

# Chpt the 3rd

## Exponential and Logarithmic Functions



### 3.4 Exponential and Logarithmic Equations



# Chpt 3-4



## Homework



3-4 p432 11, 13, 19, 31, 35, 41, 43, 53, 55, 63, 69, 77, 81, 85, 89



# Chpt 3-4



## Objectives



Use like bases to solve exponential equations.



Use logarithms to solve exponential equations.



Use the definition of a logarithm to solve logarithmic equations.



Use the one-to-one property of logarithms to solve logarithmic equations.



Solve applied problems involving exponential and logarithmic equations.



# Solving Exponential Equations



✈ There are two two primary methods for solving exponential equations:

- ✈ 1. Isolate the exponential expression.
- ✈ 2. Use the one-to-one property.

✈ There are also two primary ways of solving logarithmic equations.

- ✈ 1. Isolate the logarithmic expression, then write the equation in equivalent exponential form.
- ✈ 2. Get a single logarithmic expression with the **same base** on each side of the equation; then use the one-to-one property.

✈ One to one properties

✈  $a^x = a^y$  if and only if  $x = y$ .

✈  $\log_a x = \log_a y$  if and only if  $x = y$ .

✈ Note: In both cases the **base** must be the same.

# Solving Exponential Equations by Expressing Each Side as a Power of the Same Base



✈ Solving Using Exponential Equations with common base.

✈ If  $b^m = b^n$  then  $m = n$ .

1. Rewrite the equation in the form  $b^m = b^n$ .

✈ Note; the base ( $b$ ) is the same in both expressions.

2. Set  $m = n$ .

3. Solve the equation.

# Example: Solving Exponential Equations



Solve:

$$5^{3x-6} = 125$$

$$5^{3x-6} = 5^3$$

$$3x - 6 = 3$$

$$x = 3$$



The solution set is  $\{3\}$

$$8^{x+2} = 4^{x-3}$$

$$(2^3)^{x+2} = (2^2)^{x-3}$$

$$2^{3(x+2)} = 2^{2(x-3)}$$

$$3(x + 2) = 2(x - 3)$$

$$3x + 6 = 2x - 6$$

$$x = -12$$



The solution set is  $\{-12\}$

# Example: Solving Exponential Equations



Solve:

$$e^{x+1} = \frac{1}{e}$$

$$e^{x+1} = e^{-1}$$

$$x + 1 = -1$$



The solution set is  $\{-2\}$

$$9^x = \frac{1}{\sqrt[3]{3}}$$

$$9^x = \frac{1}{3^{\frac{1}{3}}}$$

$$9^x = 3^{-\frac{1}{3}}$$

$$(3^2)^x = 3^{-\frac{1}{3}}$$

$$3^{2x} = 3^{-\frac{1}{3}}$$

$$2x = -\frac{1}{3}$$



The solution set is  $\left\{-\frac{1}{6}\right\}$



# Using Logarithms to Solve Exponential Equations



 Solving by taking logarithm of both sides.

1. Isolate the exponential expression.
2. If the base of the exponent is 10, take the common logarithm of both sides of the equation. Use the natural logarithm on both sides of the equation for bases other than 10.
3. Simplify using:  $\ln b^x = x \ln b$  or  $\ln e^x = x$  or  $\log 10^x = x$ .
4. Solve for the variable.

## STUDY TIP

Remember that the natural logarithmic function has a base of  $e$ .



# Solving Exponential Equations



Solve:

$$5^x = 134$$

$$\ln 5^x = \ln 134$$

$$x \ln 5 = \ln 134$$

$$x = \frac{\ln 134}{\ln 5} \approx 3.043$$

 The solution set is  $\left\{ \frac{\ln 134}{\ln 5} \right\}$

$$10^x = 8000$$

$$\log 10^x = \log 8000$$

$$x \log 10 = \log 8000$$

$$x = \log 8000$$

$$x \approx 3.903$$



The solution set is  $\log 8000$ .



