \sin\theta &= \pm 1 \end{aligned}$$

Solve the equation by isolating the trigonometric function, then taking the square root of both sides. Do not forget to use ± when finding the solution.

$$\theta = \frac{\pi}{2}, \ \frac{3\pi}{2}$$

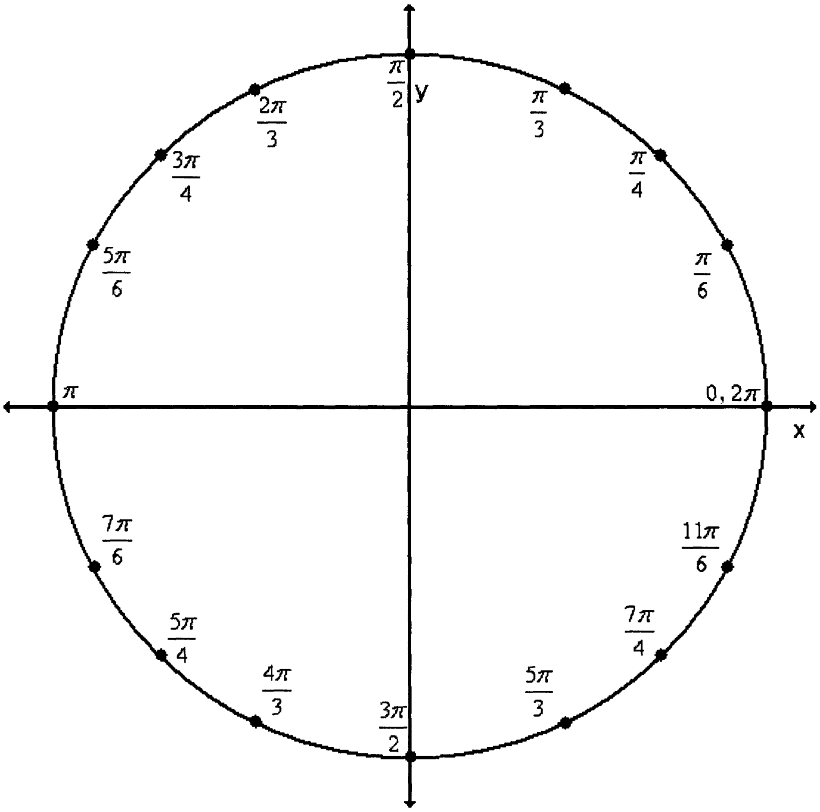
Here are the angles in the interval $(0, 2\pi]$ ***that satisfy the equation.***

Every algebraic equation that was solved previously will play into your ability to solve some of the trigonometric equations you may face.

For example, the trigonometric equation $\csc^2\theta + 3\csc\theta + 2 = 0$ ***is very similar to the quadratic equation*** $x^2 + 3x + 2 = 0$. ***This trigonometric equation would be solved in the same manner as the algebraic. Factor the equation out to*** $(\csc\theta + 1)(\csc\theta + 2) = 0$. ***Then, proceed to set each factor equal to zero and solve.***

Here are a couple of examples of trigonometric equations involving the reciprocal functions.

Solve the following equations in the interval $(0,2\pi]$**.** $\sec\theta+2=0$	
$\sec\theta+2=0$ $\sec\theta=-2$ $\frac{1}{\cos\theta}=-2$ $\cos\theta=-\frac{1}{2}$	***Begin by isolating the trigonometric function. Once that is done, raise both sides of the equation to the negative first power essentially taking the reciprocal of both sides. This yields one of the basic three trigonometric functions.***
$\theta=\frac{2\pi}{3},\ \frac{4\pi}{3}$	***The angles in the interval*** $(0,2\pi]$ ***that satisfy the equation are here.***
Solve the following equations in the interval $(0,2\pi]$**.** $3\csc^2\theta+6=10$	
$3\csc^2\theta+6=10$ $3\csc^2\theta=4$ $\csc^2\theta=\frac{4}{3}$ $\sqrt{\csc^2\theta}=\pm\sqrt{\frac{4}{3}}$ $\csc\theta=\pm\frac{2}{\sqrt{3}}$ $\frac{1}{\sin\theta}=\pm\frac{2}{\sqrt{3}}$ $\sin\theta=\pm\frac{\sqrt{3}}{2}$	***Once again, the trigonometric function is isolated.*** ***Taking the square root of both sides always results in*** ± ***answers.*** ***Since the cosecant function is really the reciprocal of the sine function, both sides are flipped over.***
$\theta=\frac{\pi}{3},\ \frac{2\pi}{3},\ \frac{4\pi}{3},\ \frac{5\pi}{3}$	***Here are the angles in the interval*** $(0,2\pi]$ ***that satisfy the equation.***



Solve each of the following trigonometric equations in the interval $(0, 2\pi]$.

