# Chapter 5

Objective: Solving Trigonometric Equations

## Analytic Trigonometry

5.5 Trigonometric Equations

## 5.5 Solving Trigonometric Equations

# Objectives:

- Find all solutions of a trigonometric equation.
- Solve equations with multiple angles.
- Solve trigonometric equations quadratic in form.
- Use factoring to separate different functions in trigonometric equations.
- Use identities to solve trigonometric equations.
- Use a calculator to solve trigonometric equations.

# Homework

- Read Sec 5.5
- Do p636 1-115 odd

# Trigonometric Equations and Their Solutions

Objective: Solving Trigonometric Equations

A **trigonometric equation** is an equation that contains a trigonometric expression with a variable, such as sin x.

The values that satisfy such an equation are its **solutions**.
(There are trigonometric equations that have no solution.)

When an equation includes multiple angles, the period of the function plays an important role in ensuring that **we do not leave out any solutions.**

It will be the rare occasion when you will find a single solution!

# Finding all Solutions of a Trigonometric Equation

Solve the equation $5\sin x = 3\sin x + \sqrt{3}$

**Step 1** Isolate the **function** on one side of the equation.

$$5\sin x = 3\sin x + \sqrt{3}$$

$$5\sin x - 3\sin x = \sqrt{3}$$

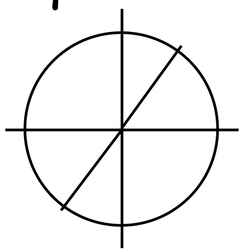
$$2\sin x = \sqrt{3}$$

$$\sin x = \frac{\sqrt{3}}{2}$$

**Step 2** Solve for x

Solutions for this equation in [0, 2π) are:

$\frac{\pi}{3}$ and $\frac{2\pi}{3}$



We are not limited to [0, 2π), and since the period of sine is 2π our solutions are:

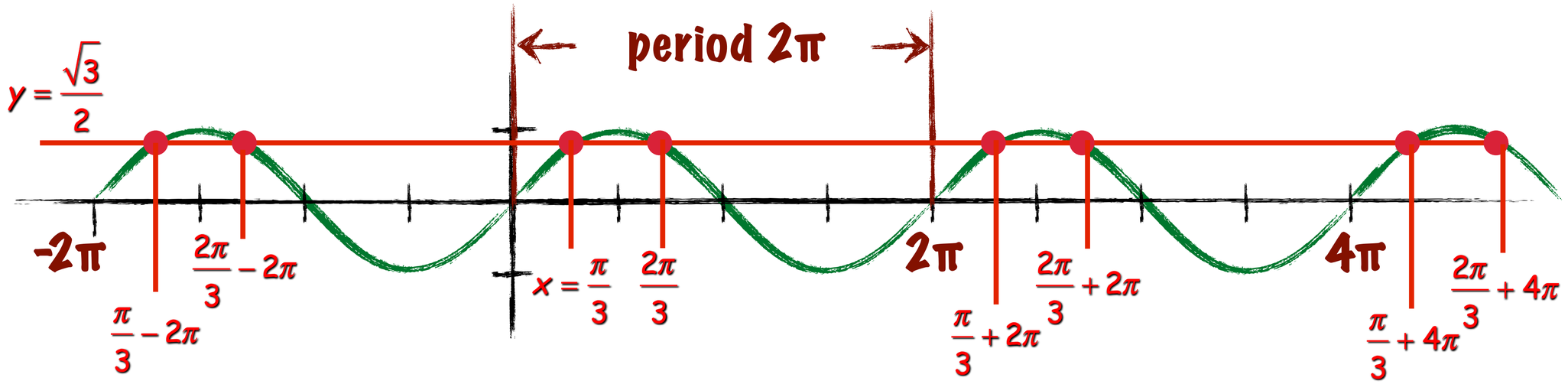
$$x = \left(\frac{\pi}{3}\right) + n2\pi, \left(\frac{2\pi}{3}\right) + n2\pi$$

# Graphical Representation



Solve the equation $5\sin x = 3\sin x + \sqrt{3}$

$$\sin x = \frac{\sqrt{3}}{2} \qquad x = \left(\frac{\pi}{3}\right) + n2\pi, \ \left(\frac{2\pi}{3}\right) + n2\pi$$



# Solving an Equation with a Multiple Angle

Objective: Solving Trigonometric Equations

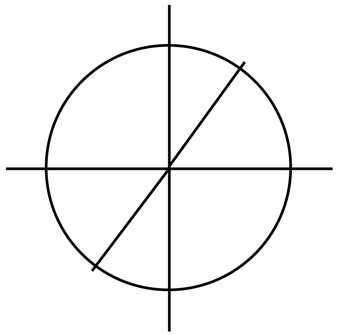
∿Solve the equation $\tan 2x = \sqrt{3},\ 0 \le x < 2\pi$

**Step 1** $\tan 2x = \sqrt{3}$

$\tan^{-1}\sqrt{3} = 2x \qquad 2x = \frac{\pi}{3}$

The period for tanx is **π**, so

$$2x = \frac{\pi}{3} + n\pi$$



But we want x!

