Chapter 5

Analytic Trigonometry

5.4 Product-to-Sum and Sum-to-Product Formulas

Chapter 5.4

Objectives

- Use the product-to-sum formulas.
- Use the sum-to-product formulas.

Chapter 5.4

Homework

- Read Sec 5-4
- Do p623 2 - 46 even

Product to Sum Formulae

Verify $\sin\alpha\sin\beta = \frac{1}{2}\left[\cos(\alpha-\beta)-\cos(\alpha+\beta)\right]$

$$= \frac{1}{2}\left[(\cos\alpha\cos\beta+\sin\alpha\sin\beta)-(\cos\alpha\cos\beta-\sin\alpha\sin\beta)\right]$$

$$= \frac{1}{2}\left[\cancel{\cos\alpha\cos\beta}+\sin\alpha\sin\beta-\cancel{\cos\alpha\cos\beta}+\sin\alpha\sin\beta\right]$$

$$= \frac{1}{2}\left[2\sin\alpha\sin\beta\right]$$

$$= \sin\alpha\sin\beta$$

Product to Sum Formulae

Verify $\cos\alpha\cos\beta = \frac{1}{2}\left[\cos(\alpha-\beta)+\cos(\alpha+\beta)\right]$

$$= \frac{1}{2}\left[(\cos\alpha\cos\beta+\sin\alpha\sin\beta)+(\cos\alpha\cos\beta-\sin\alpha\sin\beta)\right]$$

$$= \frac{1}{2}\left[\cos\alpha\cos\beta+\sin\alpha\sin\beta+\cos\alpha\cos\beta-\sin\alpha\sin\beta\right]$$

$$= \frac{1}{2}\left[2\cos\alpha\cos\beta\right]$$

$$= \cos\alpha\cos\beta$$

Product to Sum Formulae

Verify $\sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta)+\sin(\alpha-\beta)\right]$

$$= \frac{1}{2}\left[(\sin\alpha\cos\beta + \cancel{\cos\alpha\sin\beta}) + (\sin\alpha\cos\beta - \cancel{\cos\alpha\sin\beta})\right]$$

$$= \frac{1}{2}\left[2\sin\alpha\cos\beta\right]$$

$$= \sin\alpha\cos\beta$$

Product to Sum Formulae

Verify $\cos\alpha\sin\beta = \frac{1}{2}\left[\sin(\alpha+\beta) - \sin(\alpha-\beta)\right]$

$$= \frac{1}{2}\left[(\cancel{\sin\alpha\cos\beta} + \cos\alpha\sin\beta) - (\cancel{\sin\alpha\cos\beta} - \cos\alpha\sin\beta)\right]$$

$$= \frac{1}{2}\left[2\cos\alpha\sin\beta\right]$$

$$= \cos\alpha\sin\beta$$

The Product-to-Sum Formulas

Product to Sum Formulae
Sum to Product Formulae

$$\sin\alpha\sin\beta = \frac{1}{2}\left[\cos(\alpha-\beta)-\cos(\alpha+\beta)\right]$$

$$\cos\alpha\cos\beta = \frac{1}{2}\left[\cos(\alpha-\beta)+\cos(\alpha+\beta)\right]$$

$$\sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta)+\sin(\alpha-\beta)\right]$$

$$\cos\alpha\sin\beta = \frac{1}{2}\left[\sin(\alpha+\beta)-\sin(\alpha-\beta)\right]$$

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Using the Product-to-Sum Formulas

 Express the product as a sum or difference: $\sin 5x \sin 2x$

$$\sin\alpha\sin\beta = \frac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right]$$

$$\sin 5x \sin 2x = \frac{1}{2}\left[\cos(5x-2x) - \cos(5x+2x)\right]$$

$$= \frac{1}{2}\left[\cos 3x - \cos 7x\right]$$

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Using the Product-to-Sum Formulas