A) $2\sin\theta - 1 = 0$

$2\sin\theta = 1$

$\sin\theta = \frac{1}{2}$

$\boxed{\theta = \frac{\pi}{6}, \frac{5\pi}{6}}$

B) $\cos\theta + 1 = 0$

$\cos\theta = -1$

$\boxed{\theta = \pi}$

C) $\tan\theta + 1 = 0$

$\tan\theta = -1$

$\boxed{\theta = \frac{3\pi}{4}, \frac{7\pi}{4}}$

D) $4\cos^2\theta - 3 = 0$

$4\cos^2\theta = 3$

$\cos^2\theta = \frac{3}{4}$

$\cos\theta = \pm\frac{\sqrt{3}}{2}$

$\boxed{\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}}$

E) $5\tan\theta + 4 = 4$

$-4 \quad -4$

$5\tan\theta = 0$

$\tan\theta = 0$

$\tan\theta = \frac{y}{x}$ y value zero on x axis

$\boxed{\theta = \pi, 2\pi}$

F) $\csc\theta + 2 = 0$

$\csc\theta = 2$

$\sin\theta = \frac{1}{2}$

$\boxed{\theta = \frac{\pi}{6}, \frac{5\pi}{6}}$

G) $4\sin^2\theta - 2 = 0$

$4\sin^2\theta = 2$

$\sin^2\theta = \frac{1}{2}$

$\sin\theta = \pm\frac{\sqrt{2}}{2}$

$\boxed{\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}}$

H) $3\csc^2\theta - 4 = 0$

$3\csc^2\theta = 4$

$\csc^2\theta = \frac{4}{3}$

$\csc\theta = \pm\frac{2}{\sqrt{3}}$

$\sin\theta = \pm\frac{\sqrt{3}}{2}$

$\boxed{\theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}}$

I) $\sqrt{3}\cot\theta + 1 = 0$

$\sqrt{3}\cot\theta = -1$

$\cot\theta = -\frac{1}{\sqrt{3}}$

$\tan\theta = -\sqrt{3}$

$y \quad x$

$\frac{\sqrt{3}}{2} \div \frac{1}{2}$

$\frac{\sqrt{3}}{\cancel{2}} \times \frac{\cancel{2}}{1} = \sqrt{3}$

$\boxed{\theta = \frac{2\pi}{3}, \frac{5\pi}{3}}$

H) $2\sin^3\theta + \sin^2\theta - 2\sin\theta - 1 = 0$

factor by grouping

$\sin^2\theta(2\sin\theta + 1) - 1(2\sin\theta + 1) = 0$

$(\sin^2\theta - 1)(2\sin\theta + 1) = 0$

$\sin^2\theta - 1 = 0 \qquad 2\sin\theta + 1 = 0$

$\sin^2\theta = 1 \qquad \sin\theta = -\frac{1}{2}$

$\sin\theta = \pm 1 \qquad \sin\theta = -\frac{1}{2}$

$\boxed{\theta = \frac{\pi}{2}, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}}$

J) $\sin^2\theta + \sin\theta = 0$

$\sin\theta(\sin\theta + 1) = 0$

$\sin\theta = 0 \qquad \sin\theta + 1 = 0$

$\sin\theta = 0 \qquad \sin\theta = -1$

$\pi, 2\pi \qquad 3\pi/2$

$\boxed{\theta = \pi, \frac{3\pi}{2}, 2\pi}$

K) $\sec\theta - 1 = 0$

$\sec\theta = 1$

$\cos\theta = 1$

$\boxed{\theta = 2\pi}$

L) $\sec^2\theta - 3\sec\theta + 2 = 0$

$(\sec\theta - 2)(\sec\theta - 1) = 0$

$\sec\theta - 2 = 0 \qquad \sec\theta - 1 = 0$

$\sec\theta = 2 \qquad \sec\theta = 1$

$\cos\theta = \frac{1}{2} \qquad \cos\theta = 1$

$\frac{\pi}{3}, \frac{5\pi}{3} \qquad 2\pi$

$\boxed{\theta = \frac{\pi}{3}, \frac{5\pi}{3}, 2\pi}$

M) $\csc^2\theta + 3\csc\theta + 2 = 0$

$(\csc\theta + 1)(\csc\theta + 2) = 0$

$\csc\theta + 1 = 0 \qquad \csc\theta + 2 = 0$

$\csc\theta = -1 \qquad \csc\theta = -2$

$\sin\theta = -1 \qquad \sin\theta = -\frac{1}{2}$

$\frac{3\pi}{2} \qquad \frac{7\pi}{6}, \frac{11\pi}{6}$

$\boxed{\theta = \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}}$

N) $\tan^2\theta - 1 = 0$

$\tan^2\theta = 1$

$\tan\theta = \pm 1$

$\boxed{\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}}$

O) $4\cos^3\theta - 2\cos\theta = 0$

$2\cos\theta(2\cos^2\theta - 1) = 0$

$2\cos\theta = 0 \qquad 2\cos^2\theta - 1 = 0$

$\cos^2\theta = \frac{1}{2}$

$\cos\theta = 0 \qquad \cos\theta = \pm\frac{\sqrt{2}}{2}$

$\frac{\pi}{2}, \frac{3\pi}{2} \qquad \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