$$x = \frac{\pi}{6} + \frac{n\pi}{2}$$

**Step 2** Solve for x

We are looking for solutions from $0 \le x < 2\pi$,

$$x = \frac{\pi}{6} + \frac{0\pi}{2} = \frac{\pi}{6}$$

$$x = \frac{\pi}{6} + \frac{1\pi}{2} = \frac{4\pi}{6} = \frac{2\pi}{3}$$

$$x = \frac{\pi}{6} + \frac{2\pi}{2} = \frac{7\pi}{6}$$

$$x = \frac{\pi}{6} + \frac{3\pi}{2} = \frac{10\pi}{6} = \frac{5\pi}{3}$$

$$x = \frac{\pi}{6} + \frac{4\pi}{2} = \frac{13\pi}{6}$$ Oops, too big

# Graphical Representation



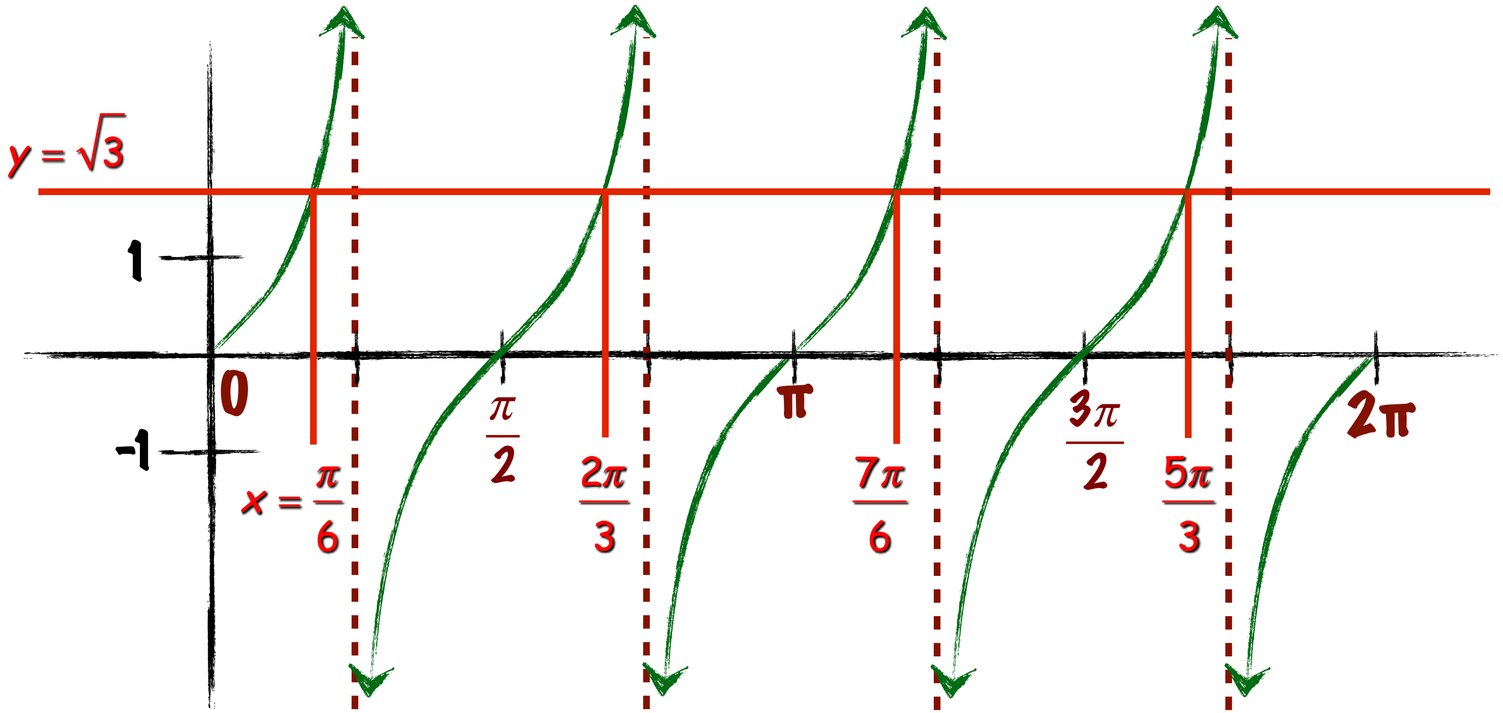
∿ Solve the equation $\tan 2x = \sqrt{3},\ 0 \le x < 2\pi$

**Remember:** We are solving for x, but graphing **tan 2x**.

What is the period of tan 2x?

$$x = \frac{\pi}{6} + \frac{n\pi}{2}$$

$$\frac{\pi}{6} \quad \frac{2\pi}{3} \quad \frac{7\pi}{6} \quad \frac{5\pi}{3}$$



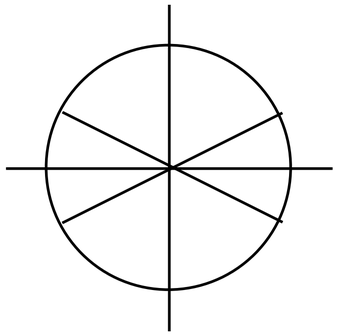
# Solving a Trigonometric Equation in Quadratic Form

Solve the equation $4\cos^2 x - 3 = 0,\ 0 \le x < 2\pi$

$$4\cos^2 x - 3 = 0$$

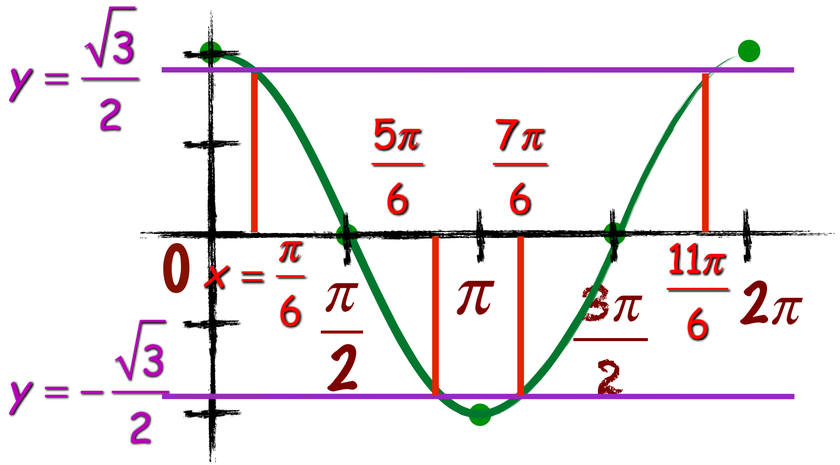
$$\cos^2 x = \frac{3}{4}$$

$$\cos x = \pm\frac{\sqrt{3}}{2}$$



$$\cos^{-1}\frac{\sqrt{3}}{2} = \frac{\pi}{6} \qquad \cos\frac{11\pi}{6} = \frac{\sqrt{3}}{2}$$

$$\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6} \qquad \cos\frac{7\pi}{6} = -\frac{\sqrt{3}}{2}$$




