# CHAPTER 8.5

## CRAMER'S RULE

# HOMEWORK

READ SEC 8.5

P628 1, 7, 11, 15, 23, 25, 29, 31, 37, 41

# OBJECTIVES

STUDENTS WILL KNOW HOW TO USE CRAMER'S RULE TO SOLVE SYSTEMS OF LINEAR EQUATIONS, AND HOW TO USE DETERMINANTS AND MATRICES TO MODEL AND SOLVE PROBLEMS.

# DETERMINANT

- We have solved systems of linear equations through the use of matrices. It turns out that the solution to the system $\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases}$

- has solution $\left( \dfrac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1}, \dfrac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1} \right)$ $x = \dfrac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1}$ and $y = \dfrac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$

- The denominator of the solution coordinates ($a_1b_2 - a_2b_1$) turns out to be the **determinant** of the coefficient matrix of the system.

# CRAMER'S RULE

- To solve the system $\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases}$

$$x = \frac{D_x}{D} = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{c_1b_2 - b_1c_2}{a_1b_2 - b_1a_2} \qquad y = \frac{D_y}{D} = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{a_1c_2 - c_1a_2}{a_1b_2 - b_1a_2}$$

- The denominator is the determinant of the coefficient matrix. $D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$

Replace x column with constants.

$$D_x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$$

Replace y column with constants.

$$D_y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

# EXAMPLE

- Use Cramer's Rule to solve $\begin{cases} 2x+3y=4 \\ 6x-5y=1 \end{cases}$

$$D=\begin{vmatrix} 2 & 3 \\ 6 & -5 \end{vmatrix}=-10-18=-28 \qquad D_x=\begin{vmatrix} 4 & 3 \\ 1 & -5 \end{vmatrix}=-20-3=-23 \qquad D_y=\begin{vmatrix} 2 & 4 \\ 6 & 1 \end{vmatrix}=2-24=-22$$

$$x=\frac{D_x}{D}=\frac{-23}{-28}=\frac{23}{28} \qquad y=\frac{D_y}{D}=\frac{-22}{-28}=\frac{11}{14}$$

- The solution is $\left(\frac{23}{28},\frac{11}{14}\right)$

# EXAMPLE

- Use Cramer's Rule to solve $\begin{cases} 5x + 4y = 12 \\ 3x - 6y = 24 \end{cases}$

$$D = \begin{vmatrix} 5 & 4 \\ 3 & -6 \end{vmatrix} = -30 - 12 = -42 \quad D_x = \begin{vmatrix} 12 & 4 \\ 24 & -6 \end{vmatrix} = -72 - 96 = -168 \quad D_y = \begin{vmatrix} 5 & 12 \\ 3 & 24 \end{vmatrix} = 120 - 36 = 84$$

$$x = \frac{D_x}{D} = \frac{-168}{-42} = 4 \qquad y = \frac{D_y}{D} = \frac{84}{-42} = -2$$

- The solution is $(4, -2)$

# HIGHER ORDER SYSTEMS

- We can generalize Cramer's Rule to any system of any number of equations (n).
- If the coefficient matrix A of a system of n equations with n variables is not 0, then the coordinates of the solution can be found by:

$$x_i = \frac{|A_i|}{|A|}$$

- where $x_i$ is the $i^{th}$ coordinate of the solution and $A_i$ is the matrix found by substituting the constants into the $i^{th}$ column.

**Cramer's Rule**

If a system of $n$ linear equations in $n$ variables has a coefficient matrix $A$ with a nonzero determinant $|A|$, the solution of the system is

$$x_1 = \frac{|A_1|}{|A|}, \quad x_2 = \frac{|A_2|}{|A|}, \quad \ldots, \quad x_n = \frac{|A_n|}{|A|}$$

where the $i$th column of $A_i$ is the column of constants in the system of equations. If the determinant of the coefficient matrix is zero, the system has either no solution or infinitely many solutions.