# Solving an Exponential Equation



Solve:

$$3^{2x-1} = 7^{x+1}$$

$$\ln 3^{2x-1} = \ln 7^{x+1}$$

$$(2x-1)\ln 3 = (x+1)\ln 7$$


$$2x\ln 3 - \ln 3 = x\ln 7 + \ln 7$$

$$2x\ln 3 - x\ln 7 = \ln 3 + \ln 7$$

$$x(2\ln 3 - \ln 7) = \ln 3 + \ln 7$$

$$x = \frac{\ln 3 + \ln 7}{2\ln 3 - \ln 7}$$

$$x = \frac{\ln 21}{\ln 9 - \ln 7} = \frac{\ln 21}{\ln \frac{9}{7}} \approx 12.1144$$

 The solution set is  $\left\{ \frac{\ln 21}{\ln \frac{9}{7}} \right\}$



# Solving Exponential Equations



Solve:

$$4e^{2x} = 16$$

$$e^{2x} = 4$$

$$\ln e^{2x} = \ln 4$$

$$2x = \ln 4$$

$$x = \frac{\ln 4}{2} \approx .693$$

$$5e^{2+x} - 8 = 14$$

$$5e^{2+x} = 22$$

$$e^{2+x} = \frac{22}{5}$$

$$\ln e^{x+2} = \ln \frac{22}{5}$$

$$x + 2 = \ln \frac{22}{5}$$

$$x = \ln \frac{22}{5} - 2 \approx -0.518$$



# Solving Exponential Equations



Solve:

$$5(4^{3x-7}) - 6 = 15 \quad 5(4^{3x-7}) = 21 \quad 4^{3x-7} = \frac{21}{5}$$



From this point you have a couple of choices.

$$\log_4 \frac{21}{5} = 3x - 7$$

$$\ln 4^{3x-7} = \ln \frac{21}{5}$$

$$3x = \log_4 \frac{21}{5} + 7$$

$$(3x - 7) \ln 4 = \ln \frac{21}{5}$$

$$x = \frac{1}{3} \left( \log_4 \frac{21}{5} + 7 \right) \quad x = \frac{1}{3} \left( \frac{\ln \frac{21}{5}}{\ln 4} + 7 \right)$$

$$3x - 7 = \frac{\ln \frac{21}{5}}{\ln 4} \quad x = \frac{1}{3} \left( \frac{\ln \frac{21}{5}}{\ln 4} + 7 \right)$$



# Quadratic Form of the Exponential



 Some exponential equations are in the form of a quadratic equation.

1. Substitute  $u$  for the exponential term ( $u$  substitution).
2. Solve the quadratic equation for  $u$ .
3. Substitute the exponential for  $u$ .
4. Solve for the variable.