$$x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$$

# Using Factoring to Separate Different Functions

Objective: Solving Trigonometric Equations

∿ Solve the equation $\sin x \tan x = \sin x, 0 \le x < 2\pi$

**Caution:** This is trickier than it looks.

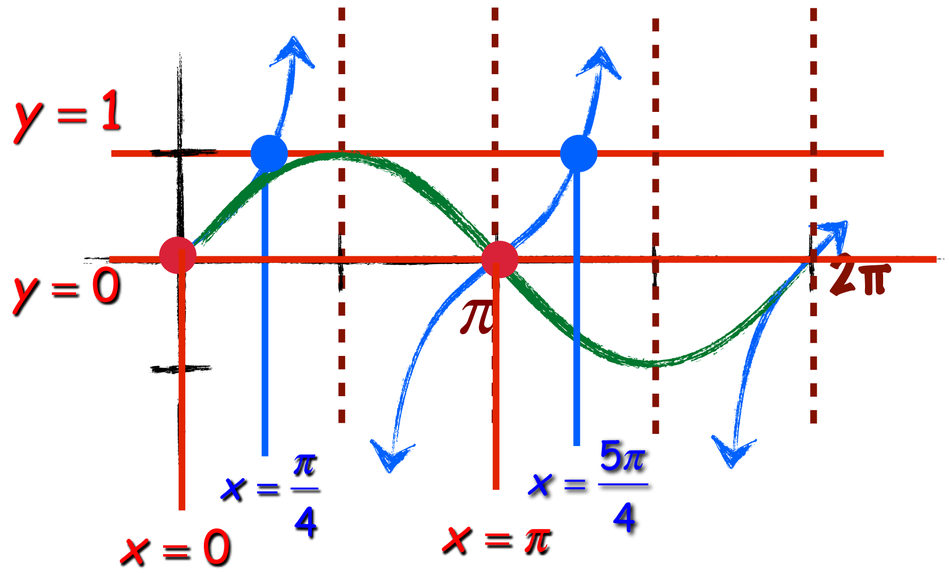
$$\sin x \tan x = \sin x, 0 \le x < 2\pi$$

$$\sin x \tan x - \sin x = 0$$

$$\sin x(\tan x - 1) = 0$$

$$\sin x = 0 \qquad \tan x = 1$$

$$x = 0, \pi \qquad x = \frac{\pi}{4}, \frac{5\pi}{4}$$




$$x = 0, \frac{\pi}{4}, \pi, \frac{5\pi}{4}$$

# Caution

- For the equation $\sin x \tan x = \sin x, 0 \leq x < 2\pi$ do not divide by sinx.
- sinx = 0 was a possible solution and dividing by sinx would lose those solutions.
- If sinx = 0 you cannot divide by sinx (0).

# Using an Identity to Solve a Trigonometric Equation

Solve the equation $\cos 2x + \sin x = 0, 0 \le x < 2\pi$

$$(\cos^2 x - \sin^2 x) + \sin x = 0$$

$$1 - 2\sin^2 x + \sin x = 0$$

$$2\sin^2 x - \sin x - 1 = 0$$

$$(2\sin x + 1)(\sin x - 1) = 0$$

$2\sin x + 1 = 0$

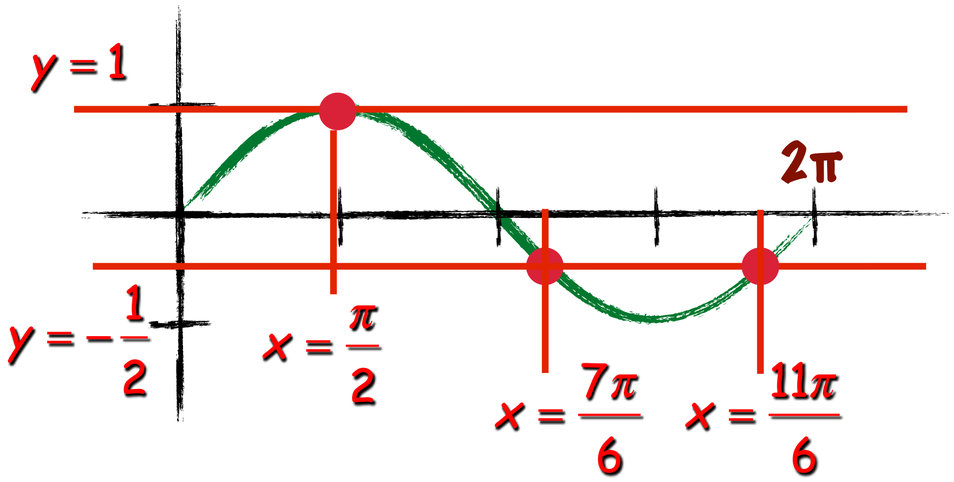
$$\sin x = -\frac{1}{2}$$

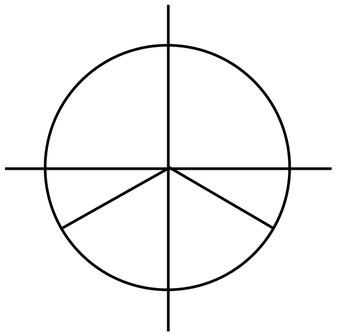
$$x = \frac{7\pi}{6}, \frac{11\pi}{6}$$

$\sin x - 1 = 0$

$$\sin x = 1$$

$$x = \frac{\pi}{2}$$



$$x = \frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$$

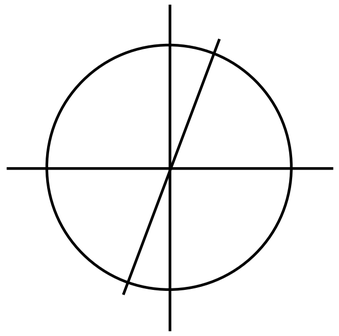
# Solving Trigonometric Equations w/Calculator

