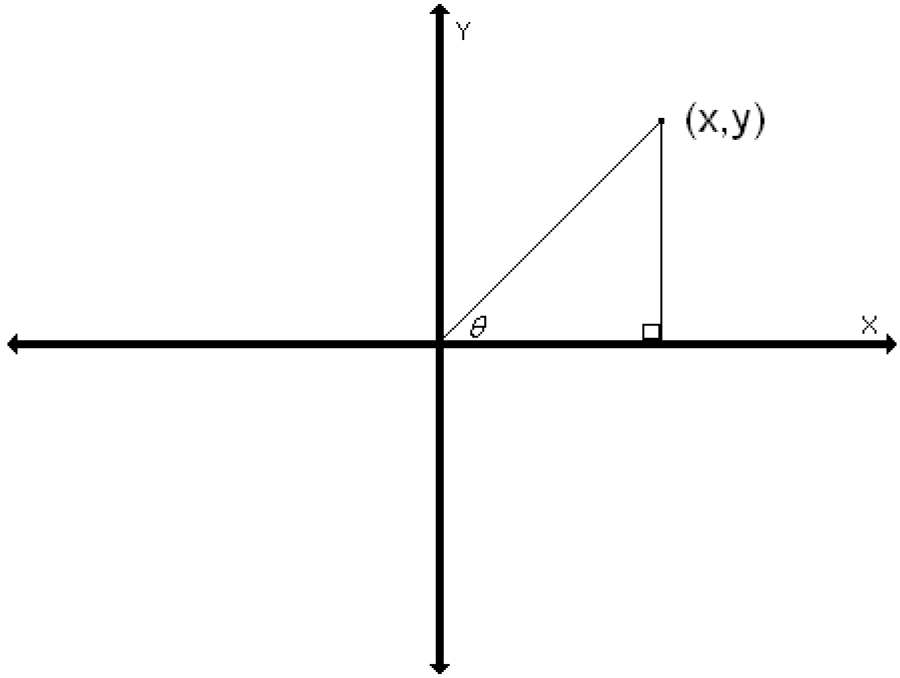
Trigonometric Functions of any Angle



When evaluating any angle θ, in standard position, whose terminal side is given by the coordinates (x,y), a reference angle is always used. Notice how a right triangle has been created. This will allow us to evaluate the six trigonometric functions of any angle.

Notice the side opposite the angle θ has a length of the y value of the given coordinates. The adjacent side has a length of the x value of the coordinates. The length of the hypotenuse is given by $\sqrt{x^2 + y^2}$.

Lets say, for the sake of argument, the length of the hypotenuse is 1 unit. This would mean the following would be true.

$$\sin\theta = y \qquad \csc\theta = \frac{1}{y}$$

$$\cos\theta = x \qquad \sec\theta = \frac{1}{x}$$

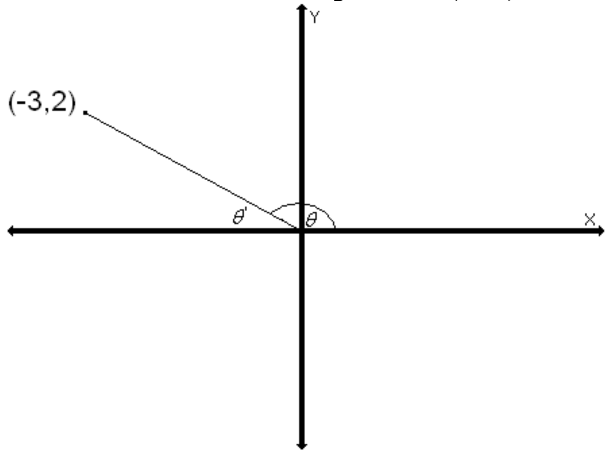
$$\tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{x}{y}$$

You must think of the sine function as giving you the y value, whereas the cosine function yields the x value. This is how we will determine whether the sine, cosine, tangent, cosecant, secant or cotangent of a given angle is a positive or negative value.