# THREE VARIABLE SYSTEM

- Given the system $\begin{cases} a_1x+b_1y+c_1z=d_1 \\ a_2x+b_2y+c_2z=d_2 \\ a_3x+b_3y+c_3z=d_3 \end{cases}$ $D=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ $\begin{cases} a_1x+b_1y+c_1z=d_1 \\ a_2x+b_2y+c_2z=d_2 \\ a_3x+b_3y+c_3z=d_3 \end{cases}$

Replace x column with constants.

$$x=\frac{D_x}{D}=\frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$$

Replace y column with constants.

$$y=\frac{D_y}{D}=\frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$$

Replace z column with constants.

$$z=\frac{D_z}{D}=\frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$$

# THREE VARIABLE SYSTEM

- The four third order determinants are found by

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

- Coefficient matrix consisting of the coefficients of the system.

$$D_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$

- x matrix replacing x column with the constants of the system.

$$D_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$$

- y matrix replacing y column with the constants of the system.

$$D_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

- z matrix replacing z column with the constants of the system.

# EXAMPLE

- Use Cramer's Rule to solve the system $\begin{cases} 3x - 2y + z = 16 \\ 2x + 3y - z = -9 \\ x + 4y + 3z = 2 \end{cases}$

$$D = \begin{vmatrix} 3 & -2 & 1 \\ 2 & 3 & -1 \\ 1 & 4 & 3 \end{vmatrix} = 3 \bullet (-1)^{1+1} \begin{vmatrix} 3 & -1 \\ 4 & 3 \end{vmatrix} + 2 \bullet (-1)^{2+1} \begin{vmatrix} -2 & 1 \\ 4 & 3 \end{vmatrix} + 1 \bullet (-1)^{3+1} \begin{vmatrix} -2 & 1 \\ 3 & -1 \end{vmatrix}$$

$$= 3(13) - 2(-10) + 1(-1) = 58$$

$$D_x = \begin{vmatrix} 16 & -2 & 1 \\ -9 & 3 & -1 \\ 2 & 4 & 3 \end{vmatrix} = 16 \bullet (-1)^{1+1} \begin{vmatrix} 3 & -1 \\ 4 & 3 \end{vmatrix} + -9 \bullet (-1)^{2+1} \begin{vmatrix} -2 & 1 \\ 4 & 3 \end{vmatrix} + 2 \bullet (-1)^{3+1} \begin{vmatrix} -2 & 1 \\ 3 & -1 \end{vmatrix}$$

$$= 16(13) - -9(-10) + 2(-1) = 116$$

$$D_y = \begin{vmatrix} 3 & 16 & 1 \\ 2 & -9 & -1 \\ 1 & 2 & 3 \end{vmatrix} = 3 \bullet (-1)^{1+1} \begin{vmatrix} -9 & -1 \\ 2 & 3 \end{vmatrix} + 2 \bullet (-1)^{2+1} \begin{vmatrix} 16 & 1 \\ 2 & 3 \end{vmatrix} + 1 \bullet (-1)^{3+1} \begin{vmatrix} 16 & 1 \\ -9 & -1 \end{vmatrix}$$

$$= 3(-25) - 2(46) + 1(-7) = -174$$

$$D_z = \begin{vmatrix} 3 & -2 & 16 \\ 2 & 3 & -9 \\ 1 & 4 & 2 \end{vmatrix} = 3 \bullet (-1)^{1+1} \begin{vmatrix} 3 & -9 \\ 4 & 2 \end{vmatrix} + 2 \bullet (-1)^{2+1} \begin{vmatrix} -2 & 16 \\ 4 & 2 \end{vmatrix} + 1 \bullet (-1)^{3+1} \begin{vmatrix} -2 & 16 \\ 3 & -9 \end{vmatrix}$$

$$= 3(42) - 2(-68) + 1(-30) = 232$$

# EXAMPLE

$$D = \begin{vmatrix} 3 & -2 & 1 \\ 2 & 3 & -1 \\ 1 & 4 & 3 \end{vmatrix} = 3(13) - 2(-10) + 1(-1) = 58$$

$$D_x = \begin{vmatrix} 16 & -2 & 1 \\ -9 & 3 & -1 \\ 2 & 4 & 3 \end{vmatrix} = 16(13) - -9(-10) + 2(-1) = 116 \qquad x = \frac{D_x}{D} = \frac{116}{58} = 2$$