Objective: Solving Trigonometric Equations

∿ Solve the equation $\tan x = 3.1044, 0 \le x < 2\pi$

$\tan^{-1} 3.1044 \approx 1.259168376$

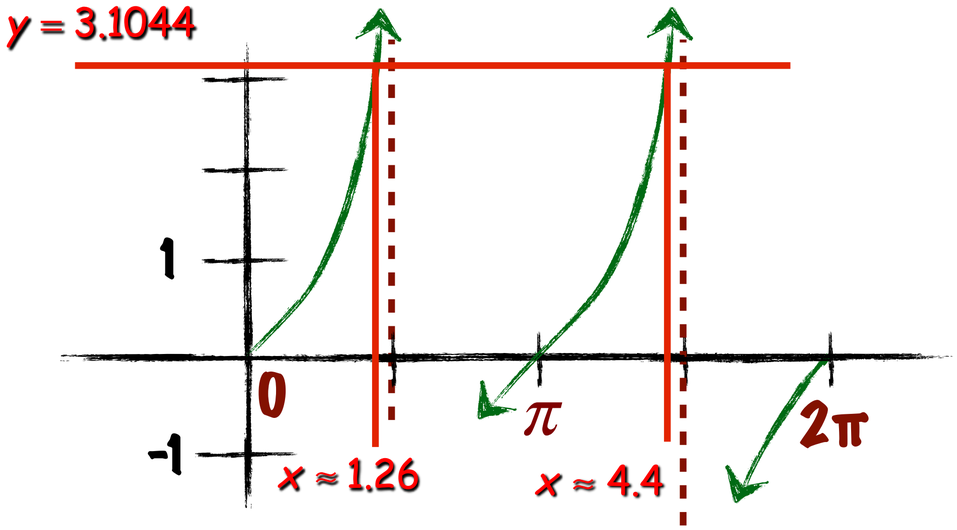
tan is also positive in QIII



$x \approx 1.259168376 + \pi$

$x \approx 4.400761029$

$x \approx 1.2592, 4.4008$



# Solving Trigonometric Equations w/Calculator

Objective: Solving Trigonometric Equations

∿Solve the equation $\sin x = -0.2315, 0 \le x < 2\pi$

Using the calculator gives us:

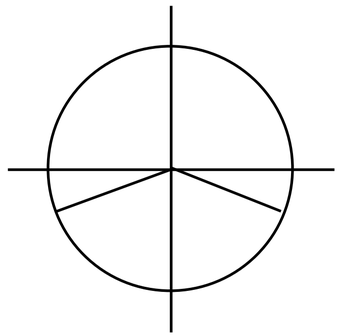
$\sin^{-1}(-0.2315) = -0.233619286$

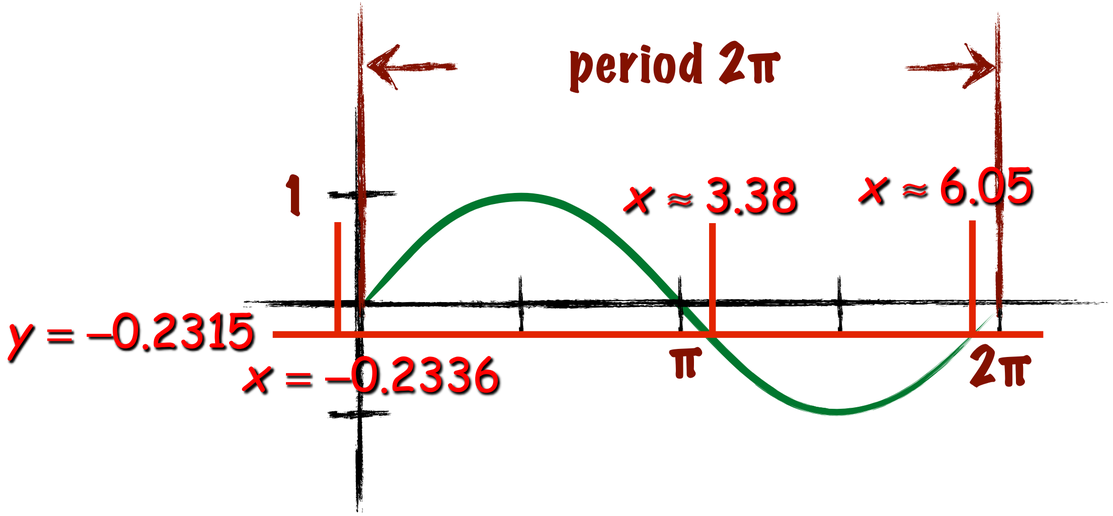
sinx is negative in QIII & QIV

$-0.233619286 + 2\pi = 6.049566021$

$0.233619286 + \pi = 3.37521194$

$x = 3.3752, 6.0496$

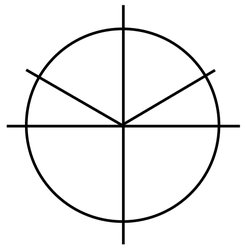




# Solving Equations with a single trigonometric function Trigonometric Equations

∿ Solve the equation $2\sin\frac{x}{2} = 1,\ 0 \le x < 2\pi$

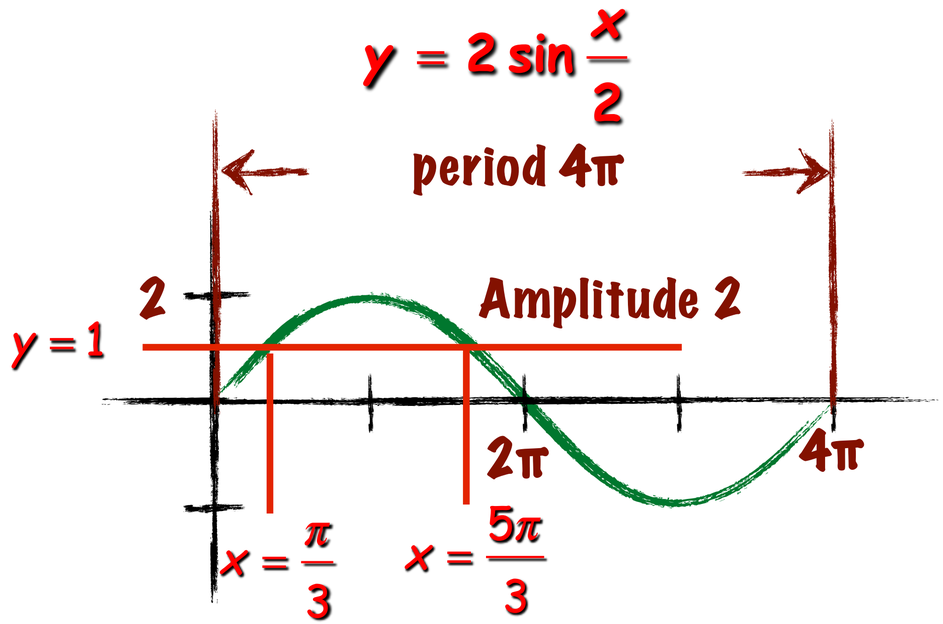
$$\sin\frac{x}{2} = \frac{1}{2}$$



$$\sin\frac{\pi}{6} = \frac{1}{2} \qquad \sin\frac{5\pi}{6} = \frac{1}{2}$$

$$\frac{x}{2} = \frac{\pi}{6} + 2\pi n \qquad \frac{x}{2} = \frac{5\pi}{6} + 2\pi n$$

