Chapter 5

Analytic Trigonometry

5.3 Double-Angle, Power-Reducing, and Half-Angle Formulas

Chapter 5

- Read Sec 5-3
- Do p614 1-77 odd

Chapter 5

Objectives

- Use the double-angle formulas.
- Use the power-reducing formulas.
- Use the half-angle formulas.

Double-Angle Formulas

$$\sin 2\alpha = \sin(\alpha + \alpha)$$

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

$$= \sin\alpha\cos\alpha + \cos\alpha\sin\alpha$$

$$= 2\sin\alpha\cos\alpha$$

$$\sin 2\alpha = 2\sin\alpha\cos\alpha$$

Double-Angle Formulas

$\mathbf{\cos 2\alpha} = \cos(\alpha + \alpha)$

$\cos(\alpha + \beta) = \cos\alpha \cos\beta + \sin\alpha \sin\beta$

$= \cos\alpha \cos\alpha - \sin\alpha \sin\alpha$

$= \cos^2\alpha - \sin^2\alpha$

$$\mathbf{\cos 2\alpha = \cos^2\alpha - \sin^2\alpha}$$

Double-Angle Formulas

$$\tan 2\alpha = \frac{\tan\alpha + \tan\alpha}{1 - \tan\alpha \tan\alpha}$$

$$= \frac{2\tan\alpha}{1-\tan^2\alpha}$$

$$\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha}$$

Double-Angle Formulas

Be Careful

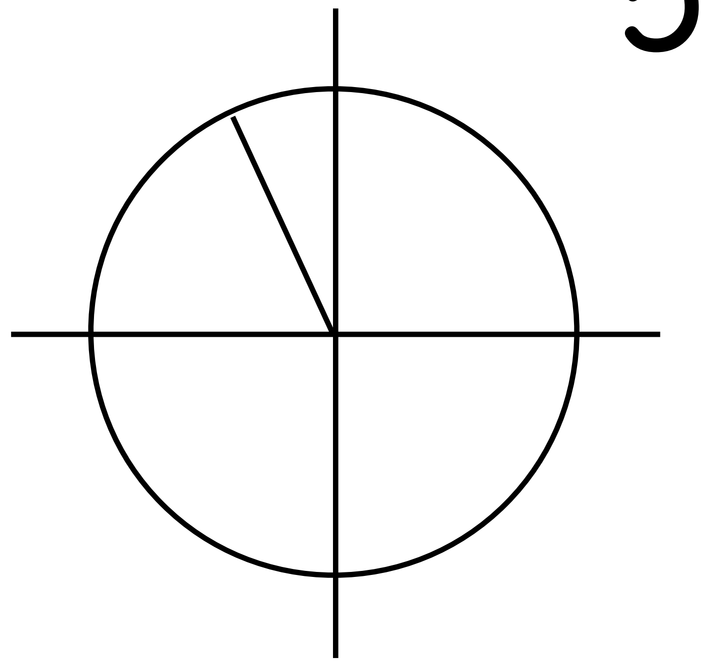
$\cos 2\theta \neq 2\cos\theta$
$\sin 2\theta \neq 2\sin\theta$
$\tan 2\theta \neq 2\tan\theta$

$\cos 2\theta \neq \cos 2\cos\theta$
$\sin 2\theta \neq \sin 2\sin\theta$
$\tan 2\theta \neq \tan 2\tan\theta$

Double-Angle Formulas

Using Double-Angle Formulas to Find Exact Values

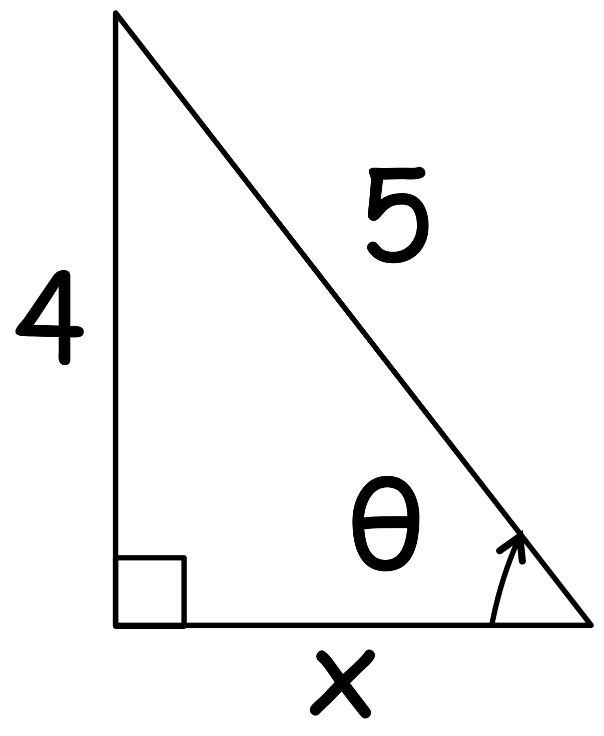
If $\sin\theta = \frac{4}{5}$ and θ lies in QII, find the exact value of $\cos 2\theta$