$$D_y = \begin{vmatrix} 3 & 16 & 1 \\ 2 & -9 & -1 \\ 1 & 2 & 3 \end{vmatrix} = 3(-25) - 2(46) + 1(-7) = -174 \qquad y = \frac{D_y}{D} = \frac{-174}{58} = -3$$

$$D_z = \begin{vmatrix} 3 & -2 & 16 \\ 2 & 3 & -9 \\ 1 & 4 & 2 \end{vmatrix} = 3(42) - 2(-68) + 1(-30) = 232 \qquad z = \frac{D_z}{D} = \frac{232}{58} = 4$$

- The solution is

$$(2, -3, 4)$$

# EXAMPLE

- Use Cramer's Rule to solve the system $\begin{cases} 3x-2y+4z=0 \\ 2x-8y=2 \\ 4x-3y-5z=1 \end{cases}$

$$D=\begin{vmatrix} 3 & -2 & 4 \\ 2 & -8 & 0 \\ 4 & -3 & -5 \end{vmatrix} = 3\bullet(-1)^{1+1}\begin{vmatrix} -8 & 0 \\ -3 & -5 \end{vmatrix} + 2\bullet(-1)^{2+1}\begin{vmatrix} -2 & 4 \\ -3 & -5 \end{vmatrix} + 4\bullet(-1)^{3+1}\begin{vmatrix} -2 & 4 \\ -8 & 0 \end{vmatrix}$$

$$=3(40)-2(22)+4(32)=204$$

$$D_x=\begin{vmatrix} 0 & -2 & 4 \\ 2 & -8 & 0 \\ 1 & -3 & -5 \end{vmatrix} = 0\bullet(-1)^{1+1}\begin{vmatrix} -8 & 0 \\ -3 & -5 \end{vmatrix} + 2\bullet(-1)^{2+1}\begin{vmatrix} -2 & 4 \\ -3 & -5 \end{vmatrix} + 1\bullet(-1)^{3+1}\begin{vmatrix} -2 & 4 \\ -8 & 0 \end{vmatrix}$$

$$=0(40)-2(22)+1(32)=-12$$

$$D_y=\begin{vmatrix} 3 & 0 & 4 \\ 2 & 2 & 0 \\ 4 & 1 & -5 \end{vmatrix} = 3\bullet(-1)^{1+1}\begin{vmatrix} 2 & 0 \\ 1 & -5 \end{vmatrix} + 2\bullet(-1)^{2+1}\begin{vmatrix} 0 & 4 \\ 1 & -5 \end{vmatrix} + 1\bullet(-1)^{3+1}\begin{vmatrix} 0 & 4 \\ 2 & 0 \end{vmatrix}$$

$$=3(-10)-2(-4)+4(-8)=-54$$

$$D_z=\begin{vmatrix} 3 & -2 & 0 \\ 2 & -8 & 2 \\ 4 & -3 & 1 \end{vmatrix} = 3\bullet(-1)^{1+1}\begin{vmatrix} -8 & 2 \\ -3 & 1 \end{vmatrix} + 2\bullet(-1)^{2+1}\begin{vmatrix} -2 & 0 \\ -3 & 1 \end{vmatrix} + 4\bullet(-1)^{3+1}\begin{vmatrix} -2 & 0 \\ -8 & 2 \end{vmatrix}$$

$$=3(-2)-2(-2)+4(-4)=-18$$

# EXAMPLE

$$D = \begin{vmatrix} 3 & -2 & 4 \\ 2 & -8 & 0 \\ 4 & -3 & -5 \end{vmatrix} = 3(40) - 2(22) + 4(32) = 204$$

$$D_x = \begin{vmatrix} 0 & -2 & 4 \\ 2 & -8 & 0 \\ 1 & -3 & -5 \end{vmatrix} = 0(40) - 2(22) + 1(32) = -12 \qquad x = \frac{D_x}{D} = \frac{-12}{204} = \frac{-1}{17}$$

$$D_y = \begin{vmatrix} 3 & 0 & 4 \\ 2 & 2 & 0 \\ 4 & 1 & -5 \end{vmatrix} = 3(-10) - 2(-4) + 4(-8) = -54 \qquad y = \frac{D_y}{D} = \frac{-54}{204} = \frac{-9}{34}$$

$$D_z = \begin{vmatrix} 3 & -2 & 0 \\ 2 & -8 & 2 \\ 4 & -3 & 1 \end{vmatrix} = 3(-2) - 2(-2) + 4(-4) = -18 \qquad z = \frac{D_z}{D} = \frac{-18}{204} = \frac{-3}{34}$$