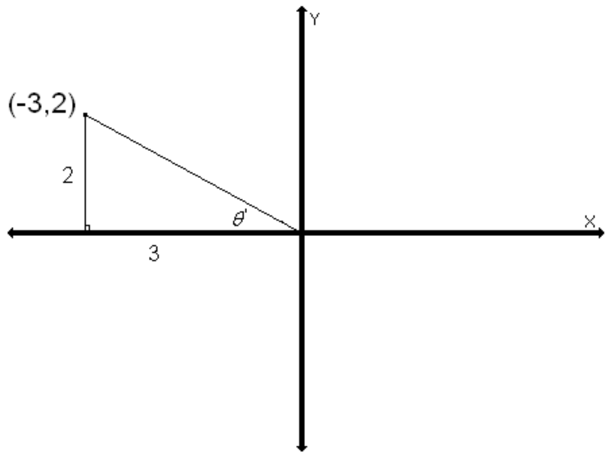
If the angle to be evaluated is in quadrant IV, for instance, the sine of the angle θ will be negative. The cosine of θ, in this instance, will be positive, while the tangent of the angle θ will be negative.

Example

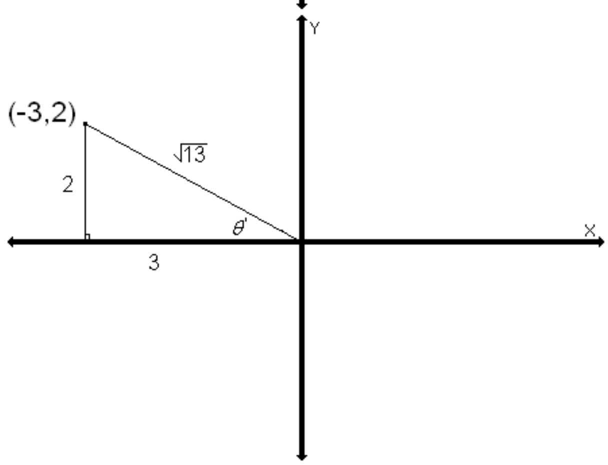
Evaluate the six trigonometric functions of an angle θ, in standard position, whose terminal side has an endpoint of (-3,2).



The angle with terminal side is first drawn. Remember, in order to evaluate the six trigonometric functions for θ, use the reference angle θ'.



From the endpoint of the terminal side of the angle, a line is drawn to the x axis. This is the reason reference angles are always drawn in relation to the x axis. It will always create a right triangle with which to work. Now all that is needed to solve the problem, is to find the length of the hypotenuse then the values of the six trigonometric functions can be found.



Using the Pythagorean Theorem, the length of the hypotenuse may be found.

$$2^2 + 3^2 = c^2$$

$$4 + 9 = c^2$$

$$13 = c^2$$

$$\sqrt{13} = c$$

$$\sin\theta = \frac{2}{\sqrt{13}} = \frac{2\sqrt{13}}{13} \qquad \csc\theta = \frac{\sqrt{13}}{2}$$

$$\cos\theta = -\frac{3}{\sqrt{13}} = -\frac{3\sqrt{13}}{13} \qquad \sec\theta = -\frac{\sqrt{13}}{3}$$