$$4^2 + x^2 = 5^2$$

$$x = -3$$

$$\cos\theta = -\frac{3}{5}$$



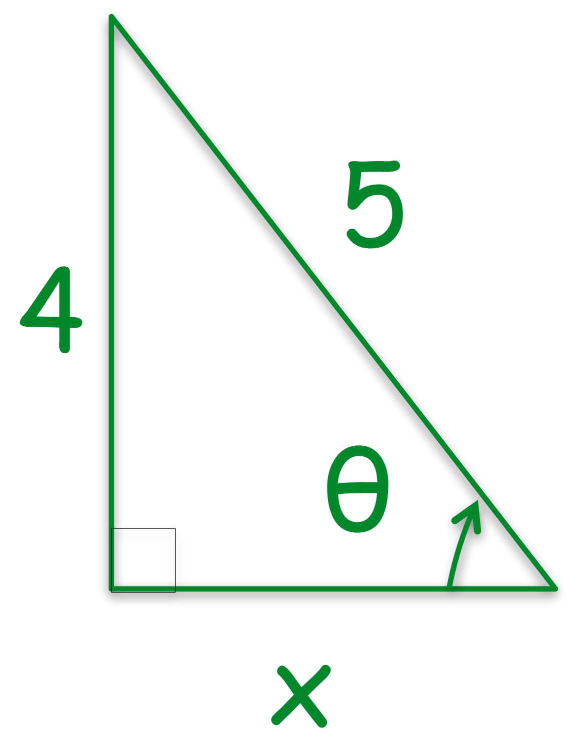
cos2α = cos² α – sin² α

$$= \left(-\frac{3}{5}\right)^2 - \left(\frac{4}{5}\right)^2 = \frac{9}{25} - \frac{16}{25} = -\frac{7}{25}$$

Double-Angle Formulas

Using Double-Angle Formulas to Find Exact Values

If $\sin\theta = \dfrac{4}{5}$ and θ lies in QII, find the exact value of $\sin 2\theta$.

