

Chapter 5

Analytic Trigonometry

5.2 Sum and Difference Formulas

Chapter 5.2

Homework

- Read Sec 5-2
- Do p603 1-77 odd

Chapter 5.2

Objectives

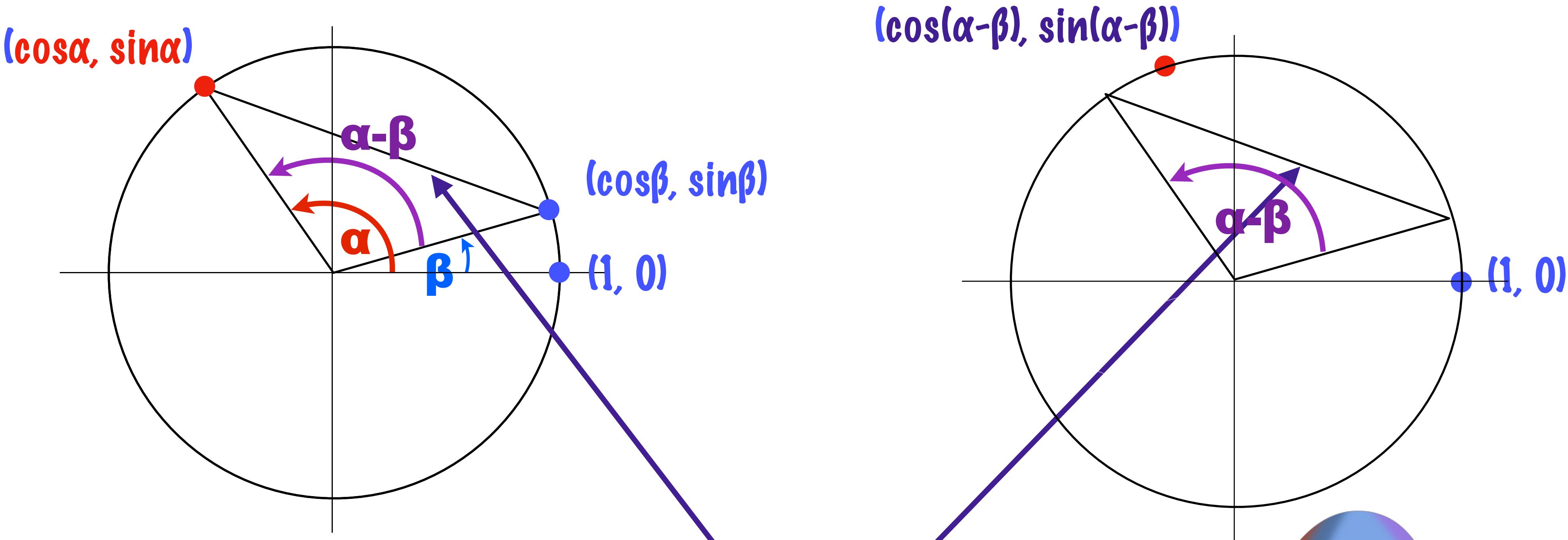
Use Sum and Difference Formulae to evaluate trig expressions, verify trig identities, and solve trig equations.



Sum and Difference Formulae

Objective: Use Sum and Difference Formulae to evaluate trig expressions and verify trig identities.

$$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$$



Same length
Do you care how we prove this?

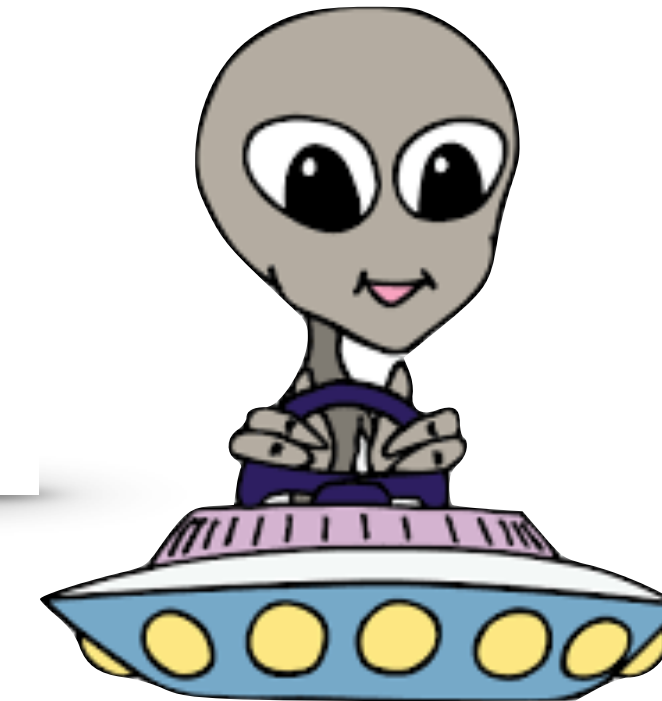


Sum and Difference Formulae

Objective: Use Sum and Difference Formulae to evaluate trig expressions and verify trig identities.

- We know that $\cos 30^\circ = \frac{\sqrt{3}}{2}$. Using $\cos 30^\circ = \cos(90 - 60)^\circ$ and the difference formula for cosines, obtain that exact value.

$$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$$



$$\cos 30^\circ = \cos(90 - 60)^\circ = \cos 90^\circ \cos 60^\circ + \sin 90^\circ \sin 60^\circ = 0 \cdot \frac{1}{2} + 1 \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}$$

- Find the exact value of $\cos 70^\circ \cos 40^\circ + \sin 70^\circ \sin 40^\circ$.

$$\cos 70^\circ \cos 40^\circ + \sin 70^\circ \sin 40^\circ = \cos(70 - 40)^\circ = \cos 30^\circ = \frac{\sqrt{3}}{2}$$

Sum and Difference Formulae

Objective: Use Sum and Difference Formulae to evaluate trig expressions and verify trig identities.

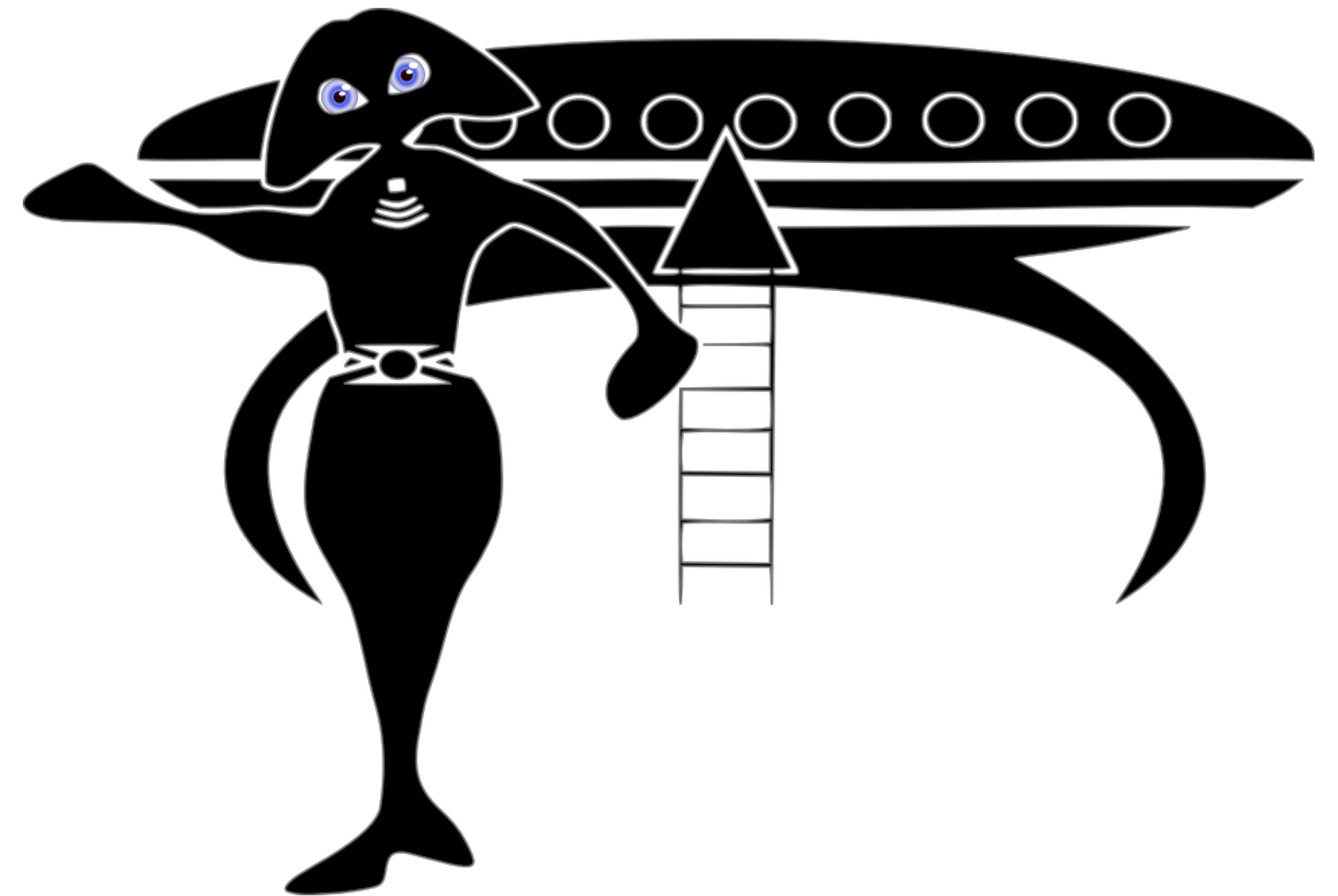
● Verify the identity $\frac{\cos(\alpha - \beta)}{\cos \alpha \cos \beta} = 1 + \tan \alpha \tan \beta$

