Solving Systems of Equations in 2 Variables Algebraically

There are three ways to a system of equations in two variables we will be covering as we look at linear functions. The first method is through graphing, while the others are more algebraic. We will be focusing on the algebraic methods here.

The first algebraic method used to solve systems of equations is the substitution method. While this method is the most difficult to master, it is also the most useful as it will be necessary to use these procedures in many other types of problems you will encounter as you study algebra.

The first step to solving a system of equations using the substitution method is to isolate and solve for one of the variable. It does not matter which. The examples below will guide you in making your decision as to which variable to solve for.

A) $\begin{cases}2x+y=9\\3x-4y=8\end{cases}$	**In example A, it is best to solve for the y value in equation one as the variable is already by itself. This means there is no coefficient, so all that has to be done is subtract 2x from both sides of the equation.**
	$2x+y=9$ $-2x \qquad -2x$ $y=9-2x$
B) $\begin{cases}3x+5y=12\\x+4y=11\end{cases}$	**In example B, it is best to solve for the x in the second equation as the variable has a coefficient of 1. Subtract 4y to both sides and we will have solved for x.**
	$x+4y=11$ $-4y \qquad -4y$ $x=11-4y$
C) $\begin{cases}12x-3y=21\\6x-5y=3\end{cases}$	**For example C, all of the variables have coefficients of something other than one. In this case, look at all of the numbers in each equation. Notice how all of the numbers in equation 1 are multiples of 3. This tells me to solve for y in the first equation. We will first subtract 12x from both sides, the divide by -3.**
	$12x-3y=21$ $-12x \qquad -12x$ $\frac{-3y}{-3}=\frac{21}{-3}-\frac{12x}{-3}$ $y=4x-7$ **Note that once divided by -3, the signs will change.**
D) $\begin{cases}2x+3y=-6\\3x+2y=25\end{cases}$	**In this particular example, there is no variable that has a coefficient of one, nor is there a case where all of the coefficients of an equation are multiples of a single number. There is however the case in equation 1 where -6 is divisible by 2 or 3. Since that is the case, we will focus attention on equation 1. Solve for y in first equation. Just be careful with the signs.**
	$2x+3y=-6$ $-2x \qquad -2x$ $\frac{3y}{3}=-\frac{2x}{3}-\frac{6}{3}$ $y=-\frac{2}{3}x-2$

Substitution Method Example

Solve the following system of equations using the substitution method: $\begin{cases} 5x+16y=15 \\ -2x-4y=1 \end{cases}$

Work	Explanation	Procedure
$eq1 \quad 5x+16y=15$ $eq2 \quad -2x-4y=1$	**Looking at the equations, it will be best to solve for the x in equation 2.**	
$-2x-4y=1$ $+4y \quad +4y$ $\frac{-2x}{-2}=\frac{4y}{-2}+\frac{1}{-2}$ $eq3 \quad x=-2y-\frac{1}{2}$	**Add 4y to both sides of the equation, then divide the entire equation by -2. We now have a new equation that will be labeled here as equation 3.**	**Step 1- chose one of the equations and solve for a variable. It does not matter which equation or variable is chosen.**
$eq1 \quad 5x+16y=15$ $eq3 \quad x=-2y-\frac{1}{2}$ $5\left(-2y-\frac{1}{2}\right)+16y=15$	**At this point, substitute our new equation, equation 3 into equation 1. Remember, equation 3 originally came from equation 2. Do not plug the new equation back into the original problem it came from.**	**Step 2- substitute the new equation into the other one. Make sure you do not substitute equation 3 back into the original problem it came from. Doing so will result in nothing but a true statement. In other words, if the new equation came from equation 1, plug it back into equation 2. If the new equation came from equation 2, plug it back into equation 1.**
$5\left(-2y-\frac{1}{2}\right)+16y=15$ $-10y-\frac{5}{2}+16y=15$ $6y-\frac{5}{2}=15$	**Distribute 5 then proceed to add like terms. We now have a decision to make. There is a fraction in the problem. It is not a difficult fraction, but let's get rid of the fractions anyway. This will make it easier to work with.**	
$2\left(6y-\frac{5}{2}\right)=2(15)$ $12y-5=30$ $+5 \quad +5$ $\frac{12y}{12}=\frac{35}{12}$ $y=\frac{35}{12}$	**Multiplying the entire equation by 2 allows us to get rid of the fraction. From this point, proceed to solve the equation using inverse operations.**	**Step 3- solve for the variable from step 2. Once this is finished, you will have solved for one of the variables in the system.**
$eq3 \quad x=-2y-\frac{1}{2}$ $y=\frac{35}{12}$ $x=-2\left(\frac{35}{12}\right)-\frac{1}{2}$ $x=-\frac{35}{6}-\frac{3}{6}$ $x=-\frac{38}{6}=-\frac{19}{3}$	**Now that we have solved for y, substitute that value back into equation 3 to solve for x. Reduce all fractions as much as possible and finally we solve for x.**	**Step 4- Substitute the solution found in step 3 back into one of the equations to solve for the remaining variable. It doesn't matter which equation is used for this, but bear in mind that equation 3 is already set up with a variable isolated.**
Solution: $\left(-\frac{19}{3},\frac{35}{12}\right)$	**We have now found the values of both x and y.**	**Step 5- write the solution to the system as an ordered pair.**

Elimination Method Example

Solve the following system of equations using the elimination method: $\begin{cases} 2x-3y=6 \\ 5x-7y=10 \end{cases}$

