The Rational Zero Test

The ultimate objective for this section of the workbook is to graph polynomial functions of degree greater than 2. The first step in accomplishing this will be to find all real zeros of the function. As previously stated, the zeros of a function are the x intercepts of the graph of that function. Also, the zeros of a function are the roots of the equation that makes up that function. You should remember, the only difference between an polynomial equation and a polynomial function is that one of them has $f_{(x)}$.

You will be given a polynomial equation such as $2x^4+7x^3-4x^2-27x-18=0$***, and be asked to find all roots of the equation.***

The Rational Zero Test states that all possible rational zeros are given by the factors of the constant over the factors of the leading coefficient.

$$\frac{factors\ \ of\ \ the\ \ constant}{factors\ \ of\ \ the\ \ leading\ \ coefficient}=all\ \ possible\ \ rational\ \ zeros$$

Let's find all possible rational zeros of the equation $2x^4+7x^3-4x^2-27x-18=0$.

We begin with the equation $2x^4+7x^3-4x^2-27x-18=0$.

The constant of this equation is 18, while the leading coefficient is 2. We do not care about the (-) sign in front of the 18.

Writing out all factors of the constant over the factors of the leading coefficient gives the following.

$$\frac{\pm 1,\ \ \pm 2,\ \ \pm 3,\ \ \pm 6,\ \ \pm 9,\ \ \pm 18}{\pm 1,\ \ \pm 2}$$

These are not all possible rational zeros. To actually find them, take each number on top, and write it over each number in the bottom. If one such number occurs more than once, there is no need to write them both.

$$\pm 1,\ \ \pm 2,\ \ \pm 3,\ \ \pm 6,\ \ \pm 9,\ \ \pm 18,\ \ \pm\frac{1}{2},\ \ \pm\frac{3}{2},\ \ \pm\frac{9}{2}$$

These are all possible rational zeros for this particular equation.

The order in which you write this list of numbers is not important. The rational zero test is meant to assist in the overall objective of finding all zeros to the polynomial equation $2x^4+7x^3-4x^2-27x-18=0$. ***Each of these numbers is a potential root of the equation. Therefore, each will eventually be tested.***

Descarte's Rule of Signs

When solving these polynomial equations use the rational zero test to find all possible rational zeros first. Synthetic division will then be used to test each one of these possible zeros, until some are found that work. When we found all possible rational zeros of the equation $2x^4 + 7x^3 - 4x^2 - 27x - 18 = 0$***, there were 18 possible solutions to this equation. It could take a very long time to test each one. Luckily there is a rule to help narrow down these choices. Descarte's Rule of Signs can help to narrow the search of possible solutions to the equation.***

Descarte's Rule of Signs

- **The number of positive zeros can be found by counting the number of sign changes in the problem. The number of positive zeros is that number, or less by an even integer.**
- **The number of negative zeros can be found by evaluating** $f_{(-x)}$**. Count the number of sign changes, and the number of negative zeros is that number, or less by an even integer.**

When using Descarte's Rule of Signs, "less by an even integer," means subtract by two until there is 1 or 0 possible zeros.

Here is an example of how to use Descarte's rule of Signs to determine the possible number of positive and negative zeros for the equation $2x^4 + 7x^3 - 4x^2 - 27x - 18 = 0$**.**

To find the number of positive roots, count the number of sign changes in $2x^4 + 7x^3 - 4x^2 - 27x - 18 = 0$.

The signs only change once in the original equation, so there is only 1 positive zero.

Evaluating $f_{(-x)}$ ***results in*** $2x^4 - 7x^3 - 4x^2 + 27x - 18 = 0$. ***To evaluate*** $f_{(-x)}$, ***substitute –x for x. When this is done, only the terms were variables are being raised to odd powers change signs.***

Here, the signs changed 3 times. That means there are either 3 or 1 negative zeros.

Knowledge of complex roots will be used in conjunction with Descarte's Rule of Signs to create a table of possible combinations. Remember, COMPLEX NUMBERS ALWAYS COME IN CONJUGATE PAIRS when solving equations.

The Remainder Theorem

When trying to find all zeros of a complex polynomial function, use the rational zero test to find all possible rational zeros. Each possible rational zero should then be tested using synthetic division. If one of these numbers work, there will be no remainder to the division problem. For every potential zero that works, there may be others that do not. Are these just useless? The answer is no. Every time synthetic division is attempted, we are actually evaluating the value of the function at the given x coordinate. When there is no remainder left, a zero of the function has just been found. This zero is an x intercept for the graph of the function. If the remainder is any other number, a set of coordinates on the graph has just been found. These coordinates would aid in graphing the function.

Let $P(x)$ **be a polynomial of positive degree n. Then for any number** c,

$$P(x) = Q(x)\cdot(x-c) + P(c),$$

Where $Q(x)$ **is a polynomial of degree n-1.**

This simply means that if a polynomial $P(x)$ ***is divided by*** $(x-c)$ ***using synthetic division, the resultant remainder is*** $P(c)$.