- The solution is

$$\left(\frac{-1}{17}, \frac{-9}{34}, \frac{-3}{34}\right)$$

# CONSISTENT / INCONSISTENT

- You can use Cramer's rule to determine if the system represented by the matrix has one solution, no solution, or infinitely many solutions.

- If $D \neq 0$, the system is consistent and has one unique solution (a point).

- If $D = 0$, **and at least one numerator determinant is 0**, the system is dependent and has infinitely many solutions (a line).

- If $D = 0$, and no numerator determinant is 0, the system is inconsistent and has no solution.

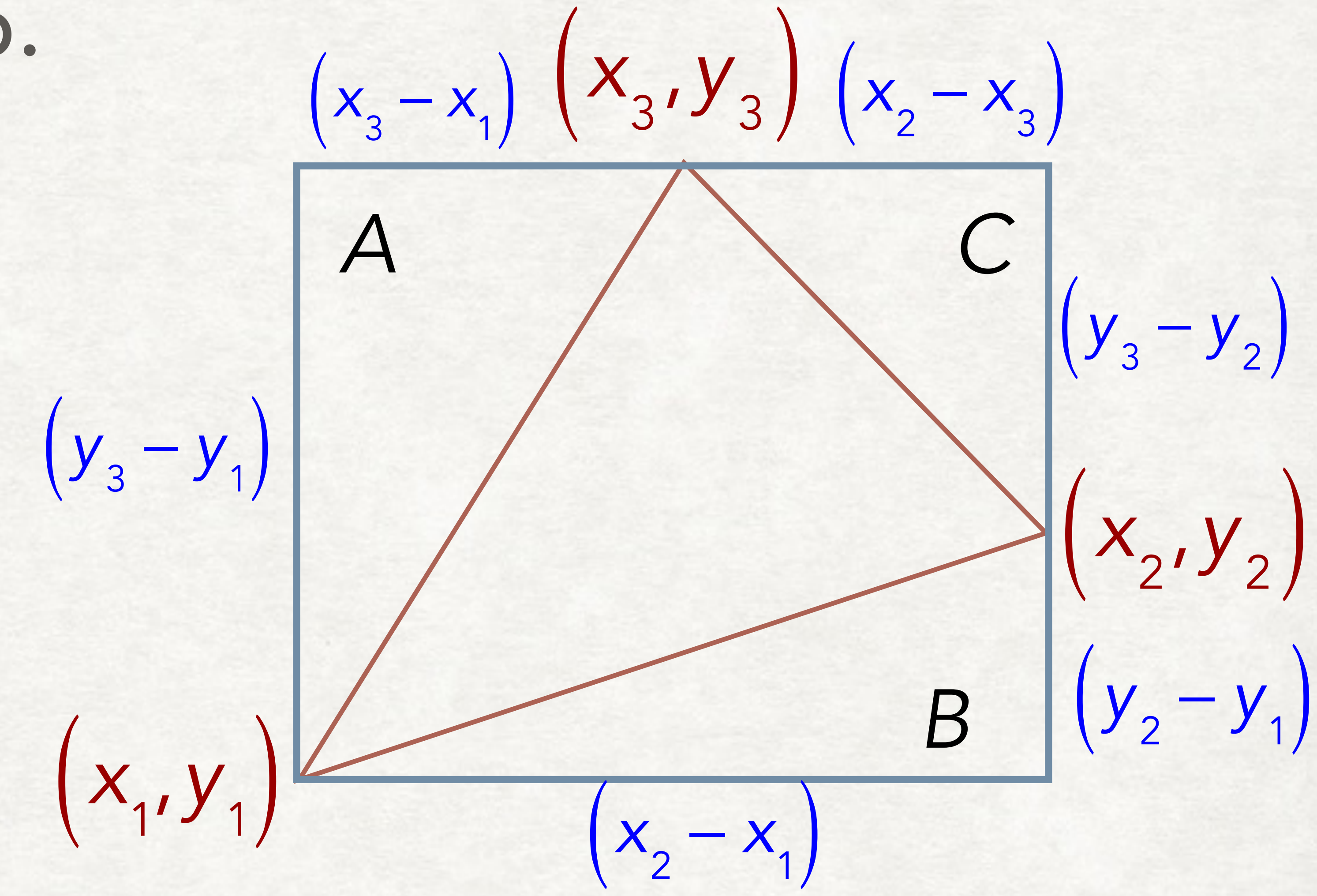
# AREA OF A TRIANGLE

- We can use matrices to find the area of a triangle given the coordinates of the vertices of the triangle.

- The derivation of the formula involves Heron's Formula and placing the triangle in a rectangle circumscribed about the triangle. It is far to involved to do here, but if you care to try to derive the formula, here is the setup.

  - Using Heron's Formula find the sum of the areas of triangles A, B, C. Simplify using a lot of algebra.

  - Find the difference in area of rectangle and sum of the triangles. Simplify with a lot of algebra.




  - Find the determinant of the matrix I am about to give you and verify that the results match the formula from above.

# AREA OF A TRIANGLE

- The area of a triangle with vertices $(x_1, y_1)$, $(x_2, y_2)$, $(x_3, y_3)$ is:

$$Area = \pm\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \quad \begin{matrix} \text{vertex 1} \\ \text{vertex 2} \\ \text{vertex 3} \end{matrix}$$

- The ± is to ensure the value is positive, area cannot be negative. We might also use the absolute value.

  - The area of a triangle with vertices $(x_1, y_1)$, $(x_2, y_2)$, $(x_3, y_3)$ is: $Area = \left|\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}\right|$

# EXAMPLE

- Find the area of a triangle with vertices (1, –1), (2, 3), (5, –3).