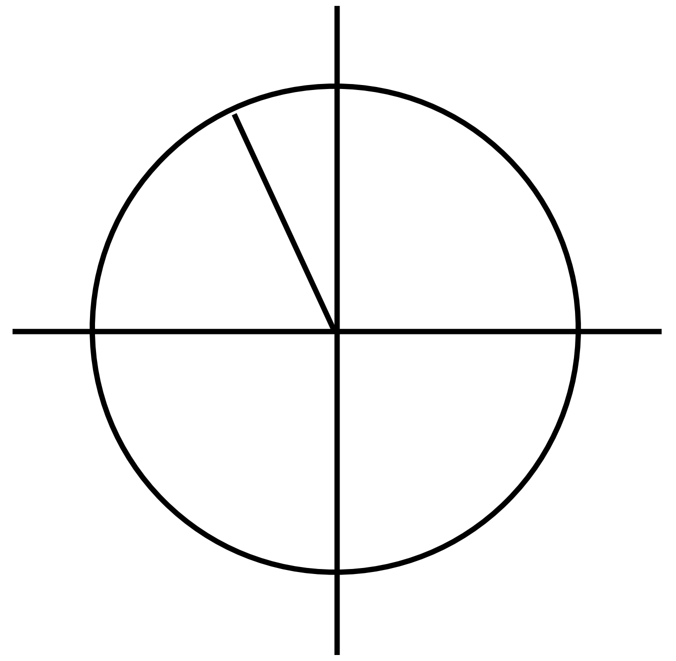


$$\cos\theta = -\frac{3}{5}$$

sin2α = 2sin α cos α

$$\sin 2\theta = 2\sin\theta\cos\theta$$

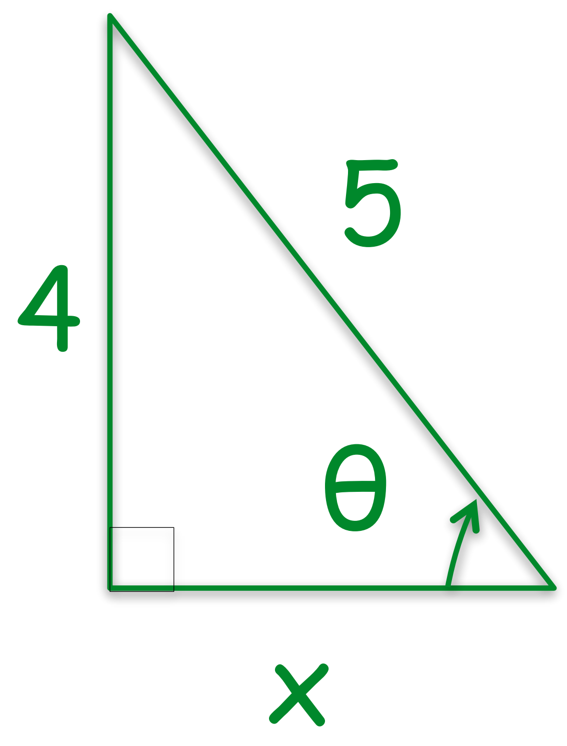
$$= 2\left(\frac{4}{5}\right)\left(-\frac{3}{5}\right) = -\frac{24}{25}$$

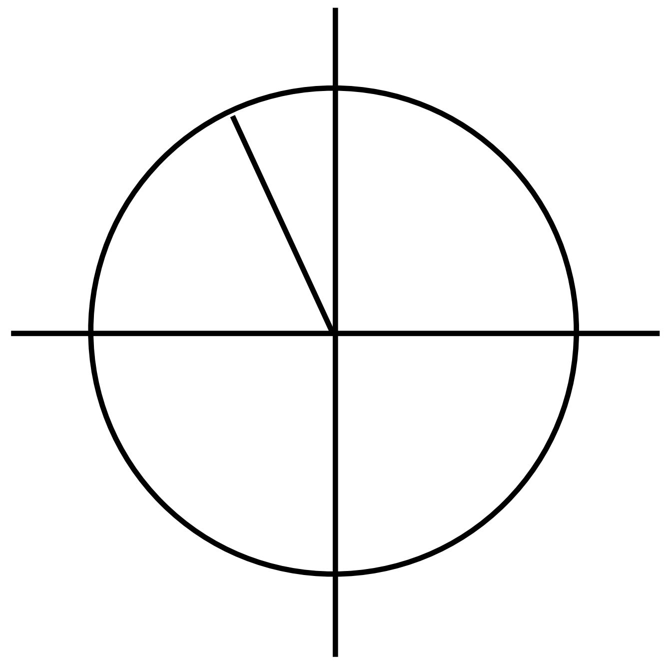


Double-Angle Formulas

Using Double-Angle Formulas to Find Exact Values

If $\sin\theta = \dfrac{4}{5}$ and θ lies in QII, find the exact value of $\tan 2\theta$





$$\cos\theta = -\frac{3}{5} \qquad \tan\theta = -\frac{4}{3}$$

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha}$$

$$\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha}$$

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$$

$$= \frac{2\left(-\frac{4}{3}\right)}{1-\left(-\frac{4}{3}\right)^2} = \frac{-\frac{8}{3}}{-\frac{7}{9}} = \frac{24}{7}$$

Of course

$$\tan 2\alpha = \frac{\sin 2\alpha}{\cos 2\alpha}$$

$$\tan 2\theta = \frac{\sin 2\theta}{\cos 2\theta}$$

$$= \frac{-\frac{24}{25}}{-\frac{7}{25}} = \frac{24}{7}$$

Double-Angle Formulas

Three Forms of the Double-Angle Formula for cos2θ

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$$

$$= \cos^2 \alpha - (1 - \cos^2 \alpha)$$

$$= \cos^2 \alpha - 1 + \cos^2 \alpha)$$

$$= 2\cos^2 \alpha - 1$$

$$\cos 2\alpha = 2\cos^2\alpha - 1$$

Double-Angle Formulas

Three Forms of the Double-Angle Formula for cos2θ

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$$

$$= 1 - \sin^2\alpha - \sin^2\alpha$$

$$= 1 - 2\sin^2\alpha$$

$$\cos 2\alpha = 1 - 2\sin^2\alpha$$

Double-Angle Formulas

Double Angle Formulae

$$\sin 2\alpha = 2 \sin\alpha \cos\alpha$$

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$$

$$\cos 2\alpha = 2\cos^2\alpha - 1$$

$$\cos 2\alpha = 1 - 2\sin^2\alpha$$

$$\tan 2\alpha = \frac{2\tan\alpha}{1 - \tan^2\alpha}$$

Verifying an Identity

Verify the identity $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

$$\sin 3\theta = \sin(2\theta + \theta)$$

$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$$= \sin 2\theta\cos\theta + \cos 2\theta\sin\theta$$