$$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$$

$$\frac{\cos(\alpha - \beta)}{\cos \alpha \cos \beta} = \frac{\cos \alpha \cos \beta + \sin \alpha \sin \beta}{\cos \alpha \cos \beta}$$

$$= \frac{\cos \alpha \cos \beta}{\cos \alpha \cos \beta} + \frac{\sin \alpha \sin \beta}{\cos \alpha \cos \beta}$$

$$= 1 + \frac{\sin \alpha}{\cos \alpha} \cdot \frac{\sin \beta}{\cos \beta} = 1 + \tan \alpha \tan \beta$$



$$\tan x = \frac{\sin x}{\cos x}$$

Sum and Difference Formulae

Objective: Use Sum and Difference Formulae to evaluate trig expressions and verify trig identities.

● Verify the identity $\cos(\alpha + \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$

$$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$$

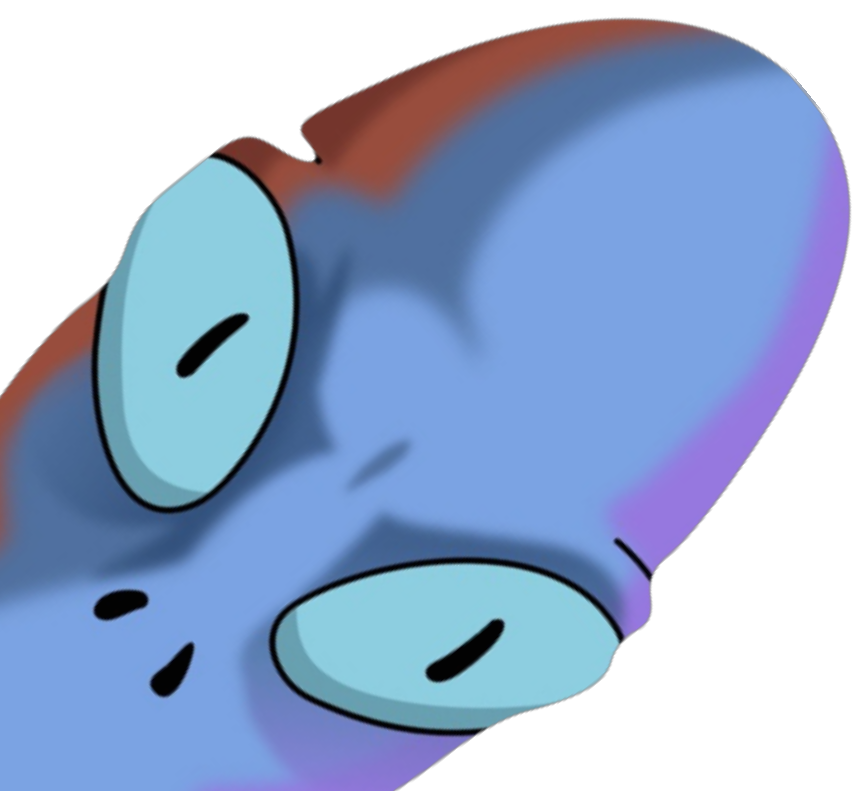
$$\cos(\alpha + \beta) = \cos(\alpha - (-\beta)) = \cos \alpha \cos(-\beta) + \sin \alpha \sin(-\beta)$$

$$= \cos \alpha \cos \beta + \sin \alpha (-\sin \beta)$$

$$\sin(-\beta) = -\sin \beta$$

$$= \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$



Sum and Difference Formulae

Objective: Use Sum and Difference Formulae to evaluate trig expressions and verify trig identities.

● Find the exact value of $\cos \frac{7\pi}{12}$

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$



$$\begin{aligned}\cos \frac{7\pi}{12} &= \cos \left(\frac{4\pi}{12} + \frac{3\pi}{12} \right) = \cos \left(\frac{\pi}{3} + \frac{\pi}{4} \right) = \cos \left(\frac{\pi}{3} \right) \cos \left(\frac{\pi}{4} \right) - \sin \left(\frac{\pi}{3} \right) \sin \left(\frac{\pi}{4} \right) \\ &= \frac{1}{2} \cdot \frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} \\ &= \frac{\sqrt{2} - \sqrt{6}}{4}\end{aligned}$$

Sum and Difference Formulae

Objective: Use Sum and Difference Formulae to evaluate trig expressions and verify trig identities.

● Verify the identity $\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$

$$\sin(\alpha - \beta) = \cos\left(\frac{\pi}{2} - (\alpha - \beta)\right) = \cos\left(\frac{\pi}{2} - \alpha + \beta\right)$$

$$\sin x = \cos\left(\frac{\pi}{2} - x\right)$$

$$= \cos\left(\left(\frac{\pi}{2} - \alpha\right) + \beta\right)$$

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

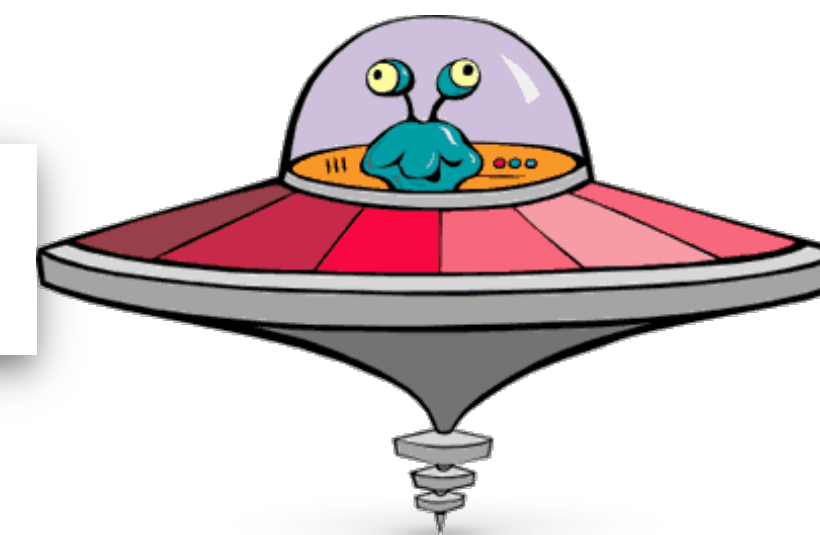
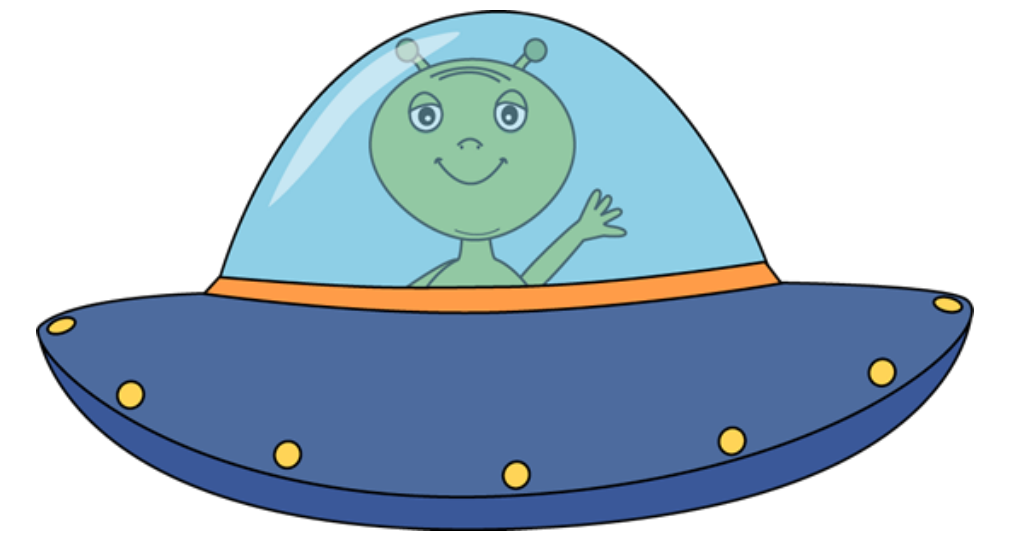
$$= \cos\left(\frac{\pi}{2} - \alpha\right) \cos(\beta) - \sin\left(\frac{\pi}{2} - \alpha\right) \sin(\beta)$$