x coordinates

y coordinates

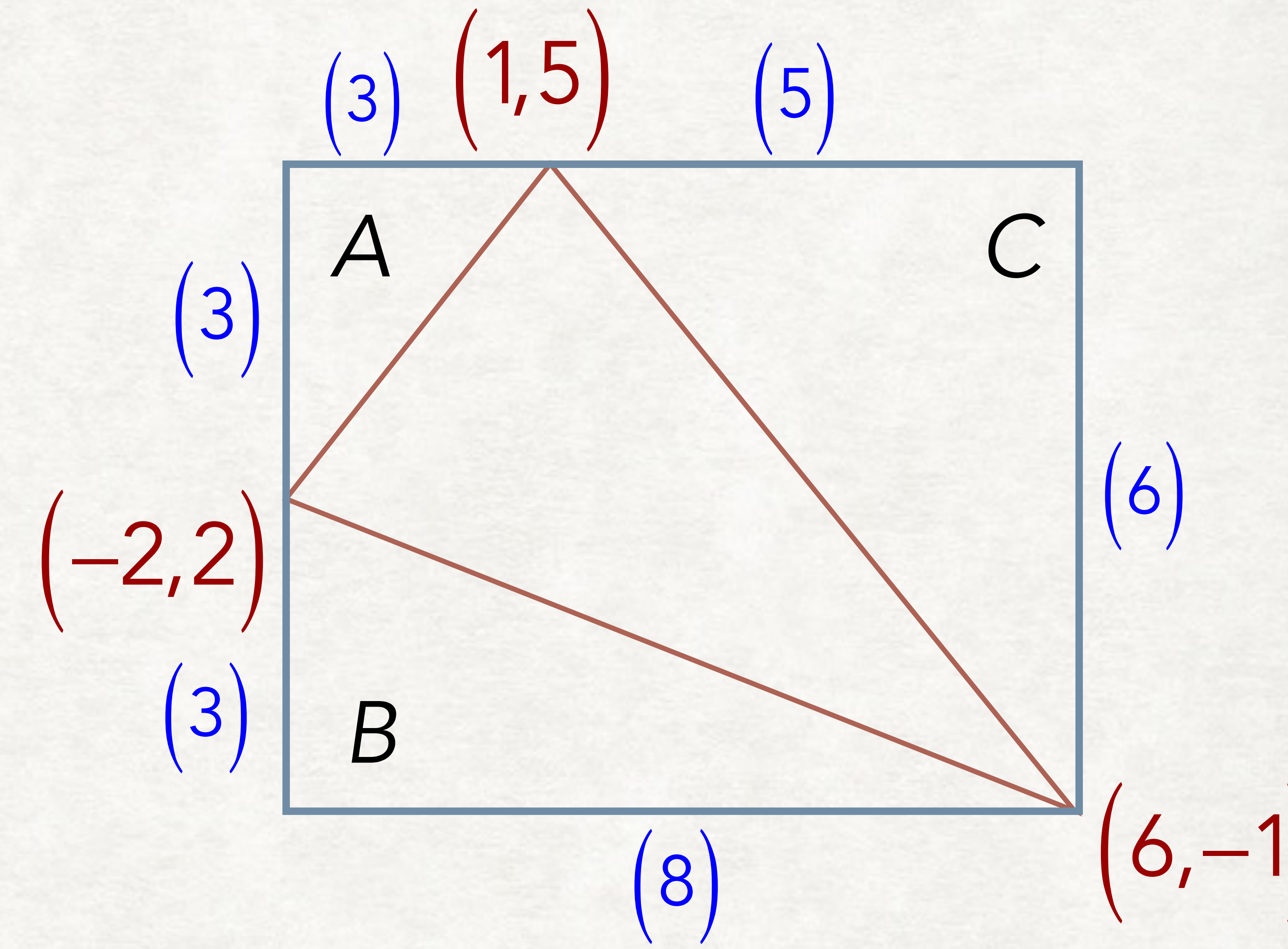
$$Area = \pm\frac{1}{2}\begin{vmatrix} 1 & -1 & 1 \\ 2 & 3 & 1 \\ 5 & -3 & 1 \end{vmatrix} = \pm\frac{1}{2}\left(1\cdot(-1)^{1+3}\begin{vmatrix} 2 & 3 \\ 5 & -3 \end{vmatrix} + 1\cdot(-1)^{2+3}\begin{vmatrix} 1 & -1 \\ 5 & -3 \end{vmatrix} + 1\cdot(-1)^{3+3}\begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix}\right)$$

$$= \pm\frac{1}{2}\left(1(-21) - 1(2) + 1(5)\right) = 9$$

- The area of a triangle with vertices (1, –1), (2, 3), (5, –3) is 9 sq units.

# EXAMPLE

- Find the area of the triangle with vertices (-2, 2), (1, 5) and (6, -1).
  - $A_{rect} = 6 \cdot 8 = 48$