Express the product as a sum or difference: $\cos 7x \cos x$

$$\cos\alpha\cos\beta = \frac{1}{2}\left[\cos(\alpha-\beta)+\cos(\alpha+\beta)\right]$$

$$\cos 7x \cos x = \frac{1}{2}\left[\cos(7x - x) + \cos(7x + x)\right]$$

$$= \frac{1}{2}\left[\cos 6x + \cos 8x\right]$$

Sum to Product Formulae

Verify $\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$

$$\sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta)+\sin(\alpha-\beta)\right]$$

$$2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} = 2\bullet\frac{1}{2}\left[\sin\left(\frac{\alpha+\cancel{\beta}}{2}+\frac{\alpha-\cancel{\beta}}{2}\right)+\sin\left(\frac{\cancel{\alpha}+\beta}{2}-\frac{\cancel{\alpha}-\beta}{2}\right)\right]$$

$$=\left[\sin\left(\frac{2\alpha}{2}\right)+\sin\left(\frac{2\beta}{2}\right)\right] \quad = \sin\alpha + \sin\beta$$

Sum to Product Formulae

Verify $\sin\alpha - \sin\beta = 2\sin\frac{\alpha-\beta}{2}\cos\frac{\alpha+\beta}{2}$

$$\sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta) + \sin(\alpha-\beta)\right]$$

$$2\sin\frac{\alpha-\beta}{2}\cos\frac{\alpha+\beta}{2} = 2\bullet\frac{1}{2}\left[\sin\left(\frac{\alpha-\cancel{\beta}}{2}+\frac{\alpha+\cancel{\beta}}{2}\right)+\sin\left(\frac{\cancel{\alpha}-\beta}{2}-\frac{\cancel{\alpha}+\beta}{2}\right)\right]$$

$$=\left[\sin\left(\frac{2\alpha}{2}\right)+\sin\left(-\frac{2\beta}{2}\right)\right]=\left[\sin(\alpha)+\sin(-\beta)\right]$$

$$=\sin\alpha-\sin\beta$$

Sum to Product Formulae

Verify $\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$

$$\cos\alpha\cos\beta = \frac{1}{2}\left[\cos(\alpha-\beta) + \cos(\alpha+\beta)\right]$$

$$2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} = 2\bullet\frac{1}{2}\left[\cos\left(\frac{\alpha+\beta}{2}-\frac{\alpha-\beta}{2}\right)+\cos\left(\frac{\alpha+\beta}{2}+\frac{\alpha-\beta}{2}\right)\right]$$

$$=\left[\cos\left(\frac{2\beta}{2}\right)+\sin\left(\frac{2\alpha}{2}\right)\right]$$

$$=\cos\alpha+\cos\beta$$

Sum to Product Formulae

 Verify $\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$

$$\sin\alpha\sin\beta = \frac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right]$$

$$-2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2} = -2\bullet\frac{1}{2}\left[\cos\left(\frac{\cancel{\alpha}+\beta}{2}-\frac{\cancel{\alpha}-\beta}{2}\right)-\cos\left(\frac{\alpha+\cancel{\beta}}{2}+\frac{\alpha-\cancel{\beta}}{2}\right)\right]$$

$$= -1\left[\cos\left(\frac{2\beta}{2}\right)-\cos\left(\frac{2\alpha}{2}\right)\right] \quad = -1\left[\cos(\beta)-\cos(\alpha)\right]$$

$$= \cos\alpha - \cos\beta$$

The Sum-to-Product Formulas

$$\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

$$\sin\alpha - \sin\beta = 2\sin\frac{\alpha-\beta}{2}\cos\frac{\alpha+\beta}{2}$$

$$\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

$$\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$

Today's Menu

Product to Sum

$$\sin\alpha\sin\beta = \frac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right]$$

$$\cos\alpha\cos\beta = \frac{1}{2}\left[\cos(\alpha-\beta) + \cos(\alpha+\beta)\right]$$

$$\sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta) + \sin(\alpha-\beta)\right]$$

$$\cos\alpha\sin\beta = \frac{1}{2}\left[\sin(\alpha+\beta) - \sin(\alpha-\beta)\right]$$