$$\cos\left(\frac{\pi}{2} - x\right) = \sin x$$

$$= \sin(\alpha) \cos(\beta) - \cos(\alpha) \sin(\beta)$$

$$\sin\left(\frac{\pi}{2} - x\right) = \cos x$$

$$\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$$



Sum and Difference Formulae

Objective: Use Sum and Difference Formulae to evaluate trig expressions and verify trig identities.

● Verify the identity $\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$

$$\sin(\alpha + \beta) = \cos\left(\frac{\pi}{2} - (\alpha + \beta)\right) = \cos\left(\frac{\pi}{2} - \alpha - \beta\right)$$

$$\sin x = \cos\left(\frac{\pi}{2} - x\right)$$

$$= \cos\left(\left(\frac{\pi}{2} - \alpha\right) - \beta\right)$$

$$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$$

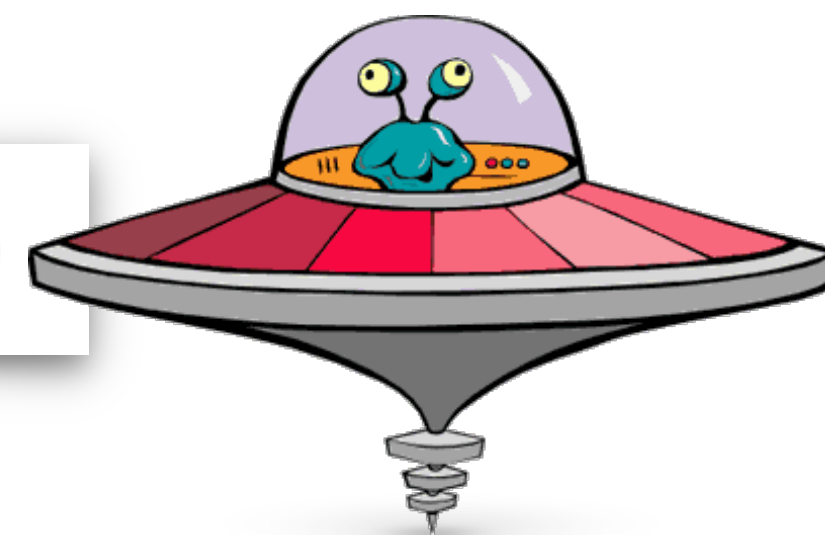
$$= \cos\left(\frac{\pi}{2} - \alpha\right) \cos(\beta) + \sin\left(\frac{\pi}{2} - \alpha\right) \sin(\beta)$$

$$\cos\left(\frac{\pi}{2} - x\right) = \sin x$$

$$= \sin(\alpha) \cos(\beta) + \cos(\alpha) \sin(\beta)$$

$$\sin\left(\frac{\pi}{2} - x\right) = \cos x$$

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$



Sum and Difference Formulae

Objective: Use Sum and Difference Formulae to evaluate trig expressions and verify trig identities.

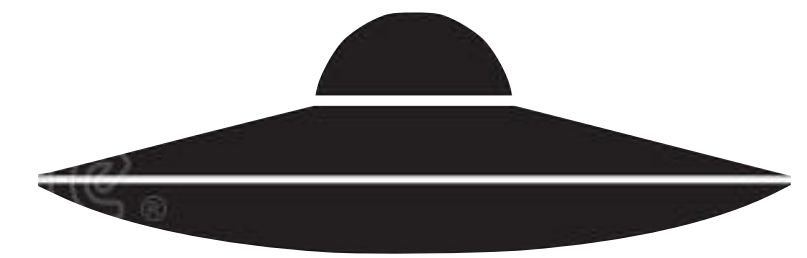
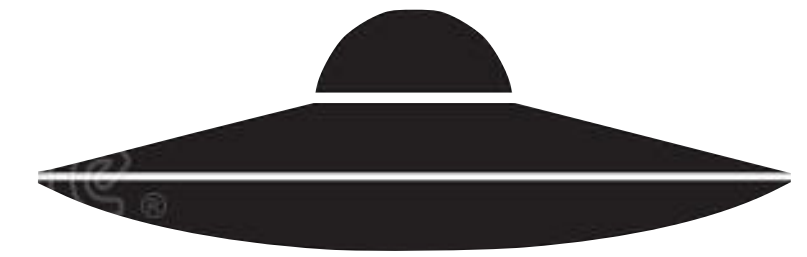
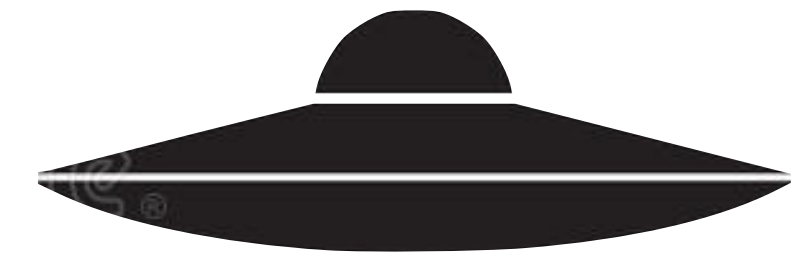
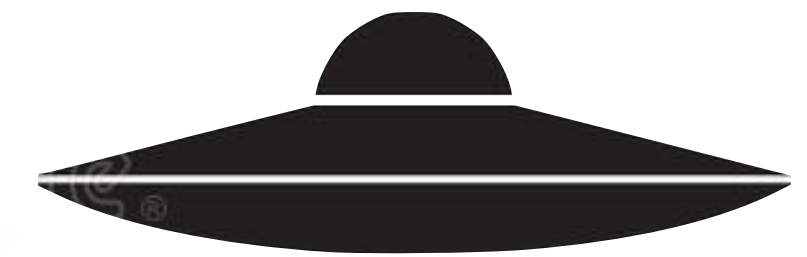
🌐 All together now!