$\sin 2\alpha = 2\sin\alpha\cos\alpha$ **$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$**

$$= (2\sin\theta\cos\theta)\cos\theta + (\cos^2\theta - \sin^2\theta)\sin\theta$$

$$= 2\sin\theta\cos^2\theta + \sin\theta\cos^2\theta - \sin^3\theta$$

$$= 3\sin\theta\cos^2\theta - \sin^3\theta$$

$$= 3\sin\theta(1 - \sin^2\theta) - \sin^3\theta$$

$$= 3\sin\theta - 3\sin^3\theta - \sin^3\theta = 3\sin\theta - 4\sin^3\theta$$

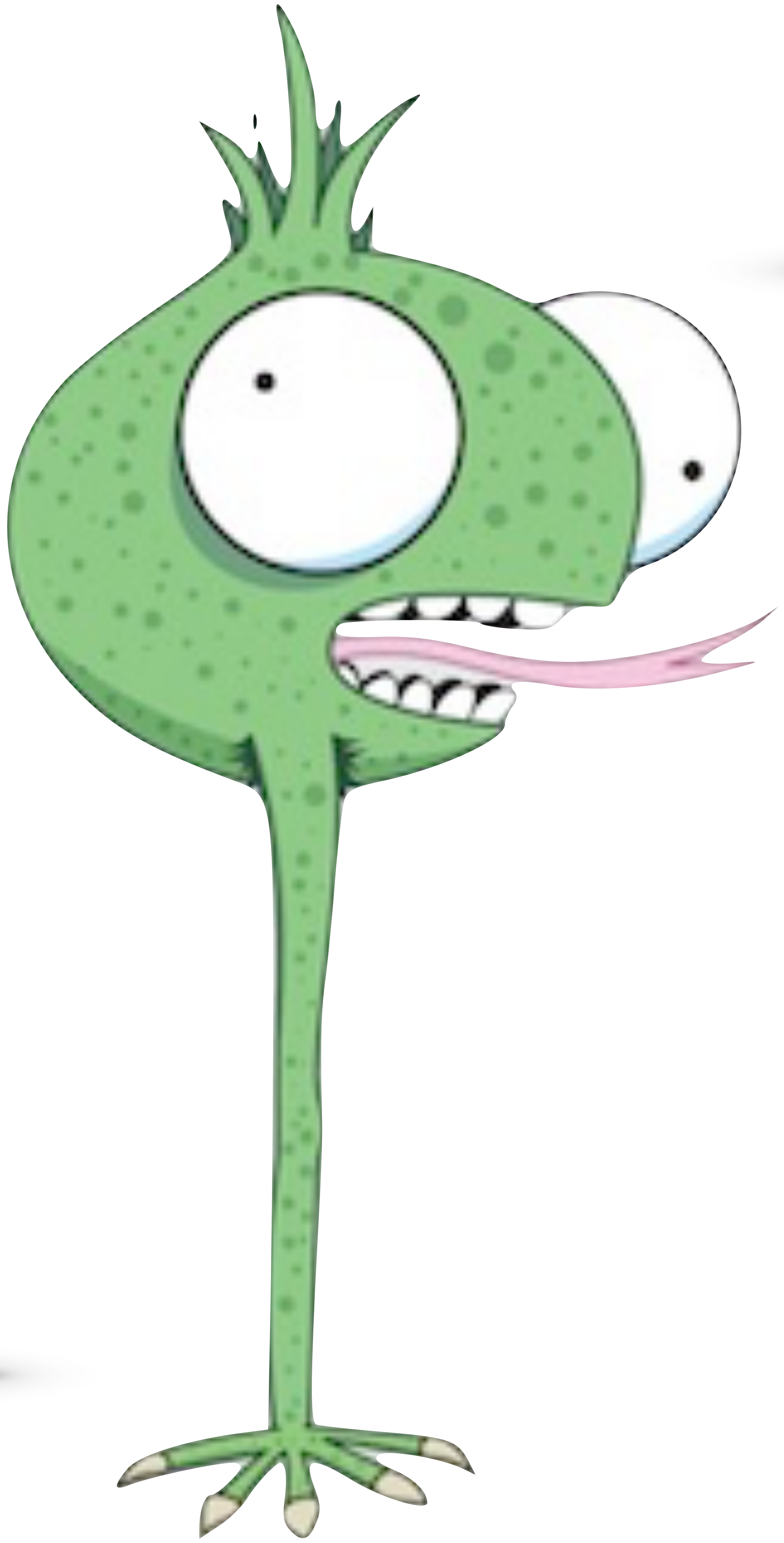
Power-Reducing Formulas

$$\cos 2\alpha = 2\cos^2\alpha - 1$$

$$2\cos^2\alpha = \cos 2\alpha + 1$$

$$\cos^2\alpha = \frac{\cos 2\alpha + 1}{2}$$

$$\cos^2\alpha = \frac{1 + \cos 2\alpha}{2}$$



$$\cos 2\alpha = 1 - 2\sin^2\alpha$$

$$2\sin^2\alpha = 1 - \cos 2\alpha$$

$$\sin^2\alpha = \frac{1 - \cos 2\alpha}{2}$$

$$\sin^2\alpha = \frac{1 - \cos 2\alpha}{2}$$

Power-Reducing Formulas

$\tan^2 \alpha$

$$\tan^2 \alpha = \frac{\sin^2 \alpha}{\cos^2 \alpha}$$

$$= \frac{\frac{1-\cos 2\alpha}{2}}{\frac{1+\cos 2\alpha}{2}} = \frac{1-\cos 2\alpha}{1+\cos 2\alpha}$$

$$\tan^2 \alpha = \frac{1-\cos 2\alpha}{1+\cos 2\alpha}$$

All Together

$$\sin 2\alpha = 2\sin\alpha\cos\alpha$$

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$$

$$\cos 2\alpha = 2\cos^2\alpha - 1$$

$$\cos 2\alpha = 1 - 2\sin^2\alpha$$

$$\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha}$$

$$\cos^2\alpha = \frac{1+\cos 2\alpha}{2}$$

$$\sin^2\alpha = \frac{1-\cos 2\alpha}{2}$$

$$\tan^2\alpha = \frac{1-\cos 2\alpha}{1+\cos 2\alpha}$$

Reducing the Power of a Trigonometric Function

Write an equivalent expression for **sin⁴ x** that does not contain powers of trigonometric functions greater than 1.

$$\sin^4 x = \left(\sin^2 x\right)^2 = \left(\frac{1-\cos 2x}{2}\right)^2 = \frac{1-2\cos 2x+\cos^2 2x}{4}$$