Sum to Product

$$\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

$$\sin\alpha - \sin\beta = 2\sin\frac{\alpha-\beta}{2}\cos\frac{\alpha+\beta}{2}$$

$$\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

$$\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$

Using the Sum-to-Product Formulas

Express the sum as a product: $\sin 7x + \sin 3x$

$$\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

$$\sin 7x + \sin 3x = 2\sin\frac{7x+3x}{2}\cos\frac{7x-3x}{2}$$

$$= 2\sin 5x\cos 2x$$

Using the Sum-to-Product Formulas

 Express the sum as a product: $\cos 3x + \cos 2x$

$$\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

$$\cos 3x + \cos 2x = 2\cos\frac{3x+2x}{2}\cos\frac{3x-2x}{2}$$

$$= 2\cos\frac{5x}{2}\cos\frac{x}{2}$$

Example

- Express the product as a sum or difference: $\sin 2x \cos 3x$

$$\sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta)+\sin(\alpha-\beta)\right]$$

$$\sin 2x \cos 3x = \frac{1}{2}\left[\sin(2x+3x)+\sin(2x-3x)\right]$$

$$= \frac{1}{2}\left[\sin 5x + \sin(-x)\right]$$

$$= \frac{1}{2}\left[\sin 5x - \sin x\right]$$

Example

Express the product as a sum or difference: $2\sin 74^\circ \cos 114^\circ$

$$\sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta)+\sin(\alpha-\beta)\right]$$

$$2\sin 74^\circ \cos 114^\circ = 2\bullet\frac{1}{2}\left[\sin(74+114)^\circ + \sin(74-114)^\circ\right]$$

$$= 1\left[\sin 188^\circ + \sin(-40^\circ)\right]$$

$$= \sin 188^\circ - \sin 40^\circ$$

Example

 Verify: $\tan x = \dfrac{\sin 3x - \sin x}{\cos 3x + \cos x}$

$$\sin\alpha - \sin\beta = 2\sin\frac{\alpha-\beta}{2}\cos\frac{\alpha+\beta}{2}$$

$$\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

$$\frac{\sin 3x - \sin x}{\cos 3x + \cos x} = \frac{2\sin\frac{3x-x}{2}\cos\frac{3x+x}{2}}{2\cos\frac{3x+x}{2}\cos\frac{3x-x}{2}} = \frac{2\sin\frac{2x}{2}\cos\frac{4x}{2}}{2\cos\frac{4x}{2}\cos\frac{2x}{2}}$$

$$= \frac{\cancel{2}\sin x\,\cancel{\cos 2x}}{\cancel{2}\,\cancel{\cos 2x}\cos x} = \tan x$$

Example

Find the exact value of sin 195° - sin 105°.

$$\sin\alpha - \sin\beta = 2\sin\frac{\alpha-\beta}{2}\cos\frac{\alpha+\beta}{2}$$

$$\sin 195^\circ - \sin 105^\circ = 2\left[\cos\left(\frac{195+105}{2}\right)^\circ \sin\left(\frac{195-105}{2}\right)^\circ\right]$$

$$= 2\left[\cos(150)^\circ \sin(45)^\circ\right]$$

$$= 2\left[-\frac{\sqrt{3}}{2}\bullet\frac{\sqrt{2}}{2}\right] = -\frac{\sqrt{6}}{2}$$

Example

 Verify $\dfrac{\sin 7x - \sin 5x}{\cos 7x - \cos 5x} = -\cot 6x$

$$\sin\alpha - \sin\beta = 2\sin\frac{\alpha-\beta}{2}\cos\frac{\alpha+\beta}{2}$$

$$\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$

$$\frac{\sin 7x - \sin 5x}{\cos 7x - \cos 5x} = \frac{2\cos\left(\frac{7x+5x}{2}\right)\sin\left(\frac{7x-5x}{2}\right)}{-2\sin\left(\frac{7x+5x}{2}\right)\sin\left(\frac{7x-5x}{2}\right)}$$

$$= \frac{\cos(6x)}{-\sin(6x)} = -\cot 6x$$