$$\tan\theta = -\frac{2}{3} \qquad \cot\theta = -\frac{3}{2}$$

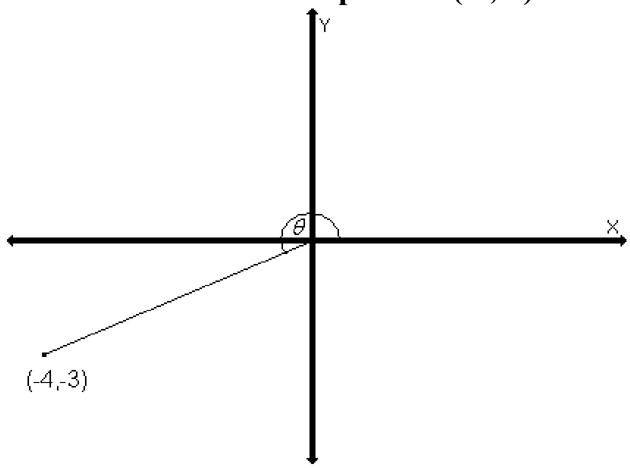
The first three functions are evaluated using Soh-Cah-Toa.

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} \qquad \cos\theta = \frac{\text{adj}}{\text{hyp}} \qquad \tan\theta = \frac{\text{opp}}{\text{adj}}$$

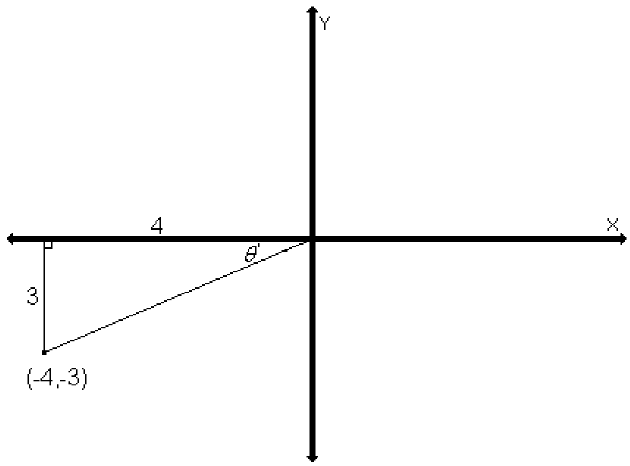
To find the second set of functions take the reciprocals of the first three. Rationalize any denominators if needed. Note the terminal side to this angle is in quadrant II. This means cosine, tangent, secant and cotangent are all negative values.

Example

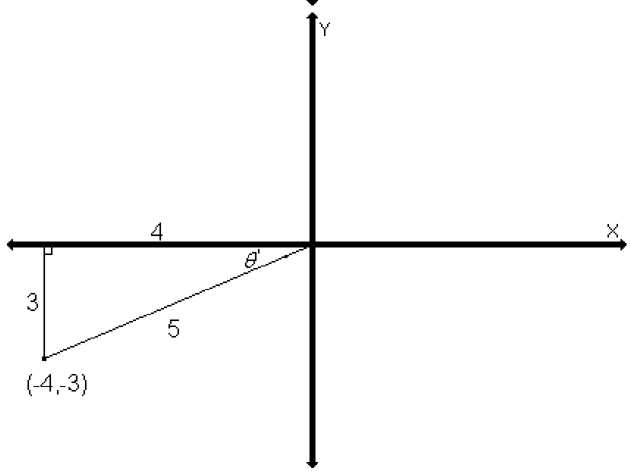
Evaluate the six trigonometric functions of the angle θ, in standard position, whose terminal side has an endpoint of (-4,-3).



Begin by drawing the angle θ in standard position whose terminal side has the endpoint of (-4,-3).



A right triangle is formed by drawing a line segment to the x axis. Now use the reference angle that is drawn in relation to the x axis to evaluate the six trigonometric functions.



Since this is obviously a 3-4-5 right triangle, there is no need to use the Pythagorean Theorem in this case.

$$\sin\theta = -\frac{3}{5} \qquad \csc\theta = -\frac{5}{3}$$

$$\cos\theta = -\frac{4}{5} \qquad \sec\theta = -\frac{5}{4}$$

$$\tan\theta = \frac{3}{4} \qquad \cot\theta = \frac{4}{3}$$

Since this angle resides in quadrant III, sine, cosine, cosecant and secant are negative values. Tangent is $\frac{y}{x}$ and cotangent is $\frac{x}{y}$. This means both tangent and cotangent will be positive values.