don't need to be in order →

$\boxed{\theta = \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}}$

P) $\csc^2\theta - 2 = 0$

$\csc^2\theta = 2$

$\csc\theta = \pm\sqrt{2}$

$\sin\theta = \pm\frac{1}{\sqrt{2}}$

$\sin\theta = \pm\frac{\sqrt{2}}{2}$

$\boxed{\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}}$

Q) $\cot^2\theta - 3 = 0$

$\cot^2\theta = 3$

$\cot\theta = \pm\sqrt{3}$

$\tan\theta = \pm\frac{1}{\sqrt{3}}$

$y \div x$

$\frac{1}{2} \div \frac{\sqrt{3}}{2}$

$\frac{1}{2} \cdot \frac{2}{\sqrt{3}} = \frac{1}{\sqrt{3}}$

$\boxed{\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}}$

R) $4\sin^2\theta-4\sin\theta+1=0$

$(2\sin\theta-1)^2=0$

$2\sin\theta-1=0$

$\sin\theta=\frac{1}{2}$

$\boxed{\theta=\frac{\pi}{6},\frac{5\pi}{6}}$

S) $2\sin\theta\cos\theta-\cos\theta=0$

$\cos\theta(2\sin\theta-1)=0$

$\cos\theta=0 \quad 2\sin\theta-1=0$

$\cos\theta=0 \quad \sin\theta=\frac{1}{2}$

$\frac{\pi}{2},\frac{3\pi}{2} \quad \frac{\pi}{6},\frac{5\pi}{6}$

order does not matter

$\boxed{\theta=\frac{\pi}{6},\frac{\pi}{2},\frac{5\pi}{6},\frac{3\pi}{2}}$

T) $9\sec^2\theta-12=0$

$9\sec^2\theta=12$

$\frac{12}{9}$

$\sec^2\theta=\frac{4}{3}$

$\sec\theta=\pm\frac{2}{\sqrt{3}}$

$\cos\theta=\pm\frac{\sqrt{3}}{2}$

$\boxed{\theta=\frac{\pi}{6},\frac{5\pi}{6},\frac{7\pi}{6},\frac{11\pi}{6}}$

U) $4\sin\theta\cos\theta+2\cos\theta-2\sin\theta-1=0$

$2\cos\theta(2\sin\theta+1)-1(2\sin\theta+1)=0$

$(2\cos\theta-1)(2\sin\theta+1)=0$

$2\cos\theta-1=0 \quad 2\sin\theta+1=0$

$\cos\theta=\frac{1}{2} \quad \sin\theta=\frac{-1}{2}$

$\boxed{\theta=\frac{\pi}{3},\frac{5\pi}{3},\frac{7\pi}{6},\frac{11\pi}{6}}$

V) $\tan^2\theta-\left(1+\sqrt{3}\right)\tan\theta+\sqrt{3}=0$

$\tan^2\theta-\tan\theta-\sqrt{3}\tan\theta+\sqrt{3}=0$

$\tan\theta(\tan\theta-1)-\sqrt{3}(\tan\theta-1)=0$

$(\tan\theta-1)(\tan\theta-\sqrt{3})=0$

$\tan\theta-1=0 \quad \tan\theta-\sqrt{3}=0$

$\tan\theta=1 \quad \tan\theta=\sqrt{3}$

$y\div x$

$\frac{\sqrt{3}}{2}\div\frac{1}{2}$

$\boxed{\theta=\frac{\pi}{4},\frac{5\pi}{4},\frac{\pi}{3},\frac{4\pi}{3}}$

Checking Progress

You have now completed the "Introduction to Trigonometry" section of the workbook. The following is a checklist so that you may check your progress. Check off each of the objectives you have accomplished.

The student should be able to…

- ☐ ***Give a graphical representation of any angle***
- ☐ ***Find positive and negative coterminal angles***
- ☐ ***Convert an angle measured in degrees to radians***
- ☐ ***Convert an angle measured in radians to degrees***
- ☐ ***Evaluate the basic trigonometric functions***
- ☐ ***Use reference angles to evaluate the basic trigonometric functions***
- ☐ ***Construct a unit circle***
- ☐ ***Use the unit circle to evaluate basic trigonometric functions***
- ☐ ***Use the unit circle to solve trigonometric equations***
- ☐
- ☐
- ☐
- ☐
- ☐
- ☐
- ☐
- ☐

MIXED REVIEW

Identifying Equations and Formulas

Identify each of the following.

A) $Ax+By=C$

The Standard Form of a Line

B) $(y-y_1)=m(x-x_1)$

Point-Slope form of a Line

C) $y=mx+b$

Slope-Intercept Form of a Line

D) $A=l\cdot w$

Area of a Rectangle

E) $A=\frac{1}{2}bh$

Area of a Triangle

F) $C=2\pi r$

Circumference of a Circle

G) $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

The Quadratic Formula

H) $m=\frac{x_2-x_1}{y_2-y_1}$

Slope of a Line

I) $a^2+b^2=c^2$

The Pythagorean Theorem

J) $d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

The Distance Formula

K) $y=a(x-h)^2+k$

Standard Form of a Parabola

L) $\log_a b=\frac{\log b}{\log a}$

Base Change Formula

M) $\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)$

Midpoint Formula

N) $(x-h)^2+(y-k)^2=r^2$

Standard Form of a Circle

O) $A=Pe^{rt}$

Interest Compounded Continuously

P) $A = P\left(1+\frac{r}{n}\right)^{nt}$

Compound Interest Formula

Q) $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$

Standard Form of a Hyperbola

R) $A = \pi r^2$

Area of a Circle

S) $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$

Standard Form of an Ellipse

T) $ax^2 + bx + c = 0$

General form of a parabola

U) $degrees \cdot \frac{\pi}{180^\circ}$

Convert degrees to Radians

Knowledge of Functions

Based on the following information, what conclusions can you make about each of the following functions?

A) $f_{(x)} = ca^{x-h} + k$, where $0 < a < 1$. What is this?

Exponential Decay

B) You have two functions, $f_{(x)}$ and $g_{(x)}$ where the following is true. $f_{\left(g_{(x)}\right)} = x$ and $g_{\left(f_{(x)}\right)} = x$.

The two functions f(x) and g(x) are inverses of each other.

C) $f_{(-x)} = f_{(x)}$

This is an even function that is symmetrical to the y-axis.

D) $f_{(-x)} = -f_{(x)}$

This is an odd function that is symmetrical to the origin

E) $f_{(x)}$ is a quadratic function where $b^2 - 4ac < 0$.

The graph of the function does not cross the x-axis.

F) This function is symmetrical to the origin.

This is an odd function.

G) This function is symmetrical to the y axis.

This is an even function

H) The graph of this function has 6 turns.

This is at least a 7th degree polynomial.

I) The function $f_{(x)}$ has no real zeros.

The function has no x-intercepts.

J) The two linear functions $f_{(x)}$ and $g_{(x)}$ have identical slopes but different y intercepts.

The lines are parallel.

K) If the range of the function is all real numbers, what can you tell me regarding intercepts?

The function must have at least one x intercept.

L) If the domain of the function is all real numbers, what can you tell me regarding intercepts?

The function must have a y intercept.

M) The function $f_{(x)}$ does not have an inverse.

$f(x)$ is not a one-to-one function

N) A graph does not pass the vertical line test.

It is not a function.

O) A function has 4 real zeros.

This function has 4 x intercepts

P) When two different functions $f_{(x)}$ and $g_{(x)}$ are graphed on the same plane, they are symmetrical to the $y = x$ axis.

The two functions, f(x) and g(x), are inverses of each other.

Q) A quadratic function that has a maximum value.

The parabola opens down.

R) A quadratic function that has a minimum.

the parabola opens up.

S) $f_{(x)} = ca^{x-h} + k$, where $a > 1$. What is this?

this is an exponential growth function

Translations of Functions

Consider the graph of a function $f_{(x)}$. Describe what happens to the function in each of the following cases. Be specific, if the function will shift to the left 4 spaces, state that.

A) $f_{(x+3)}$

function shifts left 3 spaces

add (-3) to all x values

B) $-f_{(x)}$

flips upside down

change the sign of all y values

C) $f_{(x)}-2$

shifts down 2

subtract 2 from all y values

D) $f_{(3x)}$

function shrinks horizontally by a factor of $\frac{1}{3}$.