$$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$$

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

$$\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$$

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

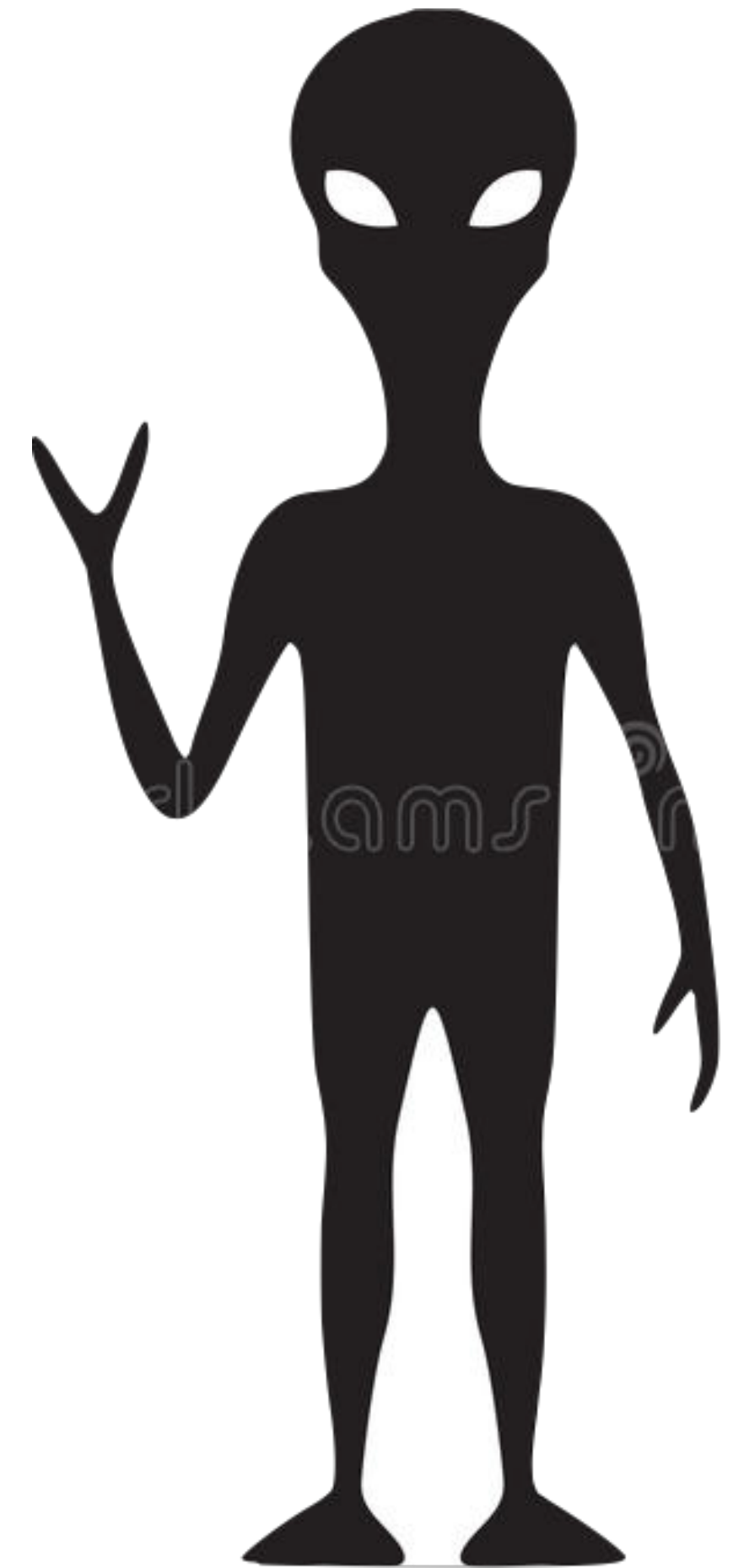


Sum and Difference Formulae

Objective: Use Sum and Difference Formulae to evaluate trig expressions and verify trig identities.

● Find the exact value of $\sin \frac{5\pi}{12}$ using the fact that $\frac{5\pi}{12} = \frac{3\pi}{12} + \frac{2\pi}{12} = \frac{\pi}{4} + \frac{\pi}{6}$

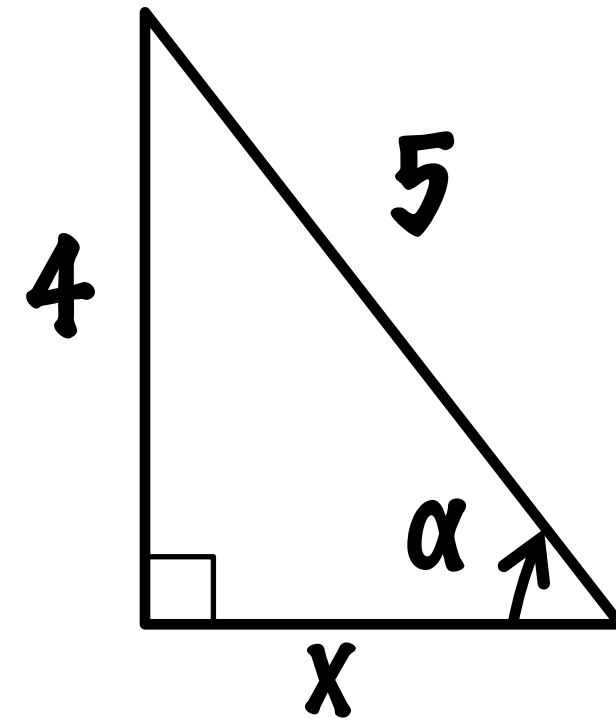
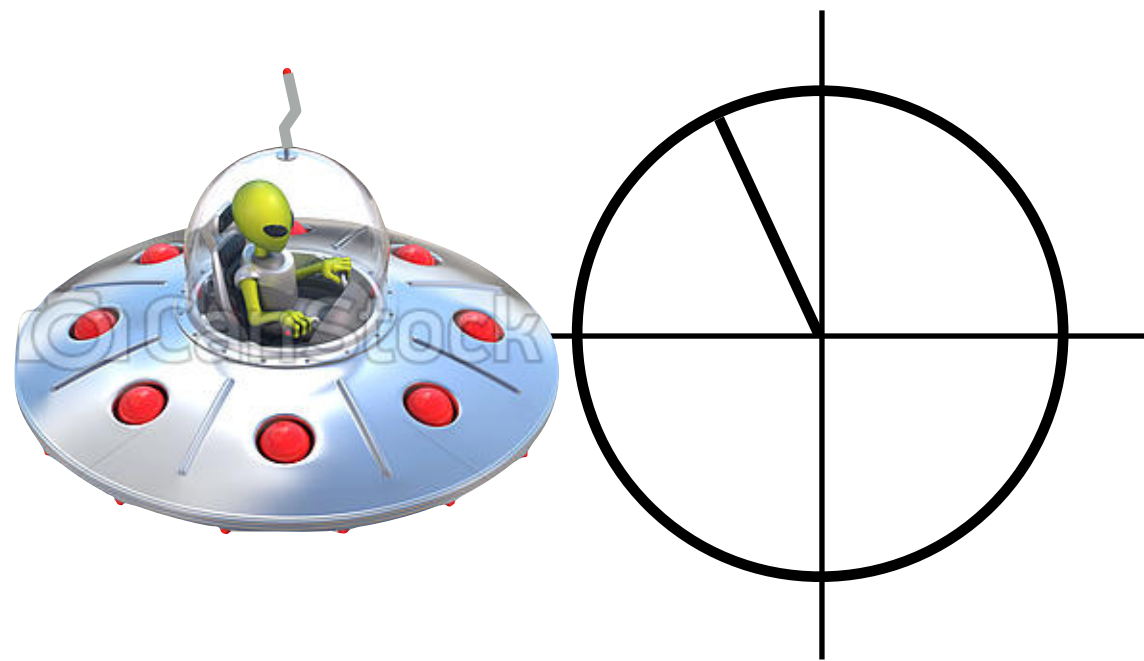
$$\begin{aligned}\sin \frac{5\pi}{12} &= \sin \left(\frac{\pi}{4} + \frac{\pi}{6} \right) = \sin \frac{\pi}{4} \cos \frac{\pi}{6} + \sin \frac{\pi}{6} \cos \frac{\pi}{4} \\ &= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{1}{2} \cdot \frac{\sqrt{2}}{2} \\ &= \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4} = \frac{\sqrt{6} + \sqrt{2}}{4}\end{aligned}$$