$$x = \frac{\pi}{3} + 4\pi n \qquad x = \frac{5\pi}{3} + 4\pi n$$



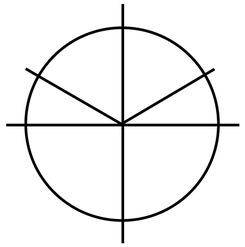

The only values within the restricted domain are: $x = \frac{\pi}{3}, \frac{5\pi}{3}$

# Solving an Equation with a Multiple Angle

Objective: Solving Trigonometric Equations

Solve the equation $\sin\frac{x}{3} = \frac{1}{2}$, $0 \le x < 2\pi$

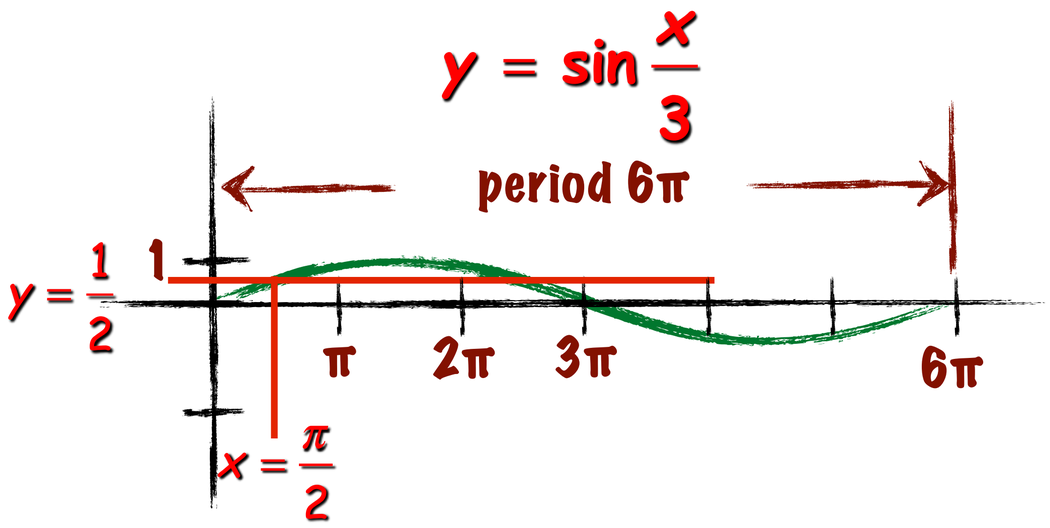
$$\sin\frac{x}{3} = \frac{1}{2}$$



$$\sin\frac{\pi}{6} = \frac{1}{2} \qquad \sin\frac{5\pi}{6} = \frac{1}{2}$$

$$\frac{x}{3} = \frac{\pi}{6} + 2\pi n \qquad \frac{x}{3} = \frac{5\pi}{6} + 2\pi n$$

$$x = \frac{\pi}{2} + 6\pi n \qquad x = \frac{5\pi}{2} + 6\pi n$$




The only value within the restricted domain is: $x = \frac{\pi}{2}$

# Example

Solve the equation $\sin x + \cos x = 1, 0 \le x < 2\pi$

$$\sin x + \cos x = 1$$

$$(\sin x + \cos x)^2 = 1^2$$

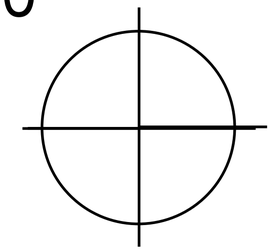
$$\sin^2 x + 2\sin x \cos x + \cos^2 x = 1$$