# Quadratic Form of the Exponential



 Solve:  $e^{2x} - 8e^x + 7 = 0$

$$\text{Let } u = e^x$$

$$\text{Then } u^2 = (e^x)^2 = e^{2x}$$

$$e^{2x} - 8e^x + 7 = 0 \Rightarrow u^2 - 8u + 7 = 0$$

$$u^2 - 8u + 7 = 0$$

$$(u - 7)(u - 1) = 0$$

$$u = 7 \text{ or } u = 1$$

$$e^x = 7 \text{ or } e^x = 1$$

$$\ln e^x = \ln 7 \quad x = \ln 7$$

or

$$\ln e^x = \ln 1 = 0 \quad x = 0$$

 The solution set is  $\{0, \ln 7\}$



# Quadratic Form of the Exponential



 Solve:  $2^{2x} + 2^x - 12 = 0$

$$\text{Let } u = 2^x$$

$$\text{Then } u^2 = (2^x)^2 = 2^{2x}$$

$$2^{2x} + 2^x - 12 = 0 \Rightarrow u^2 + u - 12 = 0$$

$$u^2 + u - 12 = 0$$

$$(u - 3)(u + 4) = 0$$

$$u = 3 \text{ or } u = -4$$

$$2^x = 3 \text{ or } \del{2^x = -4}$$


$$\ln 2^x = \ln 3 \quad x \ln 2 = \ln 3$$




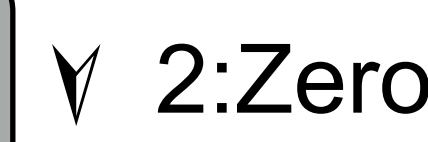



$$x = \frac{\ln 3}{\ln 2} \approx 1.5850$$

 The solution set is  $\left\{ \frac{\ln 3}{\ln 2} \right\}$





 Graph  $y = e^{2x} - 8e^x + 7$  and find the zeros.





 Set your left boundary by moving the cursor.
 
 Set your right boundary by moving the cursor.
 
 Guess?
 

 Here is your problem. I will insist on exact solutions. Your calculator suggests a zero at about 1.94591.




 Calculate  $\ln(7)$

 You must be able to solve both graphically and algebraically.



# Check your Results



-  Logarithmic expressions are defined only for logs of positive real numbers.
-  Always check proposed solutions of a logarithmic equation in the original equation.
-  Exclude from the solution set any proposed solution that produces the logarithm of a negative number or the logarithm of 0 (Extraneous solutions).



# Using the Definition of a Logarithm to Solve Logarithmic Equations



🚀 Sometimes you can convert a log equation into an exponential equation that is easier to solve.

1. Express the equation in the form  $\log_b M = x$ .
2. Use the definition of a log to rewrite the equation in exponential form.

$$\log_b M = x \Rightarrow b^x = M$$

3. Solve for the variable.
4. Check proposed solutions in the original equation. Include in the solution set only values for which  $M > 0$ .



# Example: Solving Logarithmic Equations



Solve:

$$\log_2(x-4) = 3$$

$$2^3 = (x-4)$$

$$x = 12$$

$$\log_2(12-4) = 3$$

- The solution set is  $\{12\}$ .

$$4\ln 3x = 8$$

$$\ln 3x = 2$$

$$e^2 = 3x$$

$$x = \frac{e^2}{3}$$

$$4\ln\left(3\left(\frac{e^2}{3}\right)\right) = 4\ln e^2 = 4(2) = 8$$

 The solution set is  $\left\{\frac{e^2}{3}\right\}$



# Example: Solving Logarithmic Equations



Solve:

$$\log x + \log(x-3) = 1$$

$$\log x(x-3) = 1$$

$$x(x-3) = 10$$

$$x^2 - 3x - 10 = 0$$

$$(x-5)(x+2) = 0$$

$$x = 5 \text{ or } x = -2$$

~~$$\log(-2) + \log(-2-3) = 1$$~~

$$\log 5 + \log(5-3) = 1$$

$$\log 5 + \log 2 = 1$$

$$\log 10 = 1$$

 The solution set is  $\{5\}$ .



# Example: Solving Logarithmic Equations



 Solve:

$$\log(5x+1) = \log(2x+3) + \log 2$$

$$\log(5x+1) = \log 2(2x+3)$$

$$5x+1 = 4x+6$$

$$x = 5$$

$$\log(26) = \log(13) + \log 2$$

 The solution set is  $\{5\}$ .

$$\log_2(x-3) + \log_2(x) - \log_2(x+2) = 2$$

$$\log_2\left(\frac{x(x-3)}{x+2}\right) = 2$$

$$\frac{x(x-3)}{x+2} = 4$$

$$x(x-3) = 4(x+2)$$

$$x^2 - 3x = 4x + 8$$

$$x^2 - 7x - 8 = 0$$

$$(x-8)(x+1) = 0$$

$$x = 8 \text{ or } \del{x = 1}$$

$$\log_2 5 + \log_2 8 - \log_2 10 = 2$$



# Using the One-to-One Property of Logarithms to Solve Logarithmic Equations



 Writing the equation as equal logarithms.