Sum and Difference Formulae

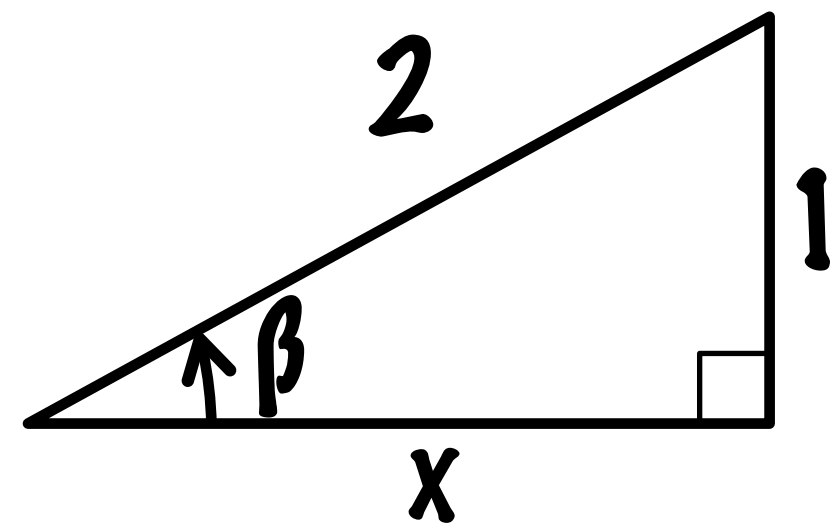
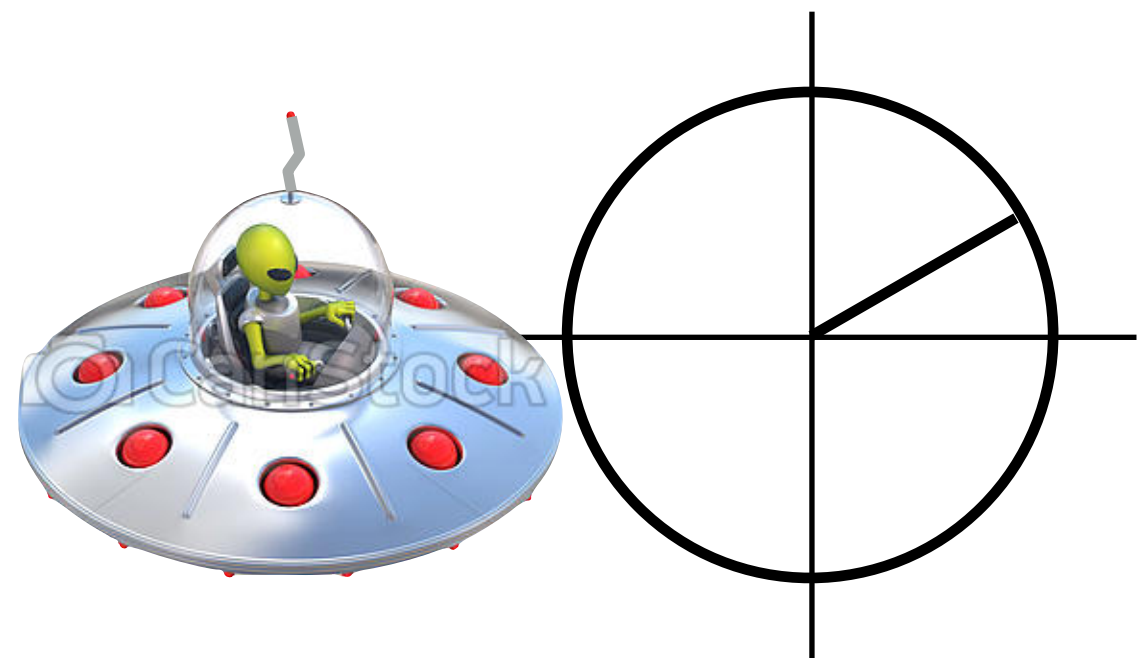
Objective: Use Sum and Difference Formulae to evaluate trig expressions and verify trig identities.

- Suppose that $\sin \alpha = \frac{4}{5}$ for a quadrant II angle α and $\sin \beta = \frac{1}{2}$ for a quadrant I angle β . Find the exact value of $\cos \alpha$ and $\cos \beta$.



$$x^2 + 4^2 = 5^2$$
$$x = -3$$

$$\cos \alpha = -\frac{3}{5}$$



$$x^2 + 1^2 = 2^2$$
$$x = \sqrt{3}$$

$$\cos \beta = \frac{\sqrt{3}}{2}$$

Sum and Difference Formulae

Objective: Use Sum and Difference Formulae to evaluate trig expressions and verify trig identities.

- Suppose that $\sin \alpha = \frac{4}{5}$ for a quadrant II angle α and $\sin \beta = \frac{1}{2}$ for a quadrant I angle β .
Find the exact value of $\cos(\alpha + \beta)$.

$$\cos \alpha = -\frac{3}{5} \quad \cos \beta = \frac{\sqrt{3}}{2}$$

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

$$\begin{aligned} &= -\frac{3}{5} \cdot \frac{\sqrt{3}}{2} - \frac{4}{5} \cdot \frac{1}{2} \\ &= -\frac{3\sqrt{3}}{10} - \frac{4}{10} = -\frac{3\sqrt{3} + 4}{10} \end{aligned}$$

Yes, this is the EXACT solution



Sum and Difference Formulae

Objective: Use Sum and Difference Formulae to evaluate trig expressions and verify trig identities.

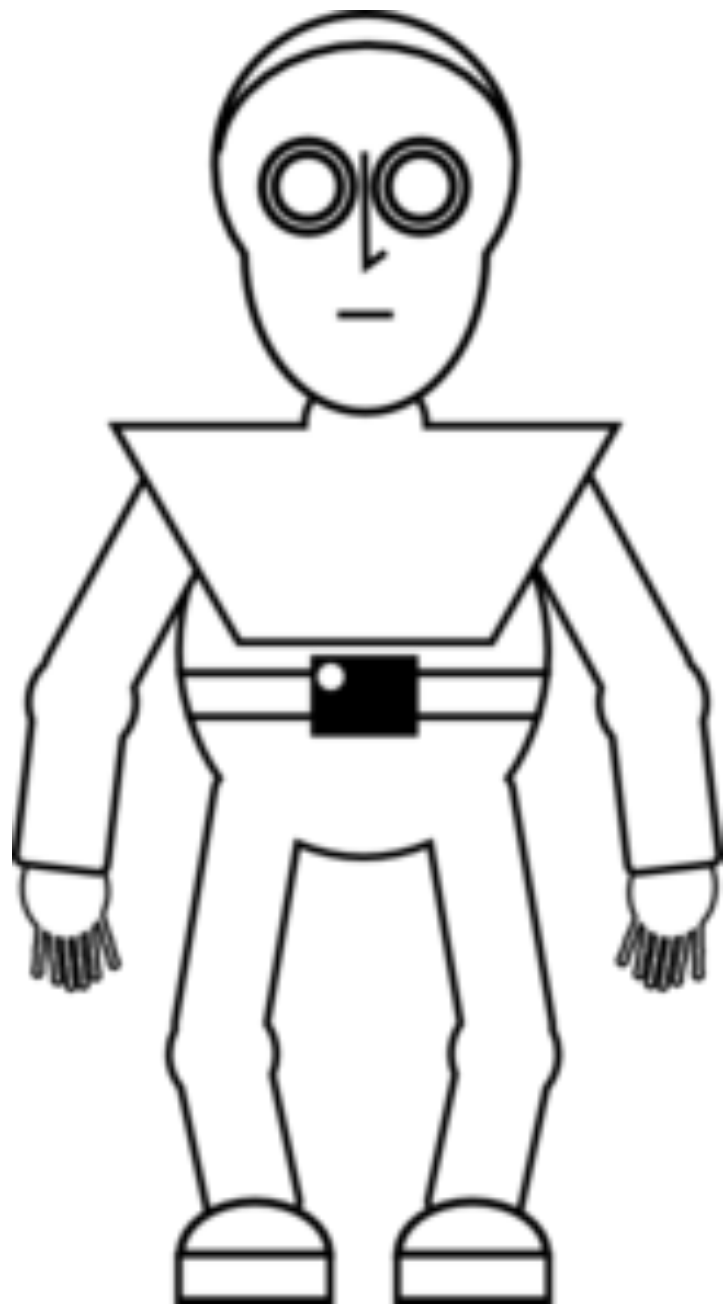
- Suppose that $\sin \alpha = \frac{4}{5}$ for a quadrant II angle α and $\sin \beta = \frac{1}{2}$ for a quadrant I angle β .
Find the exact value of $\sin(\alpha + \beta)$.

$$\cos \alpha = -\frac{3}{5} \quad \cos \beta = \frac{\sqrt{3}}{2}$$

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

$$\begin{aligned} &= \frac{4}{5} \cdot \frac{\sqrt{3}}{2} + -\frac{3}{5} \cdot \frac{1}{2} \\ &= \frac{4\sqrt{3}}{10} - \frac{3}{10} = -\frac{4\sqrt{3} - 3}{10} \end{aligned}$$

Yes, this is the EXACT solution



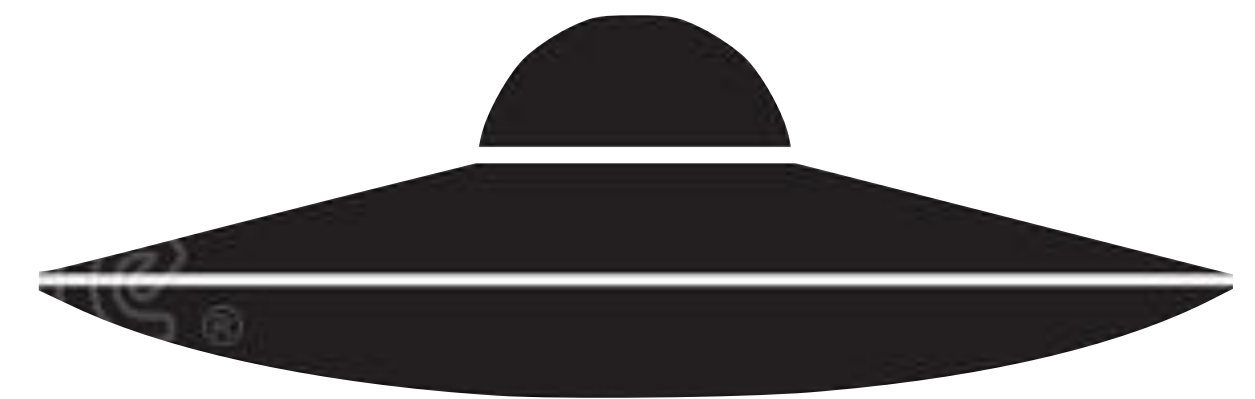
Sum and Difference Formulae

Objective: Use Sum and Difference Formulae to evaluate trig expressions and verify trig identities.