$$\sin^2 x + \cos^2 x + 2\sin x \cos x = 1$$

$$1 + 2\sin x \cos x = 1$$

$$2\sin x \cos x = 0$$

$$\sin 2x = 0$$



$$\sin 0 = 0$$

$$2x = 0 + n2\pi$$

$$x = 0 + n\pi$$

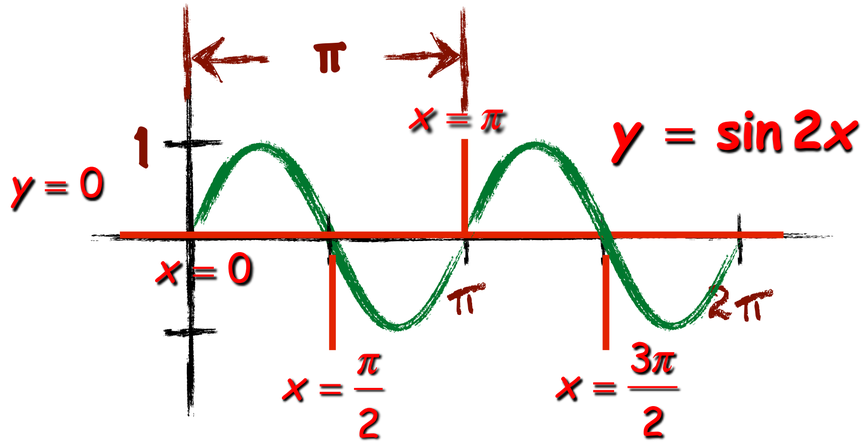
$$x = 0, \pi$$

$$\sin \pi = 0$$

$$2x = \pi + n2\pi$$

$$x = \frac{\pi}{2} + n\pi$$

$$x = \frac{\pi}{2}, \frac{3\pi}{2}$$




$$x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$$

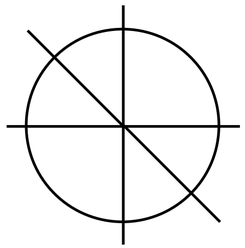
# Example

∿Solve the equation $\tan 2x = -1, 0 \le x < 2\pi$

$$\tan 2x = -1$$

$$\tan\frac{3\pi}{4} = -1$$

$$\tan\frac{7\pi}{4} = -1$$



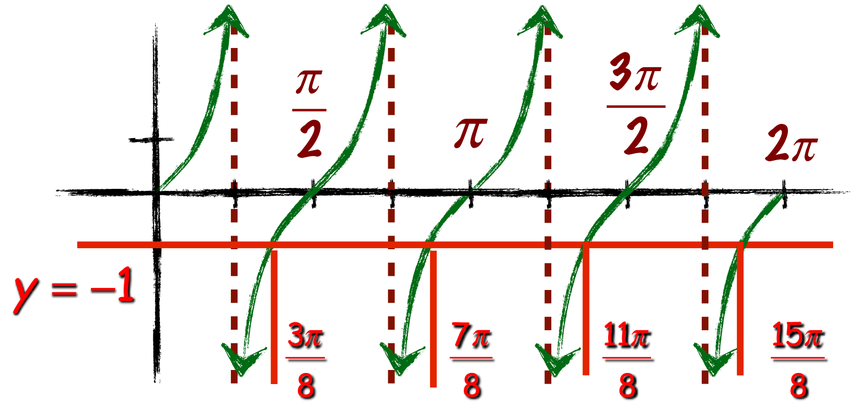
The period for tanθ is π

$$2x = \frac{3\pi}{4} + n\pi \qquad x = \frac{3\pi}{8} + \frac{n\pi}{2}$$

$$2x = \frac{7\pi}{4} + n\pi \qquad x = \frac{7\pi}{8} + \frac{n\pi}{2}$$

$$x = \frac{3\pi}{8} + \frac{0\pi}{2} \qquad x = \frac{3\pi}{8} \qquad x = \frac{3\pi}{8} + \frac{1\pi}{2} \qquad x = \frac{7\pi}{8}$$

$$x = \frac{3\pi}{8} + \frac{2\pi}{2} \qquad x = \frac{11\pi}{8} \qquad x = \frac{3\pi}{8} + \frac{3\pi}{2} \qquad x = \frac{15\pi}{8}$$




∿Repeats

$$x = \frac{3\pi}{8}, \frac{7\pi}{8}, \frac{11\pi}{8}, \frac{15\pi}{8}$$

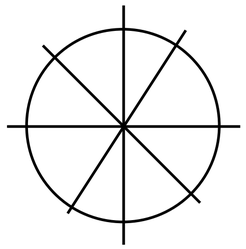
# Example

Solve the equation $\tan^2 x - \tan x - 2 = 0,\ 0 \le x < 2\pi$

$$\tan^2 x - \tan x - 2 = 0$$

$$(\tan x - 2)(\tan x + 1) = 0$$

$$\tan x = 2 \qquad \tan x = -1$$



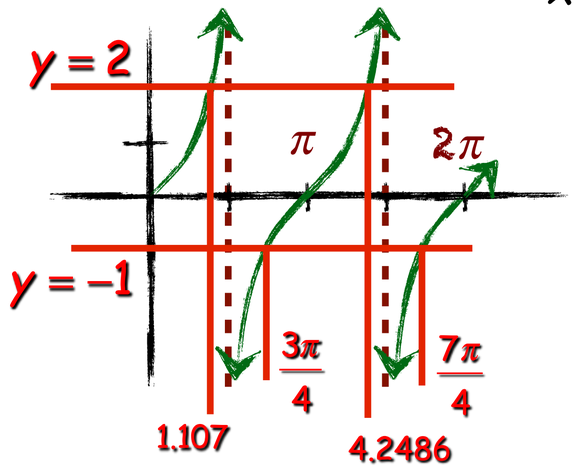
$$x = \tan^{-1} 2 \approx 1.107 \qquad x = \tan^{-1}(-1) \approx \frac{3\pi}{4}$$

The period for tan is $\pi$

$$x \approx 1.107 + n\pi \qquad x = \frac{3\pi}{4} + n\pi$$

n = 0 $\quad x = 1.107 \quad x = \frac{3\pi}{4}$

n = 1 $\quad x = 4.2486 \quad x = \frac{7\pi}{4}$




$$x \approx 1.107, \frac{3\pi}{4}, 4.2486, \frac{7\pi}{4}$$

# Example

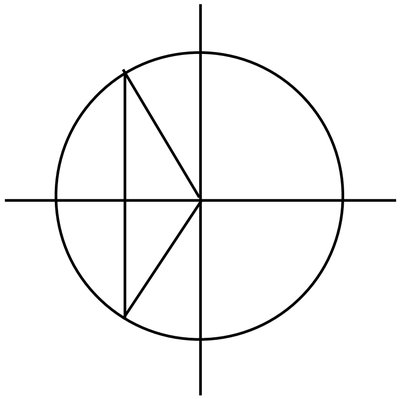
Solve the equation $4\cos^2 x + 4\cos x = -1,\ 0 \le x < 2\pi$

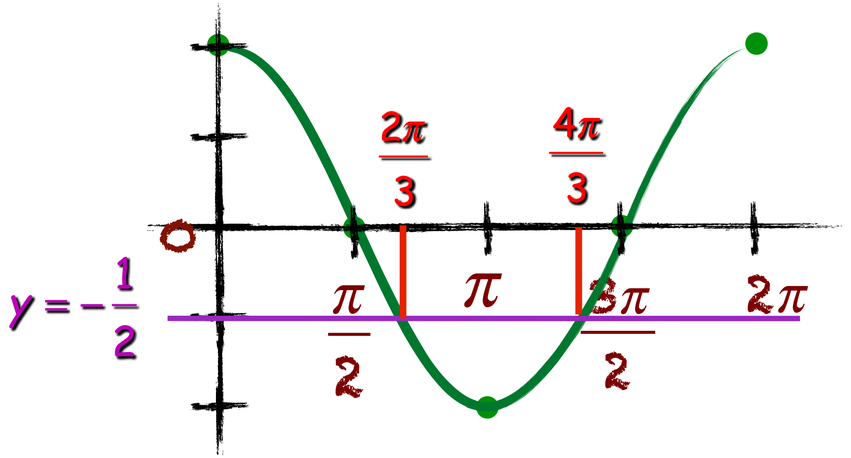
$$4\cos^2 x + 4\cos x + 1 = 0$$

$$(2\cos x + 1)^2 = 0$$

$$2\cos x = -1$$

$$\cos x = -\frac{1}{2}$$






$$x = \cos^{-1}\left(\frac{-1}{2}\right) = \frac{2\pi}{3} \qquad x = \cos^{-1}\left(\frac{-1}{2}\right) = \frac{4\pi}{3}$$