Divide all x values by 3.

E) $\frac{1}{2}f_{(x)}$

function shrinks vertically

Multiply all y values by $\frac{1}{2}$.

F) $f_{(x-1)}$

shifts right 1

add 1 to all x values

G) $f_{(x)}+5$

shifts up 5

add 5 to all y values

H) $f_{(-x)}$

flips right to left

change sign of all x values

I) $f_{\left(\frac{x}{3}\right)}$

function widens by a factor of 3.

multiply all x values by 3.

J) $4f_{(x)}$

Scale increases by a factor of 4,

Multiply all y values by 4.

K) $2f_{(x)}-4$

Scale increases by a factor of 2 and shifts down 4.

Multiply 2 to all y values, then subtract 4 from all y values

L) $-f_{(x)}+2$

flips upside down and shifts up 2,

Multiply (-1) to all y values, then add 2 to all y values.

M) $\frac{1}{3}f_{(x-2)}$

shrinks vertically to $\frac{1}{3}$ normal scale and shifts right 2.

Multiply a y values by $\frac{1}{3}$ and add 2 to all x values.

N) $-3f_{(x)}$

Stretches vertically by a factor of 3 and flips upside down.

multiply all y values by -3.

O) $f_{(x-4)}+2$

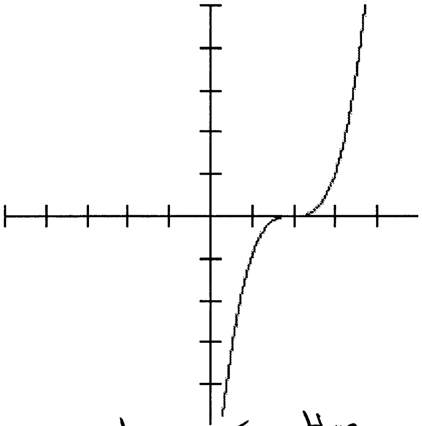
shifts right 4, up 2

add 4 to all x values and add 2 to all y values

Identifying the Graph of a Function

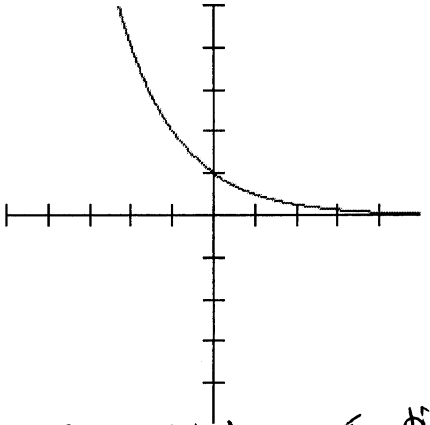
Identify each graph based on your knowledge of parent functions. Describe the type of function being illustrated.