When trying to find the zeros of the function $f_{(x)} = 2x^4 + 7x^3 - 4x^2 - 27x - 18$, ***first find all possible rational zeros. Then evaluate each one. Here is one particular example.***

$$\begin{array}{r|rrrrr} -2 & 2 & 7 & -4 & -27 & -18 \\ & 0 & -4 & -6 & 20 & 14 \\ \hline & 2 & 3 & -10 & -7 & -4 \end{array}$$

In this example, (-2) is evaluated using synthetic division to see if it was a zero of the function. It turns out that (-2) is not a zero of the function, because there is a remainder of (-4).

Therefore, Using the Remainder Theorem, it can be stated that $f(-2) = -4$.

You already saw that dividing by (-2) yields a result of (-4), giving us the statement: $f(-2) = -4$

This can be proven algebraically as follows.

$$f(-2) = 2(-2)^4 + 7(-2)^3 - 4(-2)^2 - 27(-2) - 18$$

$$f(-2) = 32 - 56 - 16 + 54 - 18$$

$$f(-2) = -4$$

Finding all Zeros of a Polynomial Function

When solving polynomial equations, use the rational zero test to find all possible rational zeros, then use Descarte's Rule of Signs to help narrow down the choices if possible. The fundamental theorem of Algebra plays a major role in this.

The Fundamental Theorem of Algebra

Every polynomial equation of degree n with complex coefficients has n roots in the complex numbers.

In other words, if you have a 5th degree polynomial equation, it has 5 roots.

Example: Find all zeros of the polynomial function $f(x) = 2x^4 + 7x^3 - 4x^2 - 27x - 18$.

$$2x^4 + 7x^3 - 4x^2 - 27x - 18 = 0$$

Begin by setting the function equal to zero.

Find all possible rational zeros.

$$\pm 1,\quad \pm 2,\quad \pm 3,\quad \pm 6,\quad \pm 9,\quad \pm 18,\quad \pm\frac{1}{2},\quad \pm\frac{3}{2},\quad \pm\frac{9}{2}$$

Once again, there are 18 possible zeros to the function. If Descarte's Rule of Signs is used, it may or may not help narrow down the choices for synthetic division.

For this equation, there is 1 possible positive zero, and either 3 or 1 possible negative zeros.

This information was found in a previous example. Based on this, a chart may be constructed showing the possible combinations. Remember, this is a 4th degree polynomial, so each row must add up to 4.

+	-	*i*
1	***3***	***0***
1	***1***	***2***

$$\pm 1,\quad \pm 2,\quad \pm 3,\quad \pm 6,\quad \pm 9,\quad \pm 18,\quad \pm\frac{1}{2},\quad \pm\frac{3}{2},\quad \pm\frac{9}{2}$$

Now set up a synthetic division problem, and begin checking each zero until a root of the equation is found..

$$\begin{array}{|rrrrr} 2 & 7 & -4 & -27 & -18 \\ 0 & & & & \\ \hline \end{array}$$

$$\begin{array}{r|rrrrr} -1 & 2 & 7 & -4 & -27 & -18 \\ & 0 & -2 & -5 & 9 & 18 \\ \hline & 2 & 5 & -9 & -18 & 0 \end{array}$$

-1 works as a zero of the function. There are now 3 zeros left. We can continue to test each zero, but we need to first rewrite the new polynomial.

$2x^3 + 5x^5 - 9x - 18$

The reason this must be done is to check using the rational zero test again. Using the rational zero test again could reduce the number of choices to work with, or the new polynomial may be factorable.

$$\begin{array}{r|rrrr} -3 & 2 & 5 & -9 & -18 \\ & 0 & -6 & 3 & 18 \\ \hline & 2 & -1 & -6 & 0 \end{array}$$

Here, we found that -3 works. The reason negative numbers are being used first here is because of the chart above. The chart says there is a greater chance of one of the negatives working rather than a positive, since there are potentially 3 negative zeros here and only one positive. Notice the new equation was used for the division.

$$2x^2 - x - 6$$

$$(2x+3)(x-2) = 0$$

$$x = -3/2 \quad and \quad x = 2$$

This is now a factorable polynomial. Solve by factoring.

We now have all zeros of the polynomial function. They are $-3/2,\quad -1,\quad -3 \quad and \quad 2$

Be aware, the remaining polynomial may not be factorable. In that case, it will be necessary to either use the quadratic formula, or complete the square.

Finding the Equation of a Polynomial Function

In this section we will work backwards with the roots of polynomial equations or zeros of polynomial functions. As we did with quadratics, so we will do with polynomials greater than second degree. Given the roots of an equation, work backwards to find the polynomial equation or function from whence they came. Recall the following example.

Find the equation of a parabola that has x intercepts of $(-3,0)$ *and* $(2,0)$.

$(-3,0)$ *and* $(2,0)$.	***Given x intercepts of -3 and 2***
$x=-3 \quad x=2$	***If the x intercepts are -3 and 2, then the roots of the equation are -3 and 2. Set each root equal to zero.***
$(x+3)\ (x-2)$	***For the first root, add 3 to both sides of the equal sign.*** ***For the second root, subtract 2 to both sides of the equal sign.***
x^2+x-6	***Multiply the results together to find a quadratic expression.***
$y=x^2+x-6$	***Set the expression equal to y, or*** $f(x)$***, to write as the equation of a parabola.***

The exercises in this section will result in polynomials greater than second degree. Be aware, you may not be given all roots with which to work.