Work	Explanation	Procedure
$eq1 \quad 2x-3y=6$ $eq2 \quad 5x-7y=10$	**When using the elimination method to solve a system of equations our goal is to add the two equations together to get a variable to cancel. In order to do this, we will multiply one or both of the equations by a number in order to get opposite coefficients so the variables cancel.**	
$-5(2x-3y=6)$ $2(5x-7y=10)$	**In looking at the problem, it may be best to get rid of the x term. In order to do this, I will need coefficients of 10 and -10. In order to accomplish this, multiply equation 1 by -5 and the second equation by 2.**	**Step 1- decide which variable to get rid of and multiply one or both of the equations by a number in order to create opposite coefficients for the variable of your choice.**
$-10x+15y=-30$ $10x-14y=20$ $y=-10$	**Adding the two equations together, the x cancels, and we are left with part of our solution. We now have a choice, we can either substitute the value of y back into one of the equations and solve for x, or we can perform the elimination method again this time getting rid of the y.**	**Step 2- add the two equations together to eliminate one of the variables. Solve for whichever variable remains.**
$eq1 \quad 2x-3y=6$ $y=-10$ $2x-3(-10)=6$ $2x+30=6$ $-30 \quad -30$ $\frac{2x}{2}=\frac{-24}{2}$ $x=-12$	**Since the numbers are smaller in equation 1, substitute -10 for y. Once this is done, we are left with a simple two step equations. Using inverse operations solve the remaining variable.**	**Step 3- when one of the solutions is found, either substitute it back into one of the equations to solve for the remaining variable, or perform the elimination method again to get rid of the other variable. We would perform the latter if our solution from step 2 came out to be a real nasty fraction that you do not want to work with.**
$(-12,-10)$	**We now have solutions for both the x and y values.**	**Step 5- write the solution to the system as an ordered pair.**

Remember, as you solve a system of equations, you are finding the point at which the two lines intersect. It is possible to have one solution as illustrated by the previous examples, it is also possible to have…

No solutions- In this case, the lines are parallel and never touch.

Infinite solutions- In this case, one line is graphed on top of the other one. This means that any point on the first line also exists on the second line. Since there are an infinite amount of points on any one line, we can conclude that there are infinite solutions.

The next page will illustrate what happens if the system is a special case.

Special Cases

As we look at the next two examples, recall our basic rules when solving an algebraic equation, inequality or system of equations.

When solving an algebraic equation, inequality or system of equations, if all of the variables cancel, and we are left with a false statement, there is no solution.
An example of a false statement is $4=9$

When solving an algebraic equation, inequality or system of equations, if all of the variables cancel and we are left with a true statement, there are infinitely many solutions.
An example of a true statement is $-6=-6$

The following is an example of a system with no solutions

Solve the system of equations: $\begin{cases} 2x+y=3 \\ 4x+2y=8 \end{cases}$

We will look at the solution to this system using either method to solve. Notice that regardless of which method is used, the answer is the same.

Substitution method

←Using the substitution method, it is best to isolate the y in equation 1, then solve the system.

$eq1 \quad 2x+y=3$

$eq2 \quad 4x+2y=8$

$$\begin{aligned} 2x+y&=3 \\ -2x\quad &-2x \\ eq3 \quad y&=3-2x \end{aligned}$$

$eq2 \quad 4x+2y=8$

$y=3-2x$

$$4x+2(3-2x)=8$$
$$4x+6-4x=8$$
$$6=8$$
$$6\neq 8$$

No Solution

Elimination Method

Using the elimination method, → multiply equation 1 by -2 to create opposite coefficients. Once that is done, solve the system.

$eq1 \quad 2x+y=3$

$eq2 \quad 4x+2y=8$

$$-2(2x+y=3)$$
$$4x+2y=8$$

$$\begin{aligned} -4x-2y&=-6 \\ 4x+2y&=8 \\ \hline 0&=2 \\ 0&\neq 2 \end{aligned}$$

No Solution

Regardless of which method is used here, the variables completely cancel out leaving us with a false statement.

The following problem is an example of a system with infinite solutions

Solve the system of equations: $\begin{cases} 2x + \ \ y = 3 \\ 4x + 2y = 6 \end{cases}$

We will look at the solution to this system using either method to solve. Notice that regardless of which method is used, the answer is the same.

Substitution method

$eq1 \quad 2x + \ \ y = 3$

$eq2 \quad 4x + 2y = 6$

←Using the substitution method, it is best to isolate the y in equation 1, then solve the system.

$$\begin{aligned} 2x + \ \ y &= 3 \\ -2x \qquad\quad &-2x \\ eq3 \qquad y &= 3 - 2x \end{aligned}$$

$eq2 \quad 4x + 2y = 8$

$y = 3 - 2x$

$4x + 2(3 - 2x) = 8$

$4x + 6 - 4x = 6$

$6 = 6$

Infinite Solutions

Elimination Method

$eq1 \quad 2x + \ \ y = 3$

$eq2 \quad 4x + 2y = 6$

Using the elimination method, → multiply equation 1 by -2 to create opposite coefficients. Once that is done, solve the system.

$-2(2x + \ \ y = 3)$

$4x + 2y = 6$

$$\begin{aligned} -4x - 2y &= -6 \\ 4x + 2y &= \ \ 6 \\ \hline 0 &= 0 \end{aligned}$$

Regardless of which method is used here, the variables completely cancel out leaving us with a true statement.

Infinite Solutions

It is important to remember that "infinite solutions" is not the same the thing as infinity. When solving a system of equations, the answer infinity would imply that every point on the Cartesian Plane is a solution to the system. "Infinite solutions" on the other hand, means that every point on line 1 exists on line 2. This means if we were to graph the two lines on the same plane, line 2 would be right on top of line 1.