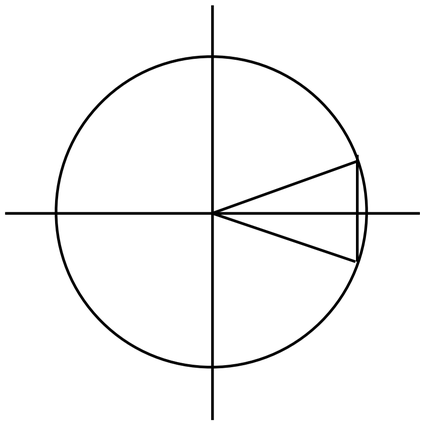
$$x = \frac{2\pi}{3}, \frac{4\pi}{3}$$

# Example

Solve the equation $5\sec^2 x = 6\sec x,\ 0^\circ \le x < 360^\circ$

$$5\sec^2 x - 6\sec x = 0$$

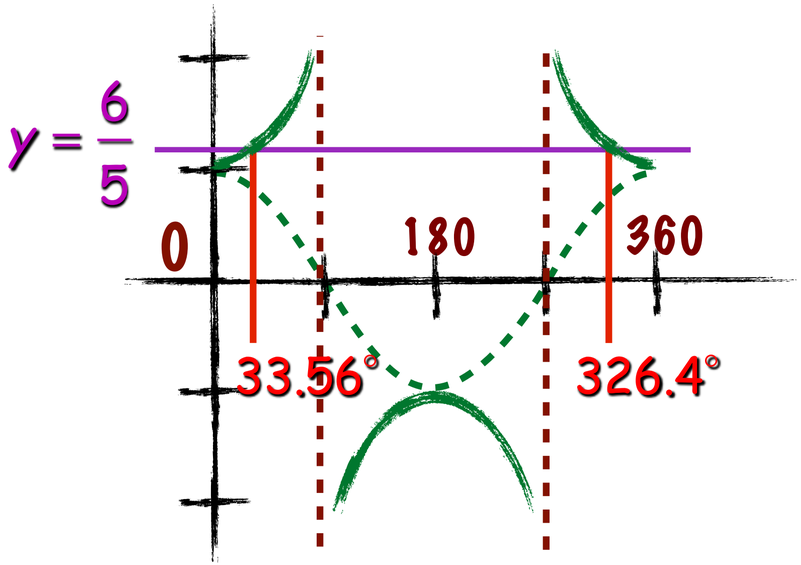
$$\sec x(5\sec x - 6) = 0$$



$\sec x = 0$ ~~$\sec x = 0$~~ $\qquad 5\sec x = 6$

$$\sec x = \frac{6}{5}$$

$$x = \cos^{-1}\left(\frac{5}{6}\right) \approx 33.5573^\circ$$




$$x \approx 33.5573, 326.4427$$

# Example

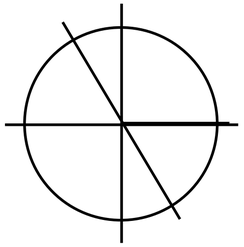
Solve the equation $\cos 2x = \cos x \quad 0 \le x < 2\pi$

$$2\cos^2 x - 1 = \cos x$$

$$2\cos^2 x - \cos x - 1 = 0$$

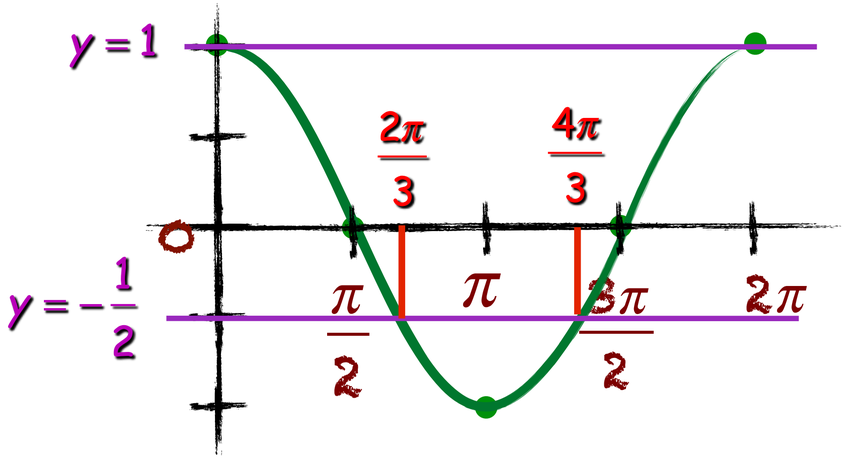
$$(2\cos x + 1)(\cos x - 1) = 0$$

$$\cos x = -\frac{1}{2} \qquad \cos x = 1$$



$$x = \cos^{-1}\left(\frac{-1}{2}\right) = \frac{2\pi}{3} \qquad x = \cos^{-1}(1) = 0$$

$$x = \frac{2\pi}{3}, \frac{4\pi}{3} \qquad x = 0$$




$$x = 0, \frac{2\pi}{3}, \frac{4\pi}{3}$$

# Example

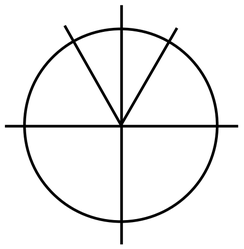
Solve the equation $4\sin x\cos x = \sqrt{3}, 0 \le x < 2\pi$

$$2(2\sin x\cos x) = \sqrt{3}$$

$$2\sin x\cos x = \frac{\sqrt{3}}{2}$$

$$\sin 2x = \frac{\sqrt{3}}{2}$$

$$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2} \qquad \sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$$