A



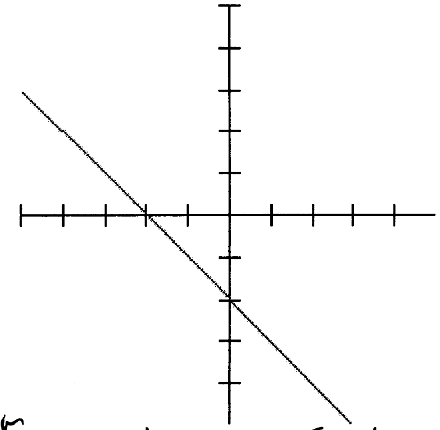
Cubic Function

B



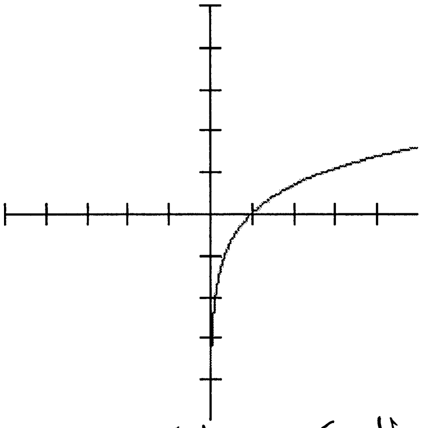
Exponential ~~Decay~~ Function

C



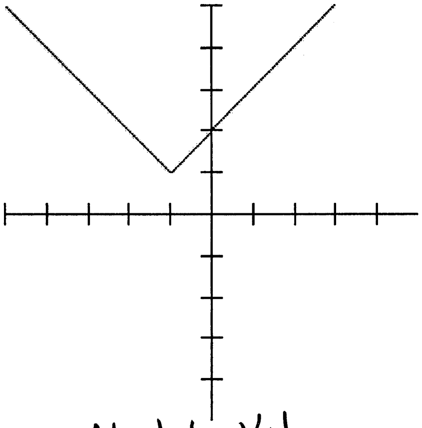
Linear Function

D



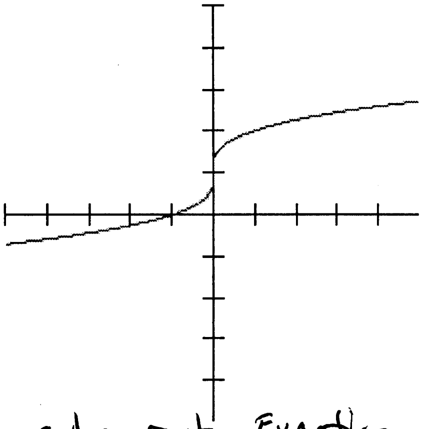
Logarithmic Function

E



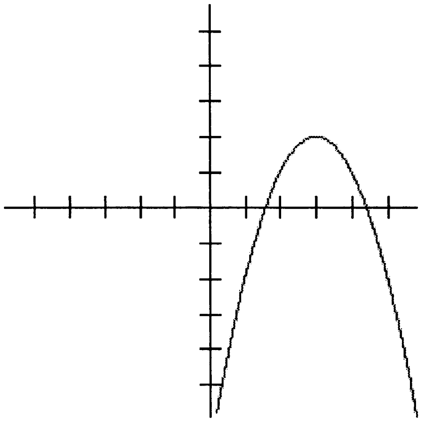
Absolute Value Function

F



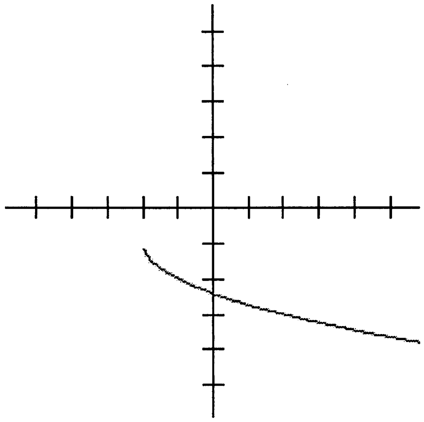
cube root Function

G



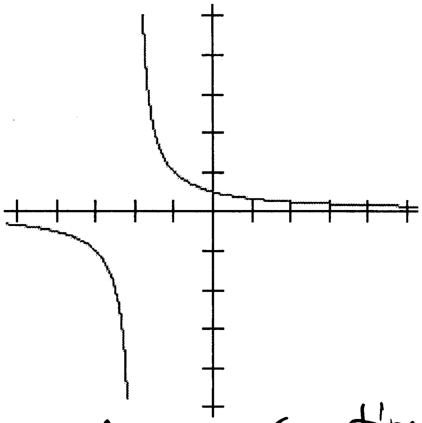
Quadratic Function

H



Radical Function

I



Rational Function

Finding the Domain of Functions

Find the domain of each of the following functions.

A) $f_{(x)} = 2(x-3)^2 + 1$

$(-\infty, \infty)$

B) $f_{(x)} = \sqrt{x+4} - 2$

$x+4 \geq 0$

$x \geq -4$

$[-4, \infty)$

C) $f_{(x)} = \dfrac{x+5}{x-2}$

$x - 2 \neq 0$ V.A. $x=2$

$x \neq 2$

$(-\infty, 2) \cup (2, \infty)$

D) $f_{(x)} = \log_3(x-3) + 2$

$x - 3 > 0$

$x > 3$

$(3, \infty)$

E) $f_{(x)} = -2^{x-3} + 1$

$(-\infty, \infty)$

F) $f_{(x)} = -|x-3|$

$(-\infty, \infty)$

G) $f_{(x)} = (x+1)^3 - 5$

$(-\infty, \infty)$

H) $f_{(x)} = \ln x - 2$

$x > 0$

$(0, \infty)$

I) $f_{(x)} = \dfrac{1}{\sqrt{x}}$

$x > 0$

Cannot be $\geq$ because 0 in denominator

$(0, \infty)$

J) $f_{(x)} = \dfrac{x+3}{x^2+3x-40}$

$x^2+3x-40=0$

$(x+8)(x-5)=0$

V.A. $x=-8$ $x=5$

$(-\infty, -8) \cup (-\infty, 5) \cup (5, \infty)$

K) $f_{(x)} = \sqrt{4-x} + 1$

$4 - x \geq 0$

$-x \geq -4$

$x \leq 4$

$(-\infty, 4]$

L) $f_{(x)} = \log_2(2-x)$

$2 - x > 0$

$-x > -2$

$x < 2$

$(-\infty, 2)$

M) $f_{(x)} = -\sqrt[3]{x+2} - 1$

$(-\infty, \infty)$

N) $f_{(x)} = \dfrac{1}{\sqrt{x^2-9}}$

$x^2 - 9 > 0$

$(x+3)(x-3) > 0$

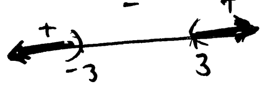
test

$\boxed{-5}$ $(-)(-)=+$

$\boxed{0}$ $(+)(-)=-$

$\boxed{5}$ $(+)(+)=+$

C.P. -3 3



$(-\infty, -3) \cup (3, \infty)$

O) $f_{(x)} = e^{x+2} - 3$

$(-\infty, \infty)$

Function Operations

Given $f_{(x)} = 2x^2 - 3$, $g_{(x)} = 5x + 2$, **and** $h_{(x)} = -x^2 + 1$ **find each of the following.**

A) $3f_{(x-3)} + 2h_{(x+2)}$

$f(x-3) = 2(x-3)^2 - 3$

18

$2(x^2 - 6x + 9) - 3$

$2x^2 - 12x + 15$

$h(x+2) = -(x+2)^2 + 1$

$-(x^2 + 4x + 4) + 1$

$-x^2 - 4x - 3$

$2h(x+2) = 2(-x^2 - 4x - 3)$

$3f(x-3) = 3(2x^2 - 12x + 15)$

$6x^2 - 36x + 45 \quad -2x^2 - 8x - 6$

$\boxed{4x^2 - 44x + 39}$

B) $3f_{(x)} + h_{(x)} - 2g_{(x)}$

$3(2x^2 - 3) + (-x^2 + 1) - 2(5x + 2)$

$6x^2 - 9 \quad -x^2 + 1 \quad -10x - 4$

$\boxed{5x^2 - 10x - 12}$

C) $f_{\left(g_{\left(h_{(3)}\right)}\right)}$

$h(3) = -(3)^2 + 1 = -8$

$-9 + 1$

$g(-8) = 5(-8) + 2$

$-40 + 2$

-38

$f(-38) = 2(-38)^2 - 3$

$\boxed{2885}$

For the following problems, find functions $f_{(x)}$ **and** $g_{(x)}$ **where** $f_{(x)} \circ g_{(x)} = h_{(x)}$ $\boxed{f(g(x))}$

A) $h_{(x)} = \sqrt{x^2 - 3} + 2$

$f(x) = \sqrt{x} + 2$

$g(x) = x^2 - 3$

B) $h_{(x)} = (3x - 4)^3$

$f(x) = x^3$

$g(x) = 3x - 4$

C) $h_{(x)} = 5(x + 3)^2 - 7$

$f(x) = 5x^2 - 7$

$g(x) = x + 3$

D) $h_{(x)} = 3\sqrt{x} + 6$

$f(x) = 3x + 6$

$g(x) = \sqrt{x}$

E) $h_{(x)} = (5x - 3)^4$

$f(x) = x^4$

$g(x) = 5x - 3$

F) $h_{(x)} = (x + 1)^3 - 2$

$f(x) = x^3 - 2$

$g(x) = x + 1$

G) $h_{(x)} = \dfrac{5(x-2)^2 - 3}{(x-2)}$

$f(x) = \dfrac{5x^2 - 3}{x}$

$g(x) = x - 2$

H) $h_{(x)} = \dfrac{3}{x - 7}$

$f(x) = \dfrac{3}{x}$

$g(x) = x - 7$

I) $h_{(x)} = -2x + \sqrt{x}$

$f(x) = -2x^2 + x$

$g(x) = \sqrt{x}$

Word Problems

Answer each of the following.