1. Express the equation in the form  $\log_b M = \log_b N$ .
2. Use the one-to-one property to rewrite the equation without logs.  
If  $\log_b M = \log_b N$ , then  $M = N$ .
3. Solve for the variable
4. Check the proposed solutions in the original equation. Include in the solution set only value form which  $M > 0$  and  $N > 0$ .

# Example: Solving a Logarithmic Equation



Solve:

$$\log_2(x-1) - \log_2(x+3) = \log_2\left(\frac{1}{x}\right)$$

$$\log_2\left(\frac{x-1}{x+3}\right) = \log_2\left(\frac{1}{x}\right)$$

$$\frac{x-1}{x+3} = \frac{1}{x}$$

$$x(x-1) = x+3$$

$$x^2 - x = x + 3$$

$$x^2 - 2x - 3 = 0$$

$$(x+1)(x-3) = 0$$

$$x = 3 \text{ or } \del{x = -1}$$

$$\log_2(2) - \log_2(6) = \log_2\left(\frac{1}{3}\right)$$



# Example: Solving a Logarithmic Equation



 Solve:

$$\ln(x-3) = \ln(7x-23) - \ln(x+1)$$

$$\ln(x-3) = \ln \frac{7x-23}{x+1}$$

$$x-3 = \frac{7x-23}{x+1}$$

$$(x-3)(x+1) = 7x-23$$

$$x^2-2x-3 = 7x-23$$

$$x^2-9x+20 = 0$$

$$(x-5)(x-4) = 0 \quad x=5 \text{ or } x=4$$

$$\ln(5-3) = \ln(7(5)-23) - \ln(5+1)$$

$$\ln 2 = \ln 12 - \ln 6$$

$$\ln(4-3) = \ln(7(4)-23) - \ln(4+1)$$

$$\ln 1 = \ln 5 - \ln 5$$

 The solution set is  $\{4, 5\}$ .

# Solving Exponential Equations



## Summary

1. Rewrite the equation in the form  $b^m = b^n$ .

- Note; the bases (b) are equal.

$$4^{5x-2} = 8^{2x+3} \Rightarrow 2^{2(5x-2)} = 2^{3(2x+3)}$$

$$2(5x-2) = 3(2x+3)$$

2. Take the  $\ln$  (or  $\log$ ) of both sides of the equation.

$$6^{2x} = 1024 \Rightarrow \ln 6^{2x} = \ln 1024$$

$$2x \ln 6 = \ln 1024$$



# Solving Logarithmic Equations



## Summary

1. Rewrite the logarithmic equation in exponential form.

$$\log_7(2x-5) = 11 \Rightarrow 7^{11} = 2x-5$$

2. Simplify the logarithmic equation to  $\log_b m = \log_b n$ .

$$\ln x + \ln(2x-1) = \ln(5x-3) - \ln 2x$$



$$\ln \frac{x}{2x-1} = \ln \frac{5x-3}{2x}$$



## Strategies for Solving Exponential and Logarithmic Equations

1. Rewrite the original equation in a form that allows the use of the One-to-One Properties of exponential or logarithmic functions.
2. Rewrite an *exponential* equation in logarithmic form and apply the Inverse Property of logarithmic functions.
3. Rewrite a *logarithmic* equation in exponential form and apply the Inverse Property of exponential functions.



# Example: Application



✈️ How long, to the nearest tenth of a year, will it take \$1000 to grow to \$3600 at 8% annual interest compounded quarterly?

$$A = P \left( 1 + \frac{r}{n} \right)^{nt}$$

$$3600 = 1000 \left( 1 + \frac{.08}{4} \right)^{4t}$$

$$3.6 = (1 + .02)^{4t}$$

$$\ln 3.6 = \ln (1.02)^{4t}$$

$$\ln 3.6 = 4t \ln (1.02)$$

$$4t = \frac{\ln 3.6}{\ln 1.02}$$

$$t = \frac{\ln 3.6}{4 \ln 1.02} \approx 16.17$$

It will take about 16.2 years for \$1000 to become \$3600.