$$\sin^2\alpha = \frac{1-\cos 2\alpha}{2}$$

$$= \frac{1-2\cos 2x+\left(\frac{1+\cos 4x}{2}\right)}{4} = \frac{2-4\cos 2x+1+\cos 4x}{8}$$

$$\cos^2\alpha = \frac{1+\cos 2\alpha}{2}$$

$$= \frac{3-4\cos 2x+\cos 4x}{8}$$

Half-Angle Formulas

Deriving a Half-Angle Formula

cos2α = 2cos²α – 1

$$\cos\left(\frac{2\alpha}{2}\right) = 2\cos^2\left(\frac{\alpha}{2}\right) - 1$$

$$\cos\alpha = 2\cos^2\left(\frac{\alpha}{2}\right) - 1$$

$$\cos\alpha + 1 = 2\cos^2\left(\frac{\alpha}{2}\right)$$

$$\frac{\cos\alpha + 1}{2} = \cos^2\left(\frac{\alpha}{2}\right)$$

$$\cos\left(\frac{\alpha}{2}\right) = \sqrt{\frac{\cos\alpha + 1}{2}} \text{ or } -\sqrt{\frac{\cos\alpha + 1}{2}}$$

$$\cos\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1 + \cos\alpha}{2}}$$

Know the Quadrant!

Half-Angle Formulas

Deriving a Half-Angle Formula

cos2α = 1 – 2sin²α

$$\cos\left(\frac{2\alpha}{2}\right) = 1 - 2\sin^2\left(\frac{\alpha}{2}\right)$$

$$\cos\alpha = 1 - 2\sin^2\left(\frac{\alpha}{2}\right)$$

$$2\sin^2\left(\frac{\alpha}{2}\right) = 1 - \cos\alpha$$

$$\sin^2\left(\frac{\alpha}{2}\right) = \frac{1-\cos\alpha}{2}$$

$$\sin\left(\frac{\alpha}{2}\right) = \sqrt{\frac{1-\cos\alpha}{2}} \text{ or } -\sqrt{\frac{1-\cos\alpha}{2}}$$

$$\sin\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1-\cos\alpha}{2}}$$

Know the Quadrant!

Half-Angle Formulas

Deriving a Half-Angle Formula

$$\tan\left(\frac{\alpha}{2}\right)$$

$$\tan\left(\frac{\alpha}{2}\right) = \frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}}$$

$$\tan\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}$$

$$= \frac{\sqrt{\frac{1-\cos\alpha}{2}}}{\sqrt{\frac{1+\cos\alpha}{2}}} = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}$$

Know the Quadrant!