A) Two families take a trip to a local amusement park. The first family purchases 2 adult tickets and 3 child tickets for \$258.00. The second family buys one adult ticket and 2 child tickets for \$153.00. How much does each type of ticket cost?

Family 1

$2A + 3C = 258$

Family 2

$1A + 2C = 153 \rightarrow$

Eq 2 $\times(-2)$

$2A + 3C = 258$

$-2A - 4C = -306$

$-C = -48$

$C = 48$

$A + 2(48) = 153$

$A + 96 = 153$

$A = 57$

$$\boxed{\begin{array}{l}\text{Adult} = \$\,57.00 \\ \text{Child} = \$\,48.00\end{array}}$$

B) An investment banker wants to invest \$50,000 among three different accounts. The first account yields 4.5% interest, while the second and third yield 7% and 11.5% respectively. The amount invested at 11.5% is \$5,000 more than the amount invested at 4.5%. How much is invested in each account if the banker earns \$3,925 in interest?

$A + B + C = 50\,000$

1000 $(.045A + .07B + .115C = 3925)$

$C = A + 5000$

$A + B + (A + 5000) = 50\,000 \rightarrow 2A + B = 45000$

$45A + 70B + 115(A + 5000) = 3\,925\,000$

$115A + 575000$

$160A + 70B = 3{,}350{,}000$

$-70(2A + B = 45000) \rightarrow -140A - 70B = -3{,}150{,}000$

$20A = 200\,000$

$A = 10\,000$

$C = 10\,000 + 5000$

$C = 15\,000$

$$\boxed{\begin{array}{l}\$\,10{,}000 \text{ at } 4.5\% \\ \$\,25{,}000 \text{ at } 7\% \\ \$\,15{,}000 \text{ at } 11.5\%\end{array}}$$

C) Last night you decided to clean out your change bowl. There were a total of 63 quarters, nickels and dimes in the bowl. There were half as many nickels as quarters. How many of each type of coin were in the bowl if the total value of the coins was \$10.30?

$Q + N + D = 63 \rightarrow 2N + N + D = 63$

Eq 1 $3N + D = 63$

$N = \frac{1}{2}Q$ or $Q = 2N$

$.25Q + .05N + .1D = 10.30 \rightarrow .5N + .05N + .1D = 10.30$

Eq 2 $.55N + .1D = 10.30$

(-10) Eq 1 $\rightarrow -30N - 10D = -630$

(100) Eq 2 $\rightarrow 55N + 10D = 1030$

$25N = 400$

$N = 16$

$Q = 2(16)$

$Q = 32$

$$\boxed{\begin{array}{l}16 \text{ Nickels} \\ 15 \text{ Dimes} \\ 32 \text{ Quarters}\end{array}}$$

D) Bob can paint a fence in 12 hours. Sally can paint the same fence in 18 hours. How long will it take them to paint the fence if they work together?

	Bob	Sally	together
time to do job	12hrs	18 hrs	
rate	$\frac{1}{12}$ job done per hr.	$\frac{1}{18}$ job done per hr.	$\frac{5}{36}$

$\frac{1}{12}+\frac{1}{18}$

$\frac{3}{36}+\frac{2}{36}$

$\frac{5}{36}$

$\frac{36}{5} = 7\frac{1}{5}$

$\frac{1}{5}$ hr(60 min)

12 min

$\boxed{\text{7hrs. 12 min}}$

or

$\boxed{7\frac{1}{5}\text{hrs}}$

E) Bob can paint a fence in 12 hours. Sally can paint the same fence in 18 hours. If Bob works alone for 4 hours, then Sally joins him, how long will it take them to finish the fence working together?

	Bob	Sally	together
time to do job	12 hrs	18 hrs.	
rate in job done per hr.	$\frac{1}{12}$	$\frac{1}{18}$	$\frac{5}{36}$ (see previous problem)

R · T = Work Done

Bob Alone → $\frac{1}{12} \cdot 4 = \frac{1}{3}$ job done

R · T = Job Done

$\frac{5}{36} \cdot T = \frac{2}{3}$ job left

$T = \frac{2}{\cancel{3}} \cdot \frac{\overset{12}{\cancel{36}}}{5} = \frac{24}{5}$

$\frac{4}{\cancel{5}} \cdot \overset{12}{\cancel{60}} = 48$ min

$\boxed{4\tfrac{4}{5}\text{ hrs or 4 hrs 48 min}}$

F) The city is repaving a road. The machine they are using can complete the job in 14 hours. After working for 6 hours, it breaks down. A second machine is brought in to finish the job. If the second machine finishes the job in 4 hours, how long would it have taken for this machine to do the entire job alone?

	Machine 1	Machine 2
Time to do job	14 hrs.	7 hrs. ↑
Rate	$\frac{1}{14}$ job per hr.	$\frac{1}{7}$ Job done per hr.

Find rate of machine 2

R · T = JD

$R \cdot 4 = \frac{4}{7}$

$R = \frac{4}{7} \cdot \frac{1}{4}$

$R = \frac{1}{7}$

R · T = Job Done

$\frac{1}{14} \cdot 6 =$

$\frac{3}{7}$ of Job Done

$\frac{4}{7}$ of Job left

$\boxed{\text{If working alone for entire job, 7 hours.}}$

Miscellaneous

Match appropriate concepts.