Once again, think of the sine of an angle θ as yielding the y value, while the cosine yields the x value when the hypotenuse is 1. Since the tangent of an angle is y over x, as $\sin\theta = y$ and $\cos\theta = x$; it is a trigonometric identity that $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$.

If the hypotenuse is any other length, the following is true.

$$\sin\theta = \frac{y}{h} \qquad \csc\theta = \frac{h}{y}$$

$$\cos\theta = \frac{x}{h} \qquad \sec\theta = \frac{h}{x}$$

$$\tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{x}{y}$$

These are the actual equations used for evaluating the six trigonometric functions. The reason we think of sine being the y value, cosine being the x value, and tangent being sine divided by cosine is to determine whether the value of a trigonometric function is positive or negative. This of course all depends on where the terminal side of the angle lies.

The following questions will require evaluating the six trigonometric functions of an angle θ given different types of information. Understand that these are the same types of questions encountered on the previous pages, just asked in a different manner.

Evaluate the six trigonometric functions of an angle θ, *in standard position, where* $\sin\theta = \dfrac{2}{3}$ *and* $\cos\theta < 0$.

This question will be done shortly. Since the sine of an angle can be thought of as the y value there are two quadrants in which sine is positive. It is therefore necessary to have one more piece of information to answer the question. There is some vital information that is needed to answer this type of question.

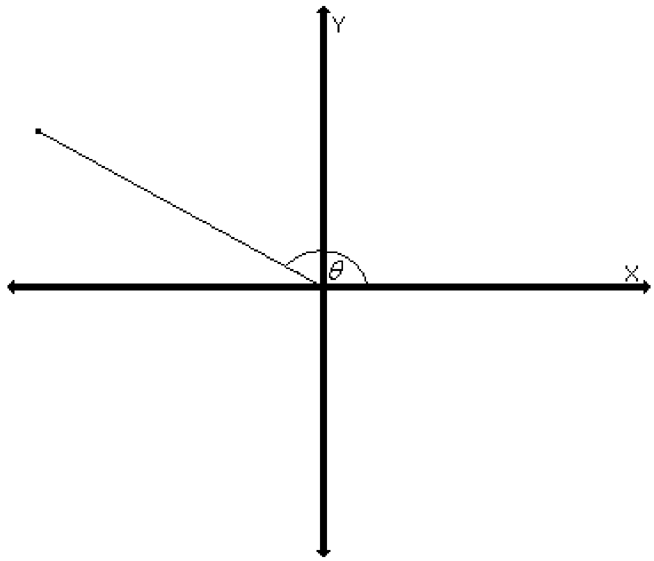
The following guidelines will help determine in which quadrant an angle lies.

- *Sine is positive in quadrants I and II.*
- *Sine is negative in quadrants III and IV.*
- *Cosine is positive in quadrants I and IV.*
- *Cosine is negative in quadrants II and III.*
- *Tangent is positive in quadrants I and III.*
- *Tangent is negative in quadrants II and IV.*

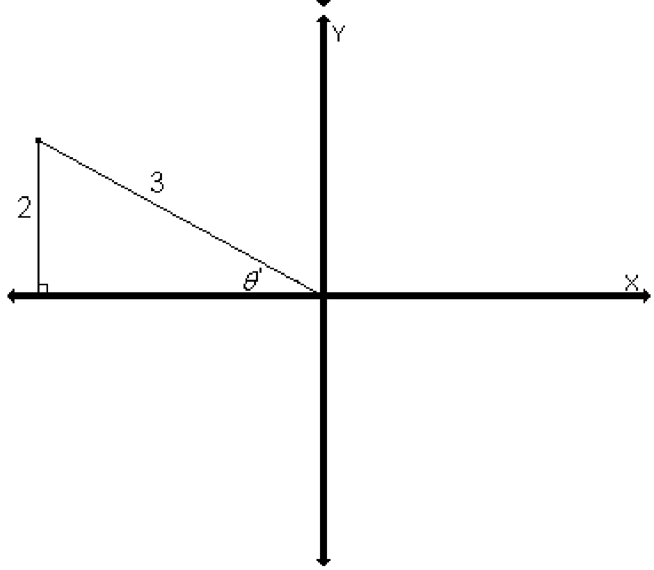
For this particular problem, $\sin\theta > 0$ and $\tan\theta < 0$, this means the angle θ must reside in quadrant II. This information tells us where to construct our triangle.