Consider the following example:

Find a polynomial function that has zeros of $0,\ 3$ *and* $2+3i$***. Although only three zeros are given here, there are actually four. Since complex numbers always come in conjugate pairs,*** $2-3i$ ***must also be a zero. Using the fundamental theorem of algebra, it can be determined that his is a 4th degree polynomial function.***

Take the zeros of $0,\ 3,\ 2\pm3i$***, and work backwards to find the original function.***

$x=0$	$x=3$	$x=2\pm3i$
$x=0$	$x=3$ $-3 \quad -3$ $x-3=0$	$x=2\pm3i$ $x-2=\pm3i$ $x^2-4x+4=9i^2$ $x^2-4x+4=-9$ $x^2-4x+13=0$
x	$(x-3)$	$(x^2-4x+13)$

The polynomial function with zeros of $0,\ 3,\ 2\pm3i$***, is equal to*** $f(x)=x(x-3)(x^2-4x+13)$***. Multiplying this out will yield the following.***

$$f(x)=x^4-7x^3+25x^2-39x$$

<u>Even vs. Odd Functions</u>

One of your many tasks in future mathematics courses will be to determine whether a function is even, odd or neither. This is very simple to do.

A function is even if $f(-x) = f(x)$

This means if a (-x) is substituted into the problem, and no signs change, the function is even.

A function is odd if $f(-x) = -f(x)$

In this case, a (-x) is substituted into the problem, and all signs change. If all signs change, this is an odd function.

If only some of the signs change, the function is neither even nor odd.

Even functions are symmetrical to the y axis.
Odd functions are symmetrical to the origin.

<u>Left and Right Behaviors of Polynomial Functions</u>

If the degree of the polynomial is even, the graph of the function will have either "both sides up", or "both sides down."

If the degree of the polynomial is odd, the graph of the function will have one side up and one side down.

As to which side is up and which is down, that all depends on the leading coefficient.

Refer to the following.

	Even Degree	***Odd Degree***
+ leading coefficient	↑↑	↓↑
- leading coefficient	↓↓	↑↓

Therefore, a 7th degree polynomial function having a leading coefficient that is negative, will rise on the left, and fall to the right.

In contrast, if the 7th degree polynomial has a positive leading coefficient, the graph of the function will fall on the left, and rise on the right hand side.

These rules are for polynomial functions in a single variable only!

When we graph these polynomial functions, the first step will be to find all zeros of the function. Once the x intercepts have been found, plot them on the x axis, and refer to the two intercepts on the ends. At this point, use the rules for left and right behaviors of functions to draw a portion of the graph.

Practice Test Polynomial Equations/Functions

Divide $(2x^3 - 4x^2 - x + 12) \div (x - 2)$ using synthetic and long division

Be able to perform poly long division/ synthetic division

Given $f(x) = 6x^3 - 2x^2 + 3x - 2$ find $f(-2)$

(Use synthetic substitution)

Find all possible rational zeros of $f(x) = 2x^5 - 4x^4 + 6x^3 - 2x + 12$

(Rational zero test)

How many zeros does the function have? $f(x) = 6x^4 - 12x^3 + 8x^2 - 7x + 2$

(Corollary to the Fundamental Theorem of Algebra)

Find the simplest polynomial function that has zeros of : $1, \quad 3, \quad and \quad -\sqrt{2}$

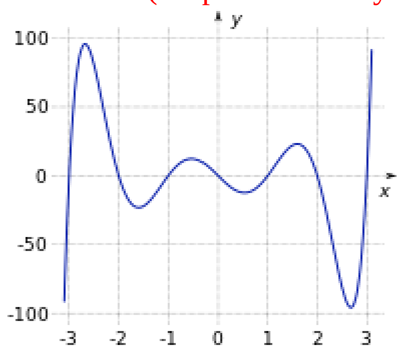
(Corollary to the Fundamental Theorem of Algebra)

Determine the left and right behaviors of the polynomial function: $f(x) = -7x^5 + 3x^4 - 2x^2 - 18$

(Properties of Polynomial Functions)

Graph the following polynomial function by finding all zeros and using the rules of left/right behavior and multiplicity. $f(x) = x^4 + 4x^3 - 16x^2 - 16x + 48$ (Properties of Polynomial Functions)

Identify whether the function graphed has an odd or even degree and a positive or negative leading coefficient. (Properties of Polynomial Functions)



A student used synthetic division for a problem shown on the right. Using the problem on the right, find the values of a, b and c.

$a =$ $b =$ $c =$

-2	3	a	5	c
	0	-6	b	6
	3	4	-3	15

a) Briefly explain the process by which you would determine whether or not $x-6$ a factor of $3x^3-16x^2-72$? (Be able to perform poly long division/ synthetic division)

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