1. Quadratic equation G

2. Zeros of the function I

3. Not a function A

4. Vertical asymptotes H

5. All possible x values of a function M

6. All possible y values of a function C

7. Vertex at (4,-3) K

8. Horizontal asymptote at y = $\frac{3}{2}$ E

9. Oblique asymptote O

10. Set x value equal to zero and solve for y N

11. The number possible (+) and (-) zeros D

12. Parabola that has no x intercepts B

13. Zeros of the numerator (rational function) R

14. Complex numbers Q

15. Polynomial function where both sides go down F

16. Polynomial function up on left, down on right J

17. A polynomial function has at most n-1 turns. P

18. Division algorithm L

~~A.~~ $x = y^2$

~~B.~~ $f(x) = 2(x+4)^2 + 5$ vertex (-4, 5) opens up

~~C.~~ Range

~~D.~~ Descarte's Rule of Signs

~~E.~~ $f(x) = \dfrac{3x+2}{2x+5}$

~~F.~~ $f(x) = -x^6 + 2x^4 - 5$

~~G.~~ $2x^2 + 3x - 4 = 0$

~~H.~~ zeros of denominator (rational function)

~~I.~~ x intercepts

~~J.~~ $f(x) = -x^3 + 2x^2 + 4x - 1$

~~K.~~ $f(x) = \frac{1}{2}(x-4)^2 - 3$

~~L.~~ $d(x)q(x) + r(x)$

~~M.~~ Domain

~~N.~~ Finding y intercept

~~O.~~ Slant asymptote

~~P.~~ n-1

~~Q.~~ Imaginary number

~~R.~~ x intercepts of rational function

Glossary

Contained in this section are some of the terms you may hear in class. An attempt has been made to make their meaning as easy to understand as possible.

Abscissa:
The horizontal, or x coordinate in an ordered pair.

Absolute Value:
The absolute value of a number is the distance between that number and the origin.

Acute Angle:
An angle that is less than 90°.

Acute Triangle:
A triangle where all interior angles are less than 90°.

Adjacent:
Something that is next to.

Algebra:
A branch mathematics that involves working with variables.

Algorithm:
A step by step procedure designed to solve a problem. Mathematical instructions used to solve a problem or obtain a desired result.

Angle:
Two rays or lines that share a common endpoint.

Arc of a Circle:
A segment of a circle, the connected section of the outer edge of a circle.

Arithmetic Mean:
The average. If you have a set of n numbers, the average can be found by finding the sum of all digits, and dividing that sum by n.

Arithmetic Sequence:
A series of numbers that has a constant difference between them, such as 4, 7, 10, 13, 16….

Asymptote:
A straight line that the graph of a relation approaches very closely as the graph is followed. Is the limiting value of the curve of the function.

Average:
Usually refers the arithmetic mean, however, can be a single value representing the center of a set of values.

Axis:
a. A straight line about which a body or a geometric figure rotates or may be supposed to rotate.
b. One of the reference lines of a coordinate system.

Axis of Symmetry:
A line of symmetry for a graph. The two sides of the graph on either side of the axis of symmetry look like mirror reflections of one another.

Binomial:
A polynomial that has two terms.

Cartesian Coordinates:
Also known as rectangular coordinates. Come as an ordered pair (x,y) or in the form (x,y,z).

Cartesian Plane:
Also called the coordinate plane. A plane that is formed by a vertical and horizontal axis usually labeled as the x and y axes.

Central Angle:
An angle whose vertex is located at the center of a circle.

Coefficient:
The constant factor of a term that is being multiplied to a variable. For example, 12 is the coefficient of the term $12x^5y^2$.

Collinear:
Lying on the same straight line. Usually referring to points.

Complementary Angles:
Two acute angles whose sum is 90°.

Complex Numbers:
The sum or difference of real number and an imaginary number in the form $a+bi$, where $b \neq 0$.

Composite Number:
A number that has factors other than 1 and itself.

Composite Function:
A function that is made by combining one or more functions together. The process involves substituting a function's formula for the variable of the second function's formula.

Compound Fraction:
A fraction that has one or more fractions in its denominator or numerator. Also called complex fractions.

Compound Inequality:
Two inequalities put together.

Compound Interest:
A method of computing interest where interest is earned not only from the principal, but interest as well.

Conjugates:
The result of writing the sum of two terms as a difference, or vice versa. For example, x+7 and x-7 are conjugates.

Conjugate Pair Theorem:
If a polynomial has real coefficients, then any complex zeros always come in conjugate pairs. i.e. $2+3i \quad and \quad 2-3i$

Consistent System of Equations:
A system of equations that has at least one solution.

Constant:

A term with no variables. For example, in the equation $3x^2 - 2x + 4 = 0$, 4 is the constant.

Constant Function:
The equation of a horizontal line. In other words, y = #.

Coordinates:
A set of numbers referring to the location of a specific point in a two or three dimensional space. An ordered pair, or ordered triple.

Coterminal Angles:
Angles when drawn in standard position who share the same terminal side.

Cubic Polynomial:
A third degree polynomial.

Degree of a Polynomial:
The greatest degree of any term in the polynomial itself.

Degree of a Term:
If the term has only one variable, the degree of the term is the exponent of that variable. If the term has more than one variable, the degree is the sum of the exponents of those variables.

Dependent Variable:
A variable whose value is dependent on one or more variables in a function. He who attempts to stand alone is dependent upon....

Descartes' Rule of Signs:
A method for determining the possible number of positive or negative zeros for a polynomial.

Domain:
The set of values of the independent variable of a function for which the function is defined. Usually, all possible x values that have a corresponding y value.

Even Function:

A function that is symmetrical to the y axis. A function is even if and only if $f_{(-x)} = f_{(x)}$.

Factor of a Polynomial:
Any polynomial that divides evenly into another polynomial.

Finite:
Having definite or definable limits.

Function:
A function is a rule that produces a correspondence between two sets of elements such that for each element in the domain, there corresponds exactly one element in the range.

Function Operations:
The process of adding, subtracting, multiplying, dividing and composing functions together.