$$\tan(\alpha + \beta)$$

$$\begin{aligned}\tan(\alpha + \beta) &= \frac{\sin(\alpha + \beta)}{\cos(\alpha + \beta)} = \frac{\sin \alpha \cos \beta + \cos \alpha \sin \beta}{\cos \alpha \cos \beta - \sin \alpha \sin \beta} = \frac{\frac{\sin \alpha \cos \beta + \cos \alpha \sin \beta}{\cos \alpha \cos \beta}}{\frac{\cos \alpha \cos \beta - \sin \alpha \sin \beta}{\cos \alpha \cos \beta}} \\ &= \frac{\frac{\sin \alpha \cos \beta}{\cos \alpha \cos \beta} + \frac{\cos \alpha \sin \beta}{\cos \alpha \cos \beta}}{\frac{\cos \alpha \cos \beta}{\cos \alpha \cos \beta} - \frac{\sin \alpha \sin \beta}{\cos \alpha \cos \beta}} = \frac{\frac{\sin \alpha}{\cos \alpha} + \frac{\sin \beta}{\cos \beta}}{1 - \frac{\sin \alpha}{\cos \alpha} \cdot \frac{\sin \beta}{\cos \beta}} = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \cdot \tan \beta}\end{aligned}$$

$$\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}$$



Sum and Difference Formulae

Objective: Use Sum and Difference Formulae to evaluate trig expressions and verify trig identities.

$$\tan(\alpha - \beta)$$

$$\tan(\alpha - \beta) = \tan(\alpha + (-\beta))$$

$$= \frac{\tan \alpha + \tan(-\beta)}{1 - \tan \alpha \tan(-\beta)}$$

$$= \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}$$

$$\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}$$

$$\tan(-\beta) = -\tan \beta$$

$$\tan(\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}$$



Sum and Difference Formulae

Objective: Use Sum and Difference Formulae to evaluate trig expressions and verify trig identities.

● Once more, all together

$$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$$

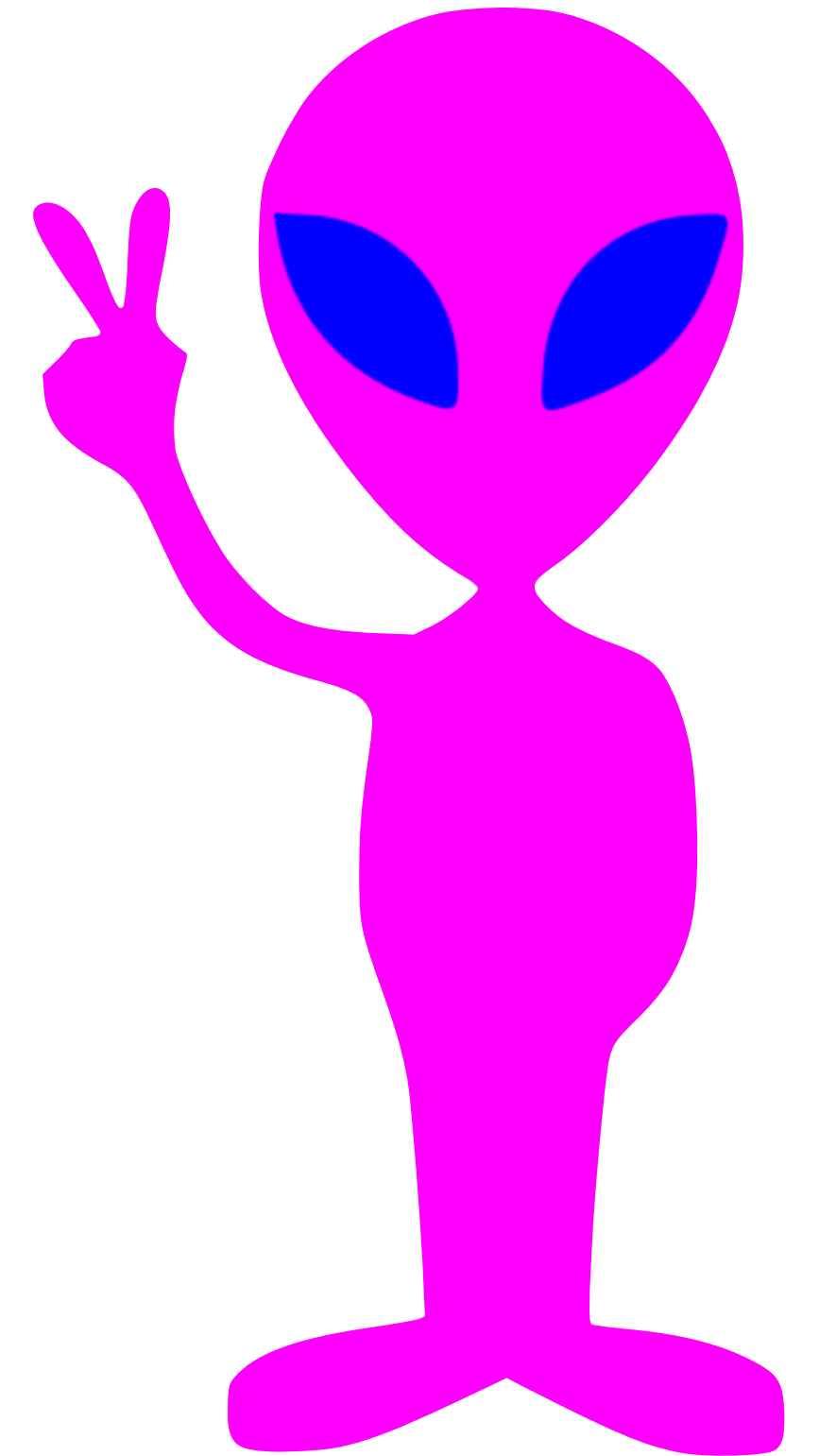
$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

$$\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$$

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

$$\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}$$

$$\tan(\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}$$



Sum and Difference Formulae

Objective: Use Sum and Difference Formulae to evaluate trig expressions and verify trig identities.