All Together Once Again

Double-Angle Formulas

$\sin 2\alpha = 2\sin\alpha\cos\alpha$

$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$

$\cos 2\alpha = 2\cos^2\alpha - 1$

$\cos 2\alpha = 1 - 2\sin^2\alpha$

$$\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha}$$

Formula Reduction

$$\cos^2\alpha = \frac{1+\cos 2\alpha}{2}$$

$$\sin^2\alpha = \frac{1-\cos 2\alpha}{2}$$

$$\tan^2\alpha = \frac{1-\cos 2\alpha}{1+\cos 2\alpha}$$

Half-Angle Formulas

$$\cos\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1+\cos\alpha}{2}}$$

$$\sin\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1-\cos\alpha}{2}}$$

$$\tan\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}$$

Caution

Keep in mind that with half angle formulae, the sign outside the radical (the sign of the half angle function) is determined by the quadrant in which $\frac{\alpha}{2}$ is found. The sign of the cos function (inside the radical) is determined by the quadrant in which α is found.

$$\cos\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1+\cos\alpha}{2}}$$

Example

Verify: $\tan\theta = \dfrac{\sin 2\theta}{1+\cos 2\theta}$

$$\frac{\sin 2\theta}{1+\cos 2\theta} = \frac{2\sin\theta\cos\theta}{1+(1-2\sin^2\theta)}$$

$$= \frac{2\sin\theta\cos\theta}{2-2\sin^2\theta} = \frac{2\sin\theta\cos\theta}{2(1-\sin^2\theta)}$$

$$= \frac{2\sin\theta\cos\theta}{2(\cos^2\theta)} = \frac{\sin\theta}{\cos\theta} = \tan\theta$$

Example

Verify: $\tan\theta = \dfrac{1-\cos 2\theta}{\sin 2\theta}$

$$\frac{1-\cos 2\theta}{\sin 2\theta} = \frac{1-\left(2\cos^2\theta - 1\right)}{2\sin\theta\cos\theta}$$

$$= \frac{2-2\cos^2\theta}{2\sin\theta\cos\theta} = \frac{2(1-\cos^2\theta)}{2\sin\theta\cos\theta}$$

$$= \frac{2\sin^2\theta}{2\sin\theta\cos\theta} = \frac{\sin\theta}{\cos\theta} = \tan\theta$$

Half-Angle Formulas

$$\tan\theta = \frac{\sin 2\theta}{1+\cos 2\theta}$$

$$\tan\theta = \frac{1-\cos 2\theta}{\sin 2\theta}$$

$$\tan\frac{\theta}{2} = \frac{\sin\frac{2\theta}{2}}{1+\cos\frac{2\theta}{2}} = \frac{\sin\theta}{1+\cos\theta}$$

$$\tan\frac{\theta}{2} = \frac{1-\cos\frac{2\theta}{2}}{\sin\frac{2\theta}{2}} = \frac{1-\cos\theta}{\sin\theta}$$

$$\tan\frac{\theta}{2} = \frac{\sin\theta}{1+\cos\theta}$$

$$\tan\frac{\theta}{2} = \frac{1-\cos\theta}{\sin\theta}$$

Example

Find the exact value of $\cos 105°$.

$$\cos 105^\circ = \cos\left(\frac{210^\circ}{2}\right)$$

105° is in QII, 210° is in QIII

$$= -\sqrt{\frac{1+\cos 210^\circ}{2}}$$

$$\cos 210^\circ = -\frac{\sqrt{3}}{2}$$

$$= -\sqrt{\frac{1+\left(-\frac{\sqrt{3}}{2}\right)}{2}} = -\sqrt{\frac{1}{2}-\frac{\sqrt{3}}{4}} = -\sqrt{\frac{2-\sqrt{3}}{4}}$$