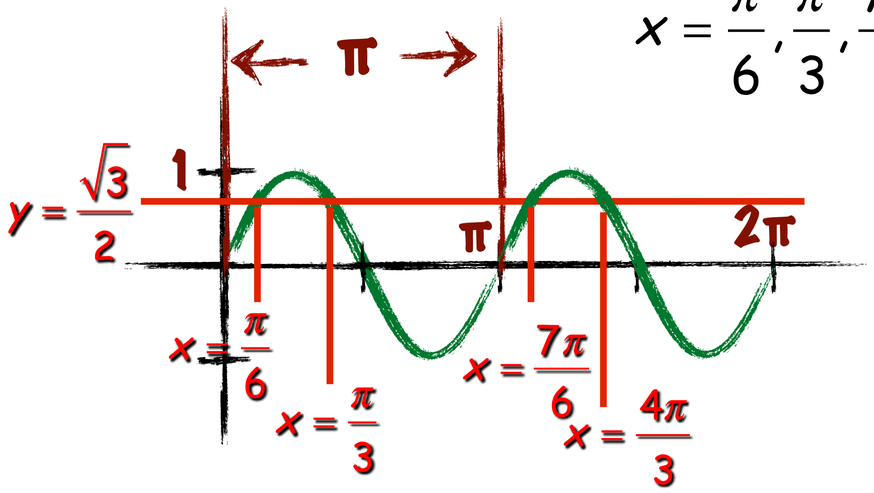


$$2x = \frac{\pi}{3} + n2\pi \qquad 2x = \frac{2\pi}{3} + n2\pi$$

$$x = \frac{\pi}{6} + n\pi \qquad x = \frac{\pi}{3} + n\pi$$

$$x = \frac{\pi}{6} + 0\pi \quad x = \frac{\pi}{6} \quad x = \frac{\pi}{6} + 1\pi \quad x = \frac{7\pi}{6}$$

$$x = \frac{\pi}{3} + 0\pi \quad x = \frac{\pi}{3} \quad x = \frac{\pi}{3} + 1\pi \quad x = \frac{4\pi}{3}$$




$$x = \frac{\pi}{6}, \frac{\pi}{3}, \frac{7\pi}{6}, \frac{4\pi}{3}$$

# Example

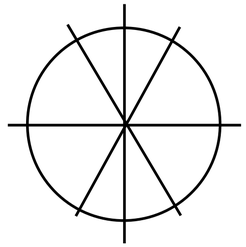
∿ Solve $3\cot^2 x - 1 = 0$ for all values of x.

$$3\cot^2 x - 1 = 0$$

$$\cot^2 x = \frac{1}{3}$$

$$\cot x = \pm\sqrt{\frac{1}{3}} = \pm\frac{1}{\sqrt{3}}$$

$$\tan x = \pm\sqrt{3}$$



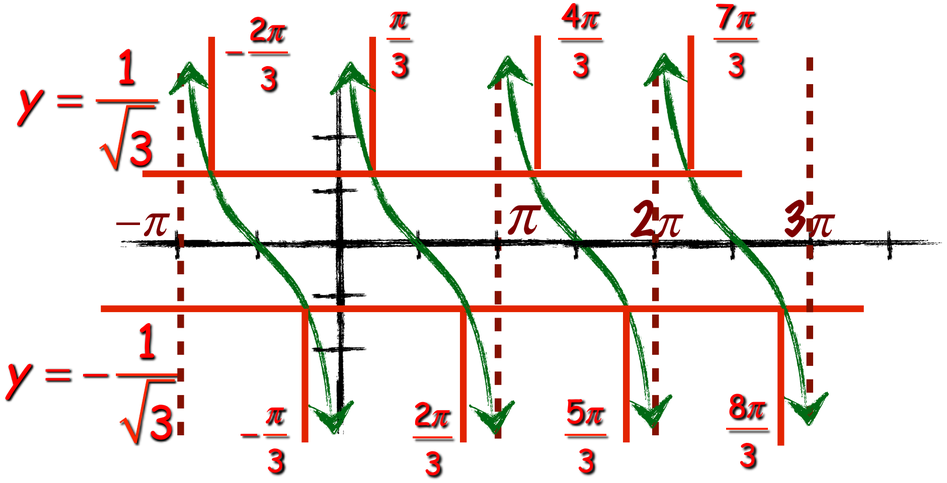
$$x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$$

$$x = \frac{\pi}{3} \pm n\pi$$

$$x = \frac{2\pi}{3} \pm n\pi$$

$$x = \frac{4\pi}{3} \pm n\pi$$

$$x = \frac{5\pi}{3} \pm n\pi$$




$$x = \frac{\pi}{3} \pm n\pi \text{ or } x = \frac{2\pi}{3} \pm n\pi$$

∿ So $\frac{4\pi}{3}$ and $\frac{5\pi}{3}$ are redundant.

# Example

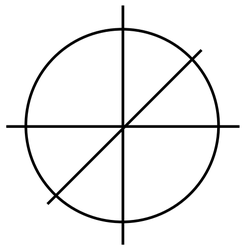
∿ Solve $\tan\left(\frac{\theta}{2}-\frac{\pi}{3}\right)-1=0$ for all values of θ.

$\tan\left(\frac{\theta}{2}-\frac{\pi}{3}\right)-1=0$

$\tan\left(\frac{\theta}{2}-\frac{\pi}{3}\right)=1$

$\left(\frac{\theta}{2}-\frac{\pi}{3}\right)=\frac{\pi}{4},\frac{5\pi}{4}\pm n\pi$

$\frac{\pi}{4}+\pi=\frac{5\pi}{4}$ ∿ So $\frac{5\pi}{4}$ is redundant.



$\frac{\theta}{2}-\frac{\pi}{3}=\frac{\pi}{4}\pm n\pi$

$\frac{\theta}{2}=\frac{\pi}{4}+\frac{\pi}{3}\pm n\pi$

$\frac{\theta}{2}=\frac{7\pi}{12}\pm n\pi$

$\theta=\frac{7\pi}{6}\pm n2\pi$