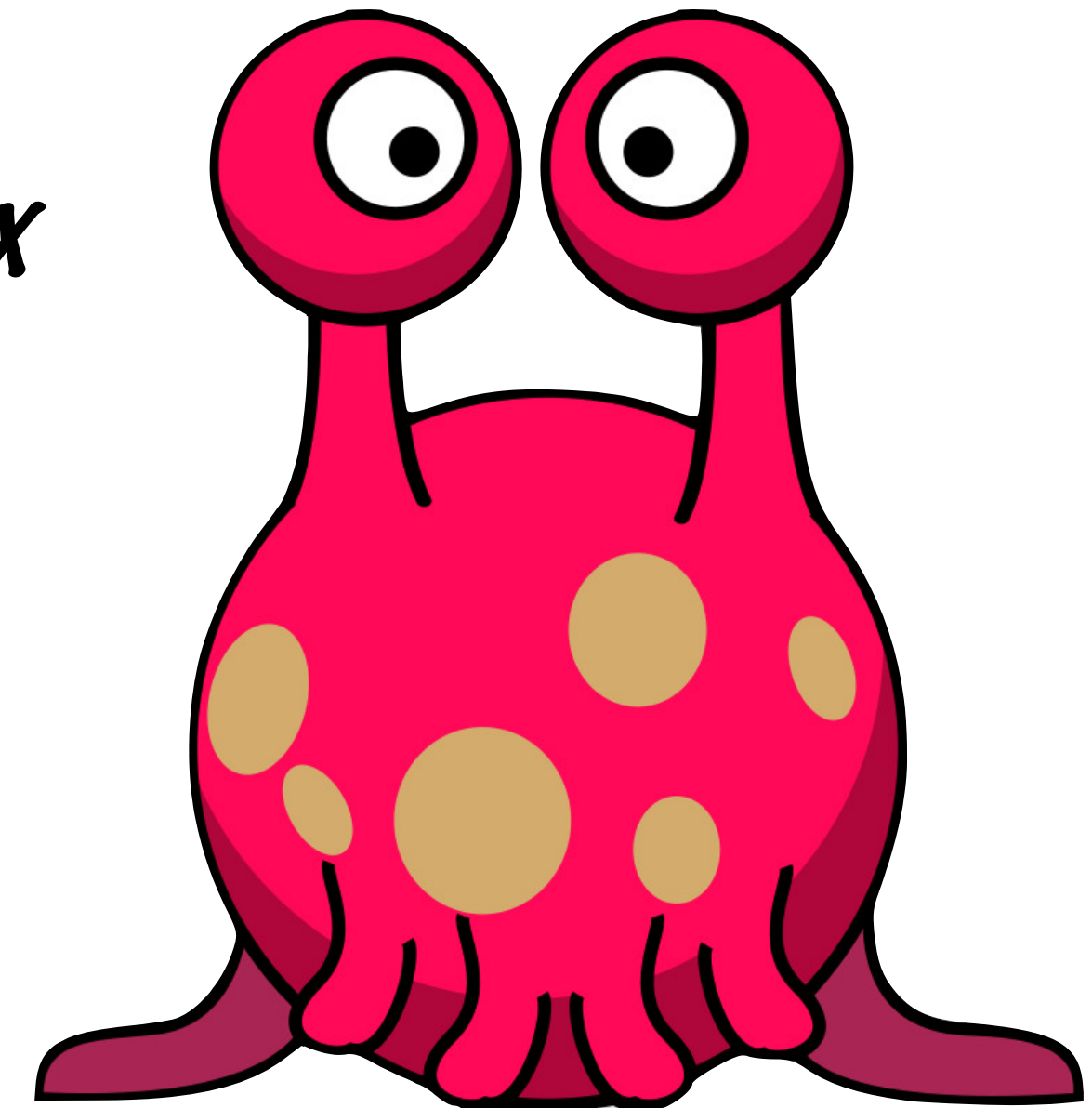
● Verify the identity $\tan(x + \pi) = \tan x$

$$\tan(x + \pi) = \frac{\tan x + \tan \pi}{1 - \tan x \tan \pi} = \frac{\tan x + 0}{1 - \tan x \cdot 0} = \frac{\tan x}{1} = \tan x$$

● Find the exact value of $\cos 100^\circ \cos 55^\circ + \sin 100^\circ \sin 55^\circ$

$$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$$

$$\begin{aligned} \cos 100^\circ \cos 55^\circ + \sin 100^\circ \sin 55^\circ &= \cos(100 - 55)^\circ \\ &= \cos 45^\circ = \frac{\sqrt{2}}{2} \end{aligned}$$



Sum and Difference Formulae

Objective: Use Sum and Difference Formulae to evaluate trig expressions and verify trig identities.

● Verify the identity $\cos(\pi - \theta) = -\cos \theta$

$$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$$

$$\cos(\pi - \theta) = \cos \pi \cos \theta + \sin \pi \sin \theta$$

$$= -1 \cos \theta + 0 \sin \theta$$

$$= -\cos \theta$$



Sum and Difference Formulae

Objective: Use Sum and Difference Formulae to evaluate trig expressions and verify trig identities.

- Find the exact value of $\sin 75^\circ$

$$\sin 75^\circ = \sin (45 + 30)^\circ$$

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$



$$= \sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2} = \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4} = \frac{\sqrt{6} + \sqrt{2}}{4}$$

- Find the exact value of $\sin 15^\circ$

$$\sin 15^\circ = \sin (45 - 30)^\circ$$

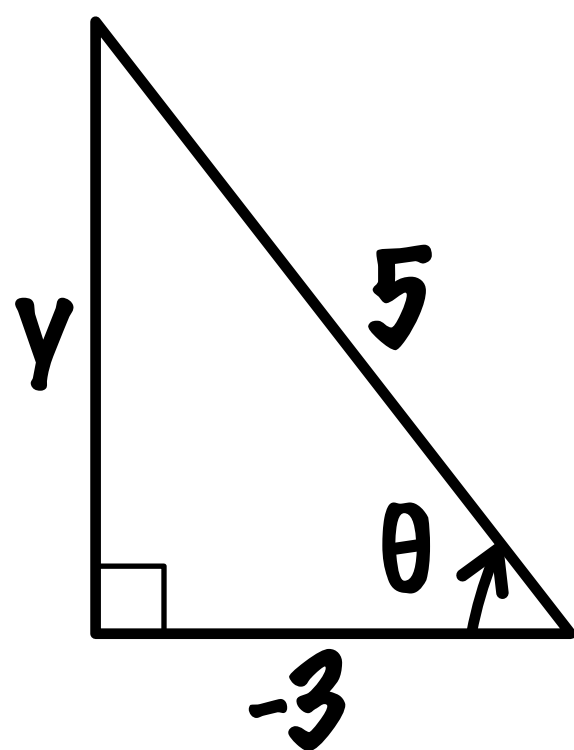
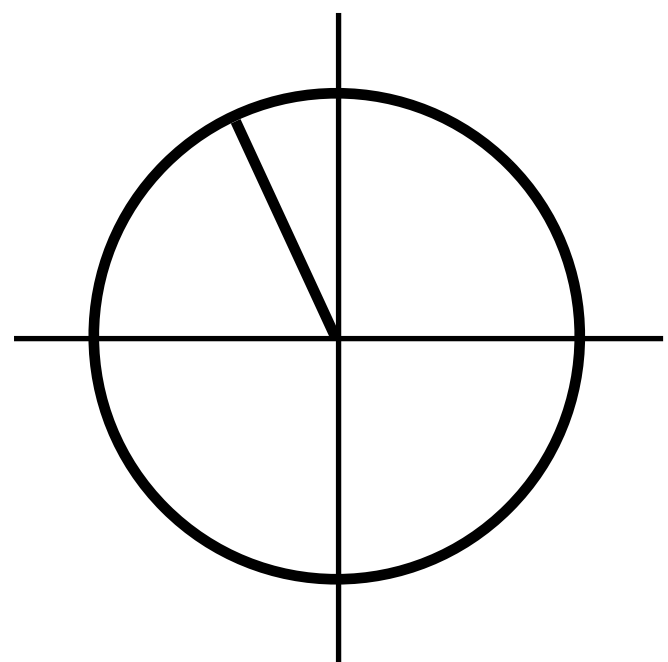
$$\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$$

$$= \sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \cdot \frac{1}{2} = \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} = \frac{\sqrt{6} - \sqrt{2}}{4}$$

Sum and Difference Formulae

Objective: Use Sum and Difference Formulae to evaluate trig expressions and verify trig identities.