    - $A_{\Delta A} = \frac{1}{2} \cdot 3 \cdot 3 = \frac{9}{2}$
    - $A_{\Delta B} = \frac{1}{2} \cdot 3 \cdot 8 = 12$
    - $A_{\Delta C} = \frac{1}{2} \cdot 5 \cdot 6 = 15$
  - $A_{\Delta} = 48 - \frac{9}{2} - 12 - 15 = \frac{33}{2}$




$$Area = \pm\frac{1}{2}\begin{vmatrix} -2 & 2 & 1 \\ 1 & 5 & 1 \\ 6 & -1 & 1 \end{vmatrix} = \pm\frac{1}{2}\left(1\cdot(-1)^{1+3}\begin{vmatrix} 1 & 5 \\ 6 & -1 \end{vmatrix} + 1\cdot(-1)^{2+3}\begin{vmatrix} -2 & 2 \\ 6 & -1 \end{vmatrix} + 1\cdot(-1)^{3+3}\begin{vmatrix} -2 & 2 \\ 1 & 5 \end{vmatrix}\right)$$

$$= \pm\frac{1}{2}\left(1(-31) - 1(-10) + 1(-12)\right) = \frac{33}{2}$$

- The area of a triangle with vertices (-2, 2), (1, 5), (6, –1) is 16.5 sq units.