Fundamental Theorem of Algebra:
Any polynomial to the nth degree has exactly n zeros.

Fundamental Theorem of Arithmetic:
All numbers have their own unique prime factorization.

Geometric Mean:
This is a sort of average. To find the geometric mean of a set of n numbers, multiply the given set of numbers, then take the nth root of their product.

Geometric Sequence:
A series of numbers in which there exists a common ratio between each term.

Greatest Common Factor:
The largest number that divides evenly into a given set of numbers.

Half-Life:
A term used to describe the decaying of a substance at an exponential rate. The length of time it takes for an amount of a particular substance to diminish by half.

Horizontal Line Test:
A test used to determine if a function is one-to-one. If a horizontal line crosses the function more than once, the function is not one-to-one.

Horizontal Reflection:
A reflection in which a two dimensional figure flips horizontally. The reflection takes place about a vertical axis.

Horizontal Translation:
A shift in which a two dimensional figure, or graph, moves horizontally to the left or right.

Imaginary Number:
A complex number with no real part. In other words, a complex number in the form $a + bi$, where $a = 0$. This leaves you with the pure imaginary number in the form bi.

Inconsistent System of Equations:
A system of equations in which there are no solutions.

Infinite:
To continue on forever, subject to no limitation, ∞.

Independent Variable:
The variable in an equation in which you may freely choose any value to substitute without considering other variables.

Initial Side of an Angle:
The ray where the measurement of an angle begins. In trigonometry, an angle is in standard position if its initial side is on the positive side of the x axis.

Interval Notation:
Using a pair of numbers rather than using an inequality to represent a specific interval. You will use parenthesis and brackets in interval notation. The smallest value is on the left of the interval.

Leading Coefficient:

The coefficient of the polynomial's first term when in the form $ax^n + bx^{n-1} + cx^{n-2}$...

Least Common Denominator:

The smallest number that can be used as a denominator for two or more fractions. This is the LCM of the denominators.

Least Common Multiple:

The smallest number that a given set of numbers can multiply to be.

Like Terms:

Terms that have the same variables, and corresponding powers. When dealing with radicals, terms that have the same index and radicand.

Limit:

The value that a function approaches as the domain variable approaches a specified value. In other words, if you are given a function, a limit says something to the affect of "What is the value of the function when x is 12?"

Mean:

Another word for average.

Midpoint:

A point on a line segment that is half way between two given points.

Mode:

The number that occurs most often in a set of data.

Monomial:

A polynomial that has only one term.

Oblique:

To be tilted at an angle. Something is oblique if it is neither vertical nor horizontal.

Odd Function:

A function that is symmetrical to the origin. A function is odd if and only if $f_{(-x)} = -f_{(x)}$.

One-Sided Limit:

Either a limit from the left, or a limit from the right.

One-to-One Function:

For every element in the range, there exists only one corresponding value in the domain. For each y value, there exists only one corresponding x value. Test using the horizontal line test.

Ordered Pair:

A set of numbers on the Cartesian plane that corresponds to the location of a particular point. In a 3-dimensional space, it is an ordered triple.

Ordinate:

The ordinate is the y coordinate in a point. In the point (2,5), 5 is the ordinate.

Origin:

Where the vertical and horizontal axis intersect on the Cartesian plane. The point (0,0).

Parent Functions:
The basic function that is used to build a more complicated one.

Period of a Periodic Function:
In trigonometry, a period is the horizontal length required for the graph of a periodic function to complete one full cycle.

Periodic Function:
A function whose graph continually repeats itself. Sine and cosine functions are examples of periodic functions.

Piecewise Function:
Pieces of different functions put together to form one graph. You only graph each of the different function in a specified domain.

Polynomial:
One or more algebraic terms joined by operation symbols.

Prime Factorization:
Completely factoring an integer to its primes. For example, $18 = 2 \cdot 3^2$.

Prime Number:
A positive number that can be divided only by 1 or itself. Remember, 1 is not a prime number.

Quadratic Equation:
An equation that contains only a second degree polynomial. A quadratic equation may be expressed in the general form $ax^2 + bx + c = 0$

Radian:
A unit used to measure angles. Found by multiplying the degree of an angle by $\frac{\pi}{180^{\circ}}$. If you were to take the length of the radius, and lay it on the arc of the circle, the number of radians in a circle, is the number of radii it takes to measure the arc of the circle.

Radicand:
The number under the $\sqrt{\ }$ symbol. In the case of $5\sqrt{3}$, 3 is the radicand.

Range:
The set of all values of the dependent variable of a function. Usually, all possible y values of a function.

Scalar Multiplication:
Multiplying one number (scalar) by another. Used in the multiplication of matricies and in vector operations.

Scalene Triangle:
A triangle where all three sides have different lengths.

Sinusoid:
A wave shaped graph as in $y = \sin x$.

Solution Set:
All values of the variable that satisfiy an equation, inequality, system of equations, etc.

Standard Position of an Angle:
An angle drawn on the x and y plane where the initial side of the angle is on the positive side of the x axis, and turns counterclockwise.

Subtend:
To cut into. To determine the measure of by marking off endpoints. For example, if you extend the legs of a central angle, the angle will subtend an arc on the circle.

Supplementary Angles:
Two angles whose sum is 180°.

System of Equations:
Two or more equations that contain common variables.

Terminal Side of an Angle:
The ray where the measurement of an angle stops. In trigonometry, angles are measured counterclockwise.

Translation:
The process of shifting a graph without changing its scale or the direction. The horizontal and vertical shifts of the graph of a function.

Vertical Line Test:
A test used to determine if a relation is a function. If a vertical line crosses the graph more than once, it is not a function.

Vertical Reflection:
A reflection in which a two dimensional figure flips over. Figures that are vertical reflections have a horizontal axis of reflection.

Vertical Translation:
A shift in which a two dimensional figure, or graph, moves vertically up or down.

***x*-intercept:**
The point at which a graph intersects the x axis.

***y*-intercept**
The point at which a graph intersects the y axis.

Zero of a Function:
The value of x for which $f_{(x)} = 0$. These are the x intercepts of the graph of the function.

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