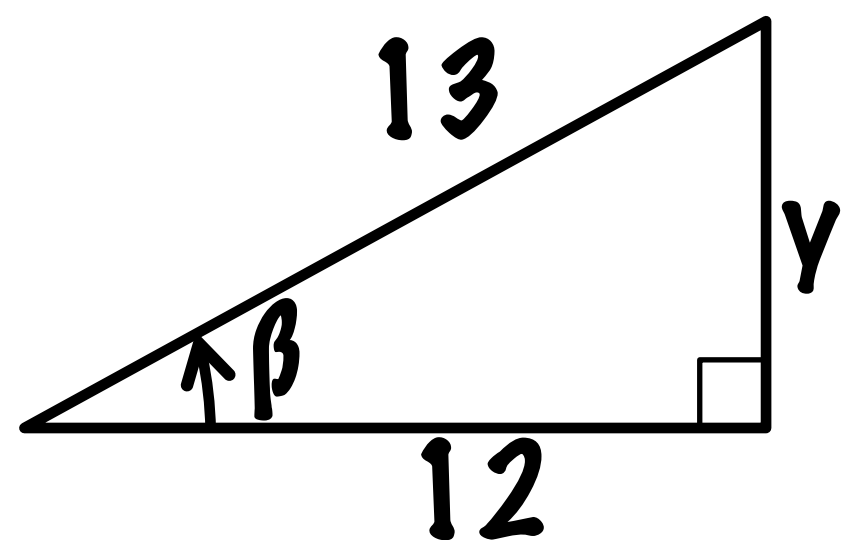
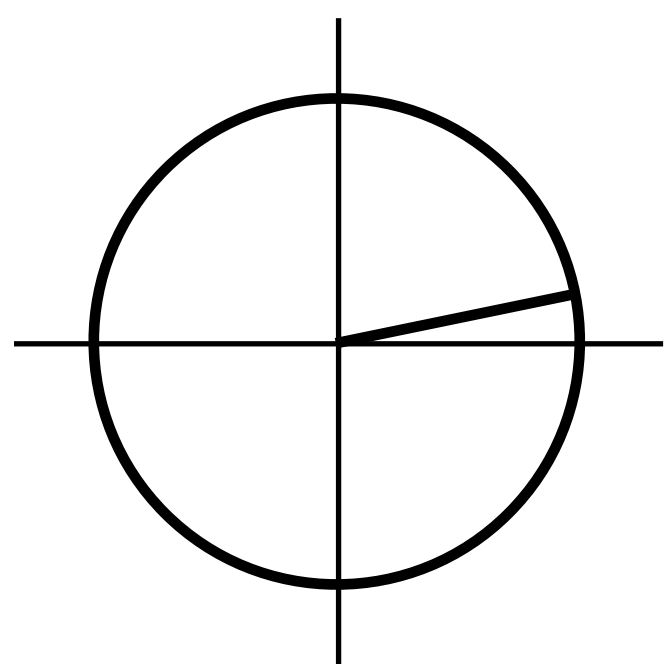
- Suppose that $\cos \theta = -\frac{3}{5}$ for a quadrant II angle θ and $\cos \beta = \frac{12}{13}$ for a quadrant I angle β . Find the exact value of $\sin \theta$, $\sin \beta$, $\sin(\theta + \beta)$, $\cos(\theta + \beta)$



$$y^2 + (-3)^2 = 5^2$$

$$y = 4$$

$$\sin \theta = \frac{4}{5}$$



$$y^2 + 12^2 = 13^2$$

$$y = 5$$

$$\sin \beta = \frac{5}{13}$$



Sum and Difference Formulae

Objective: Use Sum and Difference Formulae to evaluate trig expressions and verify trig identities.

- Suppose that $\cos \theta = -\frac{3}{5}$ for a quadrant II angle θ and $\cos \beta = \frac{12}{13}$ for a quadrant I angle β . Find the exact value of $\sin \theta$, $\sin \beta$, $\sin(\theta + \beta)$, $\cos(\theta + \beta)$

$$\sin \theta = \frac{4}{5} \quad \sin \beta = \frac{5}{13}$$

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

$$= \frac{4}{5} \cdot \frac{12}{13} + \frac{-3}{5} \cdot \frac{5}{13}$$

$$= \frac{48}{65} + \frac{-15}{65} = \frac{33}{65}$$

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

$$= \frac{-3}{5} \cdot \frac{12}{13} - \frac{4}{5} \cdot \frac{5}{13}$$

$$= \frac{-36}{65} - \frac{20}{65} = -\frac{56}{65}$$

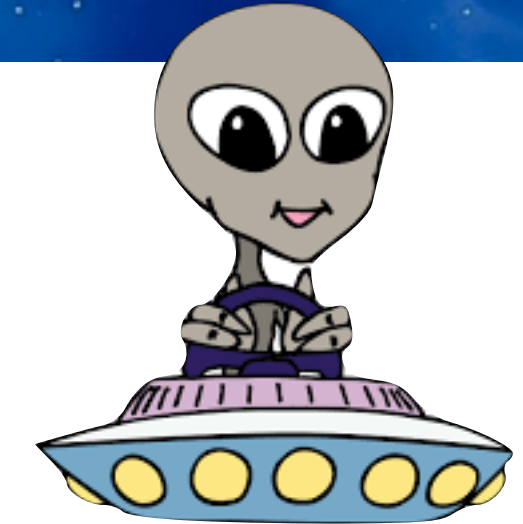


Sum and Difference Formulae

Objective: Use Sum and Difference Formulae to evaluate trig expressions and verify trig identities.

● Find the exact value of $\sin\left(\frac{2\pi}{3} - \frac{\pi}{4}\right)$

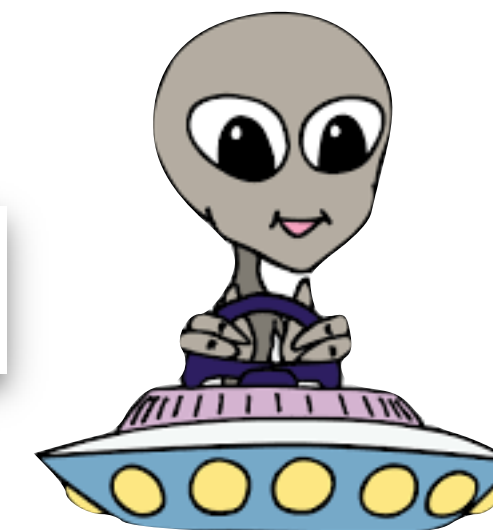
$$\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$$



$$\begin{aligned}\sin\left(\frac{2\pi}{3} - \frac{\pi}{4}\right) &= \sin\frac{2\pi}{3} \cos\frac{\pi}{4} - \cos\frac{2\pi}{3} \sin\frac{\pi}{4} = \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} - \frac{-1}{2} \cdot \frac{\sqrt{2}}{2} \\ &= \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4} = \frac{\sqrt{6} + \sqrt{2}}{4}\end{aligned}$$

● Simplify $\sin(x + 3\pi)$

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$



$$\sin(x + 3\pi) = \sin x \cos 3\pi + \cos x \sin 3\pi = \sin x(-1) + \cos x(0) = -\sin x$$

Sum and Difference Formulae

Objective: Use Sum and Difference Formulae to evaluate trig expressions and verify trig identities.

● Simplify $\tan\left(\theta + \frac{5\pi}{4}\right)$

$$\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}$$



$$\tan\left(\theta + \frac{5\pi}{4}\right) = \frac{\tan \theta + \tan \frac{5\pi}{4}}{1 - \tan \theta \tan \frac{5\pi}{4}} = \frac{\tan \theta + 1}{1 - \tan \theta (1)} = \frac{\tan \theta + 1}{1 - \tan \theta}$$

● Verify the identity $\sin\left(\frac{\pi}{2} - \theta\right) = \cos \theta$

$$\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$$



$$\sin\left(\frac{\pi}{2} - \theta\right) = \sin \frac{\pi}{2} \cos \theta - \cos \frac{\pi}{2} \sin \theta = 1 \cos \theta - 0 \sin \theta = \cos \theta$$

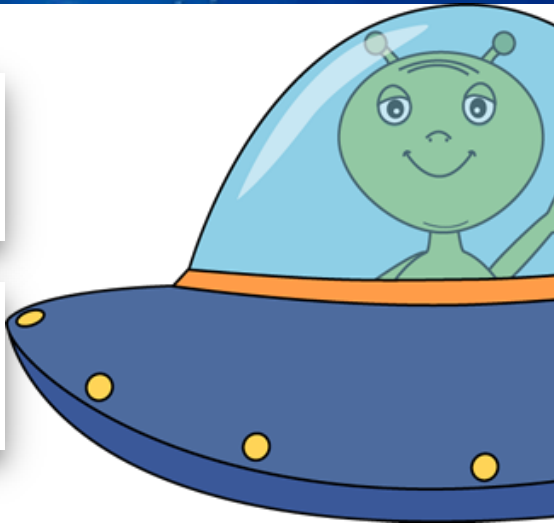
Sum and Difference Formulae

Objective: Use Sum and Difference Formulae to evaluate trig expressions and verify trig identities.

● Verify $\sin(x + y)\sin(x - y) = \cos^2 y - \cos^2 x$

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

$$\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$$



$$\sin(x + y)\sin(x - y) = (\sin x \cos y + \cos x \sin y)(\sin x \cos y - \cos x \sin y)$$

$$= (\sin x \cos y)^2 - (\cos x \sin y)^2$$

$$(a - b)(a + b) = a^2 - b^2$$

$$= \sin^2 x \cos^2 y - \cos^2 x \sin^2 y$$

$$= (1 - \cos^2 x) \cos^2 y - \cos^2 x (1 - \cos^2 y)$$

$$= (\cos^2 y - \cancel{\cos^2 y \cos^2 x}) - (\cos^2 x - \cancel{\cos^2 x \cos^2 y})$$

$$= \cos^2 y - \cos^2 x$$