Example

Find the exact value of $\cos^2 15° - \sin^2 15°$

$$\cos^2 15^\circ - \sin^2 15^\circ = \cos(2 \cdot 15^\circ)$$

$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$

$$= \cos 30^\circ$$

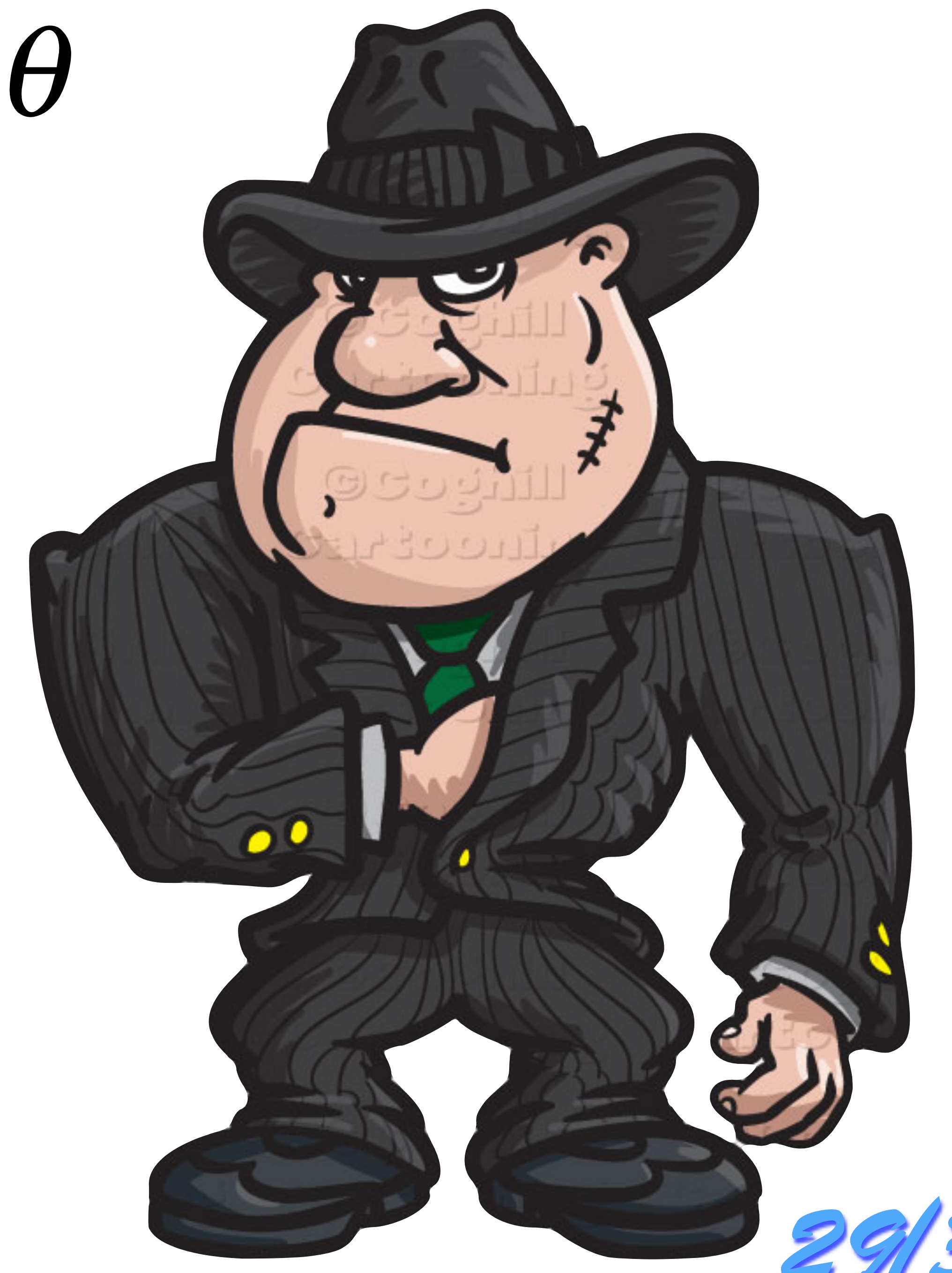
30° is in QI

$$= \frac{\sqrt{3}}{2}$$

Verify: $\tan\frac{\theta}{2} = \frac{\sec\theta}{\sec\theta\csc\theta + \csc\theta}$

$$\frac{\sec\theta}{\sec\theta\csc\theta + \csc\theta} = \frac{\frac{1}{\cos\theta}}{\frac{1}{\cos\theta\sin\theta} + \frac{1}{\sin\theta}} = \frac{\frac{1}{\cos\theta}}{\frac{1}{\cos\theta\sin\theta} + \frac{1}{\sin\theta}} \bullet \frac{\cos\theta\sin\theta}{\cos\theta\sin\theta}$$

$$= \frac{\sin\theta}{1+\cos\theta} = \tan\frac{\theta}{2}$$



Example

Verify: $\sin 4x = 4\sin x\cos^3 x - 4\sin^3 x\cos x$

$$\sin 4x = \sin 2(2x) = 2\sin 2x\cos 2x$$

$$= 2(2\sin x\cos x)(\cos^2 x - \sin^2 x)$$

$$= (4\sin x\cos x)(\cos^2 x - \sin^2 x)$$

$$= (4\sin x\cos x)(\cos^2 x) - (4\sin x\cos x)(\sin^2 x)$$

$$= (4\sin x\cos^3 x) - (4\sin^3 x\cos x)$$

Example

Verify: $\cos 3x = 4\cos^3 x - 3\cos x$

$$
\begin{aligned}
\cos 3x = \cos(2x + x) &= \cos 2x \cos x - \sin 2x \sin x \\
&= (\cos^2 x - \sin^2 x)\cos x - (2\sin x \cos x)\sin x \\
&= \cos^3 x - \sin^2 x \cos x - 2\sin^2 x \cos x \\
&= \cos^3 x - 3\sin^2 x \cos x \\
&= \cos^3 x - 3(1 - \cos^2 x)\cos x \\
&= \cos^3 x - 3(\cos x - \cos^3 x) \\
&= \cos^3 x - 3\cos x + 3\cos^3 x \\
&= 4\cos^3 x - 3\cos x
\end{aligned}
$$