# COLLINEAR POINTS

- You can also find the equation of a line on a plane. If you use the formula in the previous slide for finding the area of a triangle and the result is 0 the points do not form a triangle, but are actually collinear. We can use that fact to test for the collinearity of 3 points, $(x_1, y_1)$, $(x_2, y_2)$, $(x_3, y_3)$.

- If 3 points, $(x_1, y_1)$, $(x_2, y_2)$, $(x_3, y_3)$ are collinear, then:

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

- Determine if the points (0,1), (4, 4) and (8, 7) are collinear.

$$\begin{vmatrix} 0 & 1 & 1 \\ 4 & 4 & 1 \\ 8 & 7 & 1 \end{vmatrix} = 1 \bullet (-1)^{1+3} \begin{vmatrix} 4 & 4 \\ 8 & 7 \end{vmatrix} + 1 \bullet (-1)^{2+3} \begin{vmatrix} 0 & 1 \\ 8 & 7 \end{vmatrix} + 1 \bullet (-1)^{3+3} \begin{vmatrix} 0 & 1 \\ 4 & 4 \end{vmatrix}$$

$$= 1(-4) - 1(-8) + 1(-4) = 0$$

- Yeperdoo, the points are collinear.

# EQUATION OF A LINE

- From the previous example, if we were to replace one to the points with (x, y), we would get an equation of the line determined by the two points.

- Find the equation of the line containing the points (5,2), and (–1, 0).

$$\begin{vmatrix} x & y & 1 \\ 5 & 2 & 1 \\ -1 & 0 & 1 \end{vmatrix} = 0 \qquad \begin{vmatrix} x & y & 1 \\ 5 & 2 & 1 \\ -1 & 0 & 1 \end{vmatrix} = 1 \bullet (-1)^{1+3} \begin{vmatrix} 5 & 2 \\ -1 & 0 \end{vmatrix} + 1 \bullet (-1)^{2+3} \begin{vmatrix} x & y \\ -1 & 0 \end{vmatrix} + 1 \bullet (-1)^{3+3} \begin{vmatrix} x & y \\ 5 & 2 \end{vmatrix} = 0$$

$$= 2 + y + 5y - 2x = 0$$

$$= 2 + 6y - 2x = 0$$

- The equation of the line formed by the points (5,2) and (–1, 0) is 2x – 6y = 2.

# EQUATION OF A LINE

- It is probably better to find the determinant by using the first row.

- Find the equation of the line containing the points (-3,7), and (2, 2).

$$\begin{vmatrix} x & y & 1 \\ -3 & 7 & 1 \\ 2 & 2 & 1 \end{vmatrix} = 0 \qquad \begin{vmatrix} x & y & 1 \\ -3 & 7 & 1 \\ 2 & 2 & 1 \end{vmatrix} = x \bullet (-1)^{1+3} \begin{vmatrix} 7 & 1 \\ 2 & 1 \end{vmatrix} + y \bullet (-1)^{2+3} \begin{vmatrix} -3 & 1 \\ 2 & 1 \end{vmatrix} + 1 \bullet (-1)^{3+3} \begin{vmatrix} -3 & 7 \\ 2 & 2 \end{vmatrix} = 0$$

$$= 5x + 5y - 20 = 0$$

- The equation of the line formed by the points (-3,7), and (2, 2) is 5x + 5y = 20.

# ONE LAST POINT

- Use your calculator to find the determinant of $A = \begin{bmatrix} 3 & -2 & 4 \\ 2 & -8 & 0 \\ 4 & -3 & -5 \end{bmatrix}$

$$\det A = \begin{vmatrix} 3 & -2 & 4 \\ 2 & -8 & 0 \\ 4 & -3 & -5 \end{vmatrix} = 204$$

- Use your calculator to find the determinant of $B = \begin{bmatrix} 2 & -8 & 0 \\ 3 & -2 & 4 \\ 4 & -3 & -5 \end{bmatrix}$

$$\det B = \begin{vmatrix} 2 & -8 & 0 \\ 3 & -2 & 4 \\ 4 & -3 & -5 \end{vmatrix} = -204$$

- When you interchange consecutive rows in a matrix, the sign of the determinant will change.