Example

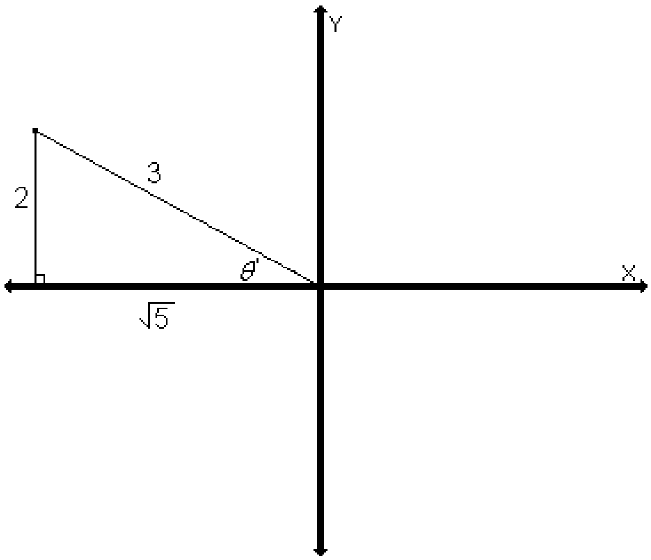
Evaluate the six trigonometric functions of an angle θ*, in standard position, where* $\sin\theta = \dfrac{2}{3}$ ***and*** $\cos\theta < 0$.



According to the information given, the sine of the angle is positive and cosine is negative. This means the terminal side of the angle to be evaluated must be in quadrant II.



From here, we will use the reference angle drawn in relation to the x axis. A right triangle is then constructed. Since the sine of an angle is opposite over hypotenuse, the 2 and the 3 can be placed on the appropriate sides of the triangle.



Using the Pythagorean Theorem, the adjacent side is found to be $\sqrt{5}$ units.

$$x^2 + 2^2 = 3^2$$

$$x^2 + 4 = 9$$

$$x^2 = 5$$

$$x = \sqrt{5}$$

$$\sin\theta = \frac{2}{3} \qquad \csc\theta = \frac{3}{2}$$

$$\cos\theta = -\frac{\sqrt{5}}{3} \qquad \sec\theta = -\frac{3}{\sqrt{5}} = -\frac{3\sqrt{5}}{5}$$

$$\tan\theta = -\frac{2}{\sqrt{5}} = -\frac{2\sqrt{5}}{5} \qquad \cot\theta = -\frac{\sqrt{5}}{2}$$

At the beginning of the problem, the value of sine was given. Therefore, we can fill in the values of sine and cosecant right away. From that point, the other values can be found using: Soh-Cah-Toa.

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} \qquad \cos\theta = \frac{\text{adj}}{\text{hyp}} \qquad \tan\theta = \frac{\text{opp}}{\text{adj}}$$

$$\sin\theta = \frac{2}{3} \qquad \csc\theta = \frac{3}{2}$$

$$\cos\theta = -\frac{\sqrt{5}}{3} \qquad \sec\theta = -\frac{3\sqrt{5}}{5}$$

$$\tan\theta = -\frac{2\sqrt{5}}{5} \qquad \cot\theta = -\frac{\sqrt{5}}{2}$$

Here are the values of the six trigonometric functions of the angle θ.