Example

Find the exact value of $\dfrac{2\tan\frac{\pi}{8}}{1-\tan^2\frac{\pi}{8}}$

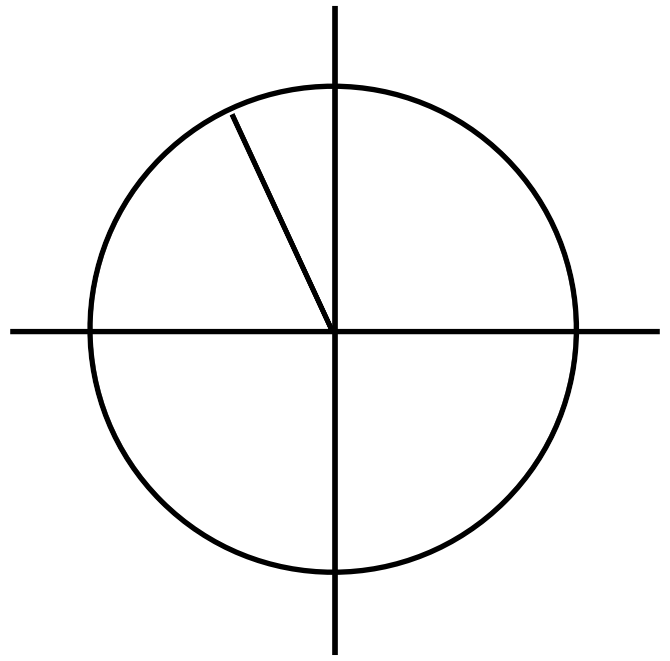
$$\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha}$$

$$\frac{2\tan\frac{\pi}{8}}{1-\tan^2\frac{\pi}{8}} = \tan 2\left(\frac{\pi}{8}\right) = \tan\left(\frac{\pi}{4}\right) = 1$$

Example

Find the exact value of $\cos\dfrac{5\pi}{8}$

$\dfrac{5\pi}{8}$ is in QII



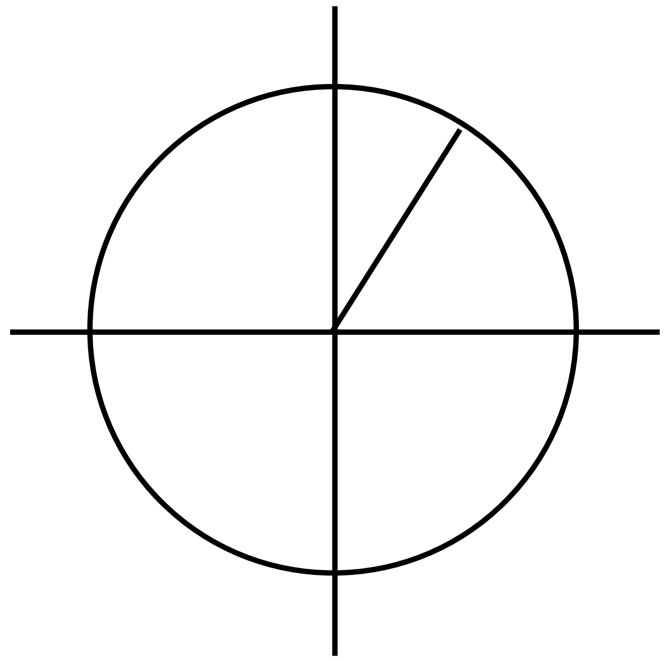
$$\cos\frac{5\pi}{8}=\cos\left(\frac{\frac{5\pi}{4}}{2}\right)=-\sqrt{\left(\frac{1+\cos\frac{5\pi}{4}}{2}\right)}=-\sqrt{\left(\frac{1+-\frac{\sqrt{2}}{2}}{2}\right)}$$

$$=-\sqrt{\frac{1}{2}-\frac{\sqrt{2}}{4}}$$

Example

Find the exact value of sin75°

75° is in QI



$$\sin 75^\circ = \sin\frac{150^\circ}{2} = \sqrt{\frac{1-\cos 150^\circ}{2}}$$

$$= \sqrt{\frac{1 - -\frac{\sqrt{3}}{2}}{2}} \qquad = -\sqrt{\frac{1}{2}+\frac{\sqrt{3